

Grätzer-Schmidt Theorem in arithmetical transfinite recursion

Soowhan Yoon

February 13, 2025

Abstract

We assess the reverse mathematical strength of the Grätzer-Schmidt theorem (GS) as a principle in second order arithmetic. The theorem GS was studied in an article by Katie Brodhead, Mushfeq Khan, Bjørn Kjos-Hanssen, William A. Lampe, Paul Kim Long V. Nguyen, and Richard A. Shore, where they establish the provability of GS in Π_1^1 comprehension ($\Pi_1^1\text{-CA}_0$) and its restrictive variant GSD in arithmetical comprehension (ACA_0). It will be shown that the arithmetical transfinite recursion (ATR_0) is sufficient to prove GS. Additionally, other variants of GS will be explored as well. Some will be proved in ACA_0 , while others will be shown equivalent to ATR_0 over ACA_0 . Then, we will discuss these results in the context of “Almost Theorems of Hyperarithmetical Analysis” (ATHA) by Shore in 2023.