On the Causal Diamond

Valentin Vankov Iliev

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"...These are mostly statements about the causal structure of spacetime where can one get, from a given starting point, along a worldline that is everywhere inside the local light cone? The causal structure is what is new in Lorentz signature General Relativity relative to the Euclidean signature case which is more visible in everyday life".
(from *Black Holes, Singularity Theorems, and All That*, Edward Witten, IAS, PiTP 2018)

A *pseudo-Euclidean space* is a bilinear space L of dimension $n \geq 2$, such that the corresponding quadratic form has index of inertia $n-1$. The structural bilinear form (x, y) is called the *inner product* of the pseudo-Euclidean space L. In this case there are nonzero vectors $x \in L$ with $(x^2) = 0$ and then the restriction of (x, y) to the one-dimensional subspace $\langle x \rangle$ is identically equal to zero, that is, $\langle x \rangle$ is a *degenerate* subspace of L. A nonzero vector $x \in L$ is said to be *space-like* (respectively, *light-like*, *time-like*) if $(x^2) > 0$ (respectively, $(x^2) = 0$, $(x^2) < 0$). The sets $V = \{x \in L \mid (x^2) = 0\}$, $\text{int}(V) = \{x \in L \mid (x^2) < 0\}$, and $\text{ext}(V) = \{x \in L \mid (x^2) > 0\}$, are said to be *light cone*, *interior of the light cone*, and *exterior of the light cone*, respectively. For any time-like vector $x \in L$ we set $\tau(x) = \sqrt{-(x^2)}$ and call this number *time duration of* x.

Any Euclidean hyperplane E through the origin divides the pseudo-Euclidean space L into two open half-spaces, thus producing a division of the light cone V (respectively, its interior $\mathrm{int}(V)$) into two *poles* V_+ and V_- (respectively, $\mathrm{int}(V)_+$ and $\mathrm{int}(V)_-$) which do not depend on the choice of the hyperplane. The two normal unit vectors e_n and $-e_n$ to E are time-like $((e_n^2) = -1)$. The coordinate cross $E \oplus \langle e_n \rangle = L$, $E^\perp = \langle e_n \rangle$, defines *space coordinates* (after choosing an orthonormal basis $e_1, \ldots, e_{n-1}$ in E) and, after fixing e_n, the *time coordinate* (in $\langle e_n \rangle$) as well as the positive sign pole via the relation $e_n \in \mathrm{int}(V)_+$. Then the pole $\mathrm{int}(V)_+$ is said to be the *future cone* of the origin O and the pole $\mathrm{int}(V)_-$ is said to be the *past cone* of the origin O. The vectors in $\mathrm{int}(V)_+$ (respectively, in $\mathrm{int}(V)_-$) are called *future directed* (respectively, *past directed*). The points of the affine space (L, L) are called *events*. The *future cone* (respectively, the *past cone*) of the event $x \in L$ is $V_+(x) = x + \mathrm{int}(V)_+$ (respectively, $V_-(x) = x + \mathrm{int}(V)_-$).

Let us consider the simplest possible pseudo-Euclidean space L with $\dim(L) = 2$. In this case there exists an orthonormal basis e_1, e_2 of L: $(e_1^2) = 1$, $(e_2^2) = -1$, $(e_1, e_2) = 0$, that is, the scalar square of the vector $x = x_1 e_1 + x_2 e_2$ has the form $(x^2) = x_1^2 - x_2^2$. Its polar bilinear form is $(x, y) = x_1 y_1 - x_2 y_2$. There exists a basis φ_1, φ_2, consisting of light-like vectors $\varphi_1 = \frac{e_1 + e_2}{2}$, $\varphi_2 = \frac{e_1 - e_2}{2}$. We have $(\varphi_1^2) = (\varphi_2^2) = 0$, $(\varphi_1, \varphi_2) = \frac{1}{2}$, and the scalar square of the vector $\xi = \xi_1 \varphi_1 + \xi_2 \varphi_2$ is the quadratic form $(\xi^2) = \xi_1 \xi_2$. Its polar bilinear form is $(\xi, \eta) = \frac{1}{2}(\xi_1 \eta_2 + \xi_2 \eta_1)$. The light cone V consists of two *light-like lines*: $V = \langle \varphi_1 \rangle \cup \langle \varphi_2 \rangle$. Its interior $\mathrm{int}(V)$ contains the coordinate vector e_2 (the time vector) and its exterior $\mathrm{ext}(V)$ contains the coordinate vector e_1 (the space vector).

Theorem

Let L be a pseudo-Euclidean space and let $L' \subset L$ be a non-degenerate subspace. Then the subspace L' is either Euclidean or pseudo-Euclidean.

Given two vectors $x, y \in L$, we denote by $G(x, y)$ their Gram determinant: $G(x, y) = (x^2)(y^2) - (x, y)^2$.

Theorem

Let x, y be time-like vectors in a pseudo-Euclidean space.
(i) *The reverse of Cauchy-Schwarz inequality is satisfied:*
$G(x, y) \leq 0$.
(ii) *The reverse of triangle inequality is satisfied:*
$\tau(x + y) \geq \tau(x) + \tau(y)$.
In the above inequalities the equality holds if and only if x and y are proportional.

Pseudo-Euclidean Spaces: Time-like World-lines

Let L be a pseudo-Euclidean space and let $[a, b] \subset \mathbb{R}$, $a < b$, be a closed interval. A curve $\alpha\colon [a, b] \to L$ is said to be smooth *world-line* if in the representation $\alpha(t) = \sum_{i=1}^{n} \alpha_i(t)e_i$ ($e_1, \ldots, e_n$ is an orthonormal basis for L) all $\alpha_i(t)$, $i = 1, \ldots, n$, are continuously differentiable functions in $t \in [a, b]$. Any smooth world-line α represents "travel" from the point $x = \alpha(a)$ to the point $y = \alpha(b)$. The world-line α is said to be *time-like* if the velocity vectors $v^{(\alpha)}(t) = \frac{\mathrm{d}\alpha}{\mathrm{d}t}(t) = \sum_{i=1}^{n} \frac{\mathrm{d}\alpha_i}{\mathrm{d}t}(t)e_i$ are time-like vectors for all $t \in I$. If, in addition, all velocity vectors $v^{(\alpha)}(t)$ are future directed, the world-line α is called *future directed*. If the "traveller" is material, then its world-line is time-like and future directed.

Pseudo-Euclidean Spaces: Time-like World-lines

Theorem

Let x, y be two different points in L and let $\alpha\colon [a,b] \to L$ be a smooth, time-like, and future-directed world-line with $x = \alpha(a)$ and $y = \alpha(b)$. The following two statements are equivalent and hold:
(i) *The displacement vector $y - x$ is time-like and future-directed.*
(ii) *If $a < a' < b' < b$ and $x' = \alpha(a')$, $y' = \alpha(b')$, then the displacement vector $y' - x'$ is time-like and future-directed.*

Corollary

Any smooth, time-like, and future-directed world-line from x to y is contained in the intersection $\operatorname{int} V(x)_+ \cap \operatorname{int} V(y)_-$.

In accord with the previous theorem, the displacement vector $e = y - x$ is time-like and future-directed. In terms of the metric topology of L, the closure of the open set $\operatorname{int} V(x)_+ \cap \operatorname{int} V(y)_-$ is denoted by $\Diamond(x, y)$ and called *causal diamond corresponding to the events x and y*. This closure includes parts of the future cone of x and the past cone of y and their intersection is a conic section of rank $n - 1$. According to the above corollary, the compact $\Diamond(x, y)$ is
"... a stage, and all the men and women merely players; They have their exits and their entrances; And one man in his time plays many parts, His acts being seven ages... That ends this strange eventful history, Is second childishness and mere oblivion; Sans teeth, sans eyes, sans taste, sans everything." (William Shakespeare, As You Like It).

The world-line α is said to be *causal path* if the velocity vectors $v^{(\alpha)}(t) = \frac{\mathrm{d}\alpha}{\mathrm{d}t}(t) = \sum_{i=1}^{n} \frac{\mathrm{d}\alpha_i}{\mathrm{d}t}(t) e_i$ are time-like or light-like vectors for all $t \in I$. If, in addition, all velocity vectors $v^{(\alpha)}(t)$ are future directed, the casual path α is called *future directed*. It turns out that all casual paths from x to y are contained in the casual diamond $\Diamond(x, y)$. Moreover, the set of all causal paths is compact: For any sequence $\alpha_1, \alpha_2, \ldots,$ of casual paths from x to y there exists a convergent subsequence. The compactness roughly translates physically to having no naked singularities in L.
The situation changes dramatically if we trow out a point z from the casual diamond $\Diamond(x, y)$. Then it is not compact and the set of all causal paths is not compact, the removed point z being a naked singularity.

I would like to finish with a citation from Eduard Witten lectures *Black Holes, Singularity Theorems, and All That*, IAS,PiTP 2018, where he recapitulated: "... We are studying causal paths because they are the key to understanding black holes, singularities, and all that..."

I. R. Shafarevich, A. O. Remizov, Linear Algebra and Geometry, Springer-Verlag Berlin Heidelberg 2013.

E. Witten, Black Holes, Singularity Theorems, and All That, IAS,PiTP 2018.

Thank you!