Hilbert series for exterior algebras and for some relatively free algebras

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Definitions and notations

Definition

Let $A = \bigoplus_{i \geq 0} A^i$ *be a finitely generated graded algebra over* $\mathbb{C}$ *such that* $A^0 = \mathbb{C}$ *or* $A^0 = 0$. *The Hilbert series of* A *is the formal power series*

$$H(A,t) = \sum_{i \geq 0} (\dim A^i) t^i.$$

The Hilbert series $H(A,t)$ gives information about the lowest degree of the generators in a minimal generating set of A and the maximal number of generators in each degree.

Definitions and notations

- Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \in (\mathbb{N}_0)^n$ be a non-negative integer partition. By V_λ we denote the irreducible $\mathrm{GL}(n)$-module with highest weight λ.

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We consider the following class of algebras:

- Let $A = \bigoplus_{i \geq 0} A^i$ be a finitely generated graded algebra such that each homogeneous component A^i is a polynomial $\mathrm{GL}(n)$-module.
- A general class of examples is given by $T(W)/I$, where $T(W)$ is the tensor algebra of W and I is a $\mathrm{GL}(n)$-invariant ideal.

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Question

Determine $H(A^G, t)$ for G being one of $SL(n)$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, or $\mathrm{Sp}(2d)$ (in the case $n = 2d$).

Hilbert series and multiplicity series

Let A have the following decomposition as a $\mathrm{GL}(n)$-module:

$$A = \bigoplus_{i \geq 0} A^i = \bigoplus_{i \geq 0} \bigoplus_{\lambda} m_i(\lambda) V_\lambda.$$

Following a work of Benanti, Boumova, Drensky, Genov, and Koev for $S(W)$, we introduce the following Hilbert series of A:

$$H(A, x_1, \ldots, x_n, t) = \sum_{i \geq 0} \chi_{A^i}(x_1, \ldots, x_n) t^i =$$

$$\sum_{i \geq 0} \left(\sum_{\lambda} m_i(\lambda) S_\lambda(x_1, \ldots, x_n) \right) t^i,$$

where $\chi_{A^i}(x_1, \ldots, x_n)$ is the character of the $\mathrm{GL}(n)$-module A^i and $S_\lambda(x_1, \ldots, x_n)$ is the Schur polynomial corresponding to λ.

Hilbert series and multiplicity series

The Hilbert series of A

$$H(A, x_1, \ldots, x_n, t) = \sum_{i \geq 0} \left(\sum_{\lambda} m_i(\lambda) S_\lambda(x_1, \ldots, x_n) \right) t^i$$
$$\in \mathbb{Z}[[x_1, \ldots, x_n]]^{S_n}[[t]],$$

where S_n denotes the symmetric group in n variables. Following BBDGK, we introduce the **multiplicity series** of A by

$$M(A, x_1, \ldots, x_n, t) = \sum_{i \geq 0} \left(\sum_{\lambda} m_i(\lambda) x_1^{\lambda_1} \cdots x_n^{\lambda_n} \right) t^i.$$

By a change of variables $v_1 = x_1$, $v_2 = x_1 x_2$, $\ldots$, $v_n = x_1 \cdots x_n$ one can rewrite the above series as

$$M'(A, v_1, \ldots, v_n, t) = \sum_{i \geq 0} \left(\sum_{\lambda} m_i(\lambda) v_1^{\lambda_1 - \lambda_2} \cdots v_{n-1}^{\lambda_{n-1} - \lambda_n} v_n^{\lambda_n} \right) t^i.$$

The algebra A^G for $G = \mathrm{SL}(n)$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, or $\mathrm{Sp}(2d)$

Theorem (BBDGK)

For the Hilbert series of $A^{\mathrm{SL}(n)}$ we obtain

$$H(A^{\mathrm{SL}(n)}, t) = M'(A, 0, \ldots, 0, 1, t).$$

Theorem

Let $n = 2d$. For the Hilbert series of $A^{\mathrm{Sp}(2d)}$ we obtain

$$H(A^{\mathrm{Sp}(2d)}, t) = M'(A, 0, 1, 0, 1, \ldots, 0, 1, t).$$

The algebra A^G for $G = \mathrm{SL}(n)$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, or $\mathrm{Sp}(2d)$

Theorem

For the Hilbert series of $A^{\mathrm{O}(n)}$ we obtain

$$H(A^{\mathrm{O}(n)}, t) = M_n(t),$$

where

$$M_1(x_2, \dots, x_n, t) =$$
$$\frac{1}{2}\left(M(A, -1, x_2, \dots, x_n, t) + M(A, 1, x_2, \dots, x_n, t)\right),$$
$$M_2(x_3, \dots, x_n, t) = \frac{1}{2}(M_1(-1, x_3, \dots, x_n, t) + M_1(1, x_3, \dots, x_n, t)),$$
$$\dots\dots\dots$$
$$M_n(t) = \frac{1}{2}(M_{n-1}(-1, t) + M_{n-1}(1, t)).$$

The algebra A^G for $G = \mathrm{SL}(n)$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, or $\mathrm{Sp}(2d)$

Theorem

For the Hilbert series of $A^{\mathrm{SO}(n)}$ we obtain

$$H(A^{\mathrm{SO}(n)}, t) = M'_n(t),$$

where

$$M'_1(v_2, \ldots, v_n, t) =$$
$$\frac{1}{2}(M'(A, -1, v_2, \ldots, v_n, t) + M'(A, 1, v_2, \ldots, v_n, t)),$$
$$M'_2(v_3, \ldots, v_n, t) = \frac{1}{2}(M'_1(-1, v_3, \ldots, v_n, t) + M'_1(1, v_3, \ldots, v_n, t)),$$
$$\ldots\ldots\ldots$$
$$M'_{n-1}(v_n, t) = \frac{1}{2}(M'_{n-2}(-1, v_n, t) + M'_{n-2}(1, v_n, t)),$$
$$M'_n(t) = M'_{n-1}(1, t).$$

Applications: Computing $H(\Lambda(W)^G, t)$

- Let $A = \Lambda(W) = \bigoplus_i \Lambda^i(W)$, where W is a p-dimensional polynomial $\mathrm{GL}(n)$-module. Let $\alpha_1 = (\alpha_{11}, \ldots, \alpha_{1n})$, ..., $\alpha_p = (\alpha_{p1}, \ldots, \alpha_{pn})$ denote the weights of W (with possible repetitions). Then, for each i

$$\chi_{\Lambda^i(W)}(x_1, \ldots, x_n) = \sum_{1 \le s_1 < \cdots < s_i \le n} (x_1^{\alpha_{s_1 1}} \cdots x_n^{\alpha_{s_1 n}}) \cdots (x_1^{\alpha_{s_i 1}} \cdots x_n^{\alpha_{s_i n}}).$$

Therefore,

$$H(\Lambda(W), x_1, \ldots, x_n, t) = \sum_{i \ge 0} \chi_{\Lambda^i(W)}(x_1, \ldots, x_n) t^i = \prod_{j=1}^{p} (1 + x_1^{\alpha_{j1}} \cdots x_n^{\alpha_{jn}} t).$$

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- Let $W = S^k V$, where $V = \mathbb{C}^n$ is the natural $\mathrm{GL}(n)$-module. Then

$$H(\Lambda(S^k V), x_1, \dots, x_n, t) = \prod_{i_1 + \cdots + i_n = k} (1 + x_1^{i_1} \cdots x_n^{i_n} t).$$

Applications: Computing $H(\Lambda(W)^G, t)$

A generalization of a lemma of Berele.

Lemma

Let $X = \{x_1, \ldots, x_n\}$ and let $H(A, X, t)$ denote the Hilbert series of A. Let

$$g(X, t) = H(A, X, t) \prod_{i<j} (x_i - x_j) = \sum_{i \geq 0} \Big(\sum_{r_{i_j} \geq 0} \alpha_i(r_{i_1}, \ldots, r_{i_n}) x_1^{r_{i_1}} \cdots x_n^{r_{i_n}} \Big) t^i,$$

for some $\alpha_i(r_{i_1}, \ldots, r_{i_n}) \in \mathbb{C}$. Then the multiplicity series of A is given by

$$M(A; x_1, \ldots, x_n, t) = \frac{1}{x_1^{n-1} x_2^{n-2} \cdots x_{n-2}^2 x_{n-1}} \sum_{i \geq 0} \Big(\sum_{r_{i_j} > r_{i_{j+1}}} \alpha_i(r_{i_1}, \ldots, r_{i_n}) x_1^{r_{i_1}} \cdots x_n^{r_{i_n}} \Big) t^i,$$

where the sum is over all $r_i = (r_{i_1}, \ldots, r_{i_n})$ such that $r_{i_1} > r_{i_2} > \cdots > r_{i_n}$.

Applications: Computing $H(\Lambda(W)^G, t)$

Table : Hilbert series for $n = 2$

k	$H(\Lambda(S^k V)^{\mathrm{SL}(2)}, t)$
3	$1 + t^2 + t^4$
4	$1 + t^5$
5	$1 + t^2 + t^4 + t^6$
6	$1 + t^3 + t^4 + t^7$
7	$1 + t^2 + t^4 + t^6 + t^8$
8	$1 + t^4 + t^5 + t^9$
9	$1 + t^2 + 2t^4 + 2t^6 + t^8 + t^{10}$
10	$1 + t^3 + t^4 + t^7 + t^8 + t^{11}$
11	$1 + t^2 + 2t^4 + 3t^6 + 2t^8 + t^{10} + t^{12}$
12	$1 + 2t^4 + 2t^5 + 2t^8 + 2t^9 + t^{13}$
13	$1 + t^2 + 2t^4 + 4t^6 + 4t^8 + 2t^{10} + t^{12} + t^{14}$
14	$1 + t^3 + 2t^4 + 4t^7 + 4t^8 + 2t^{11} + t^{12} + t^{15}$

Applications: Computing $H(\Lambda(W)^G, t)$

Table : Hilbert series for $n = 3$

k	$H(\Lambda(S^k V)^{\mathrm{SL}(3)}, t)$
3	$1 + t^3 + t^7 + t^{10}$
4	$1 + t^6 + t^9 + t^{15}$
5	$1 + t^3 + t^6 + t^9 + t^{12} + t^{15} + t^{18} + t^{21}$
6	$1 + t^5 + 2t^6 + t^7 + t^8 + 6t^9 + 7t^{10} + 6t^{11} + 8t^{12} + 13t^{13} + 16t^{14} + 13t^{15} + 8t^{16} + 6t^{17} + 7t^{18} + 6t^{19} + t^{20} + t^{21} + 2t^{22} + t^{23} + t^{28}$

Example (Hilbert series for $n = 4$ and $n = 5$)

$$H(\Lambda(S^3 V)^{\mathrm{SL}(4)}, t) = 1 + t^4 + t^8 + t^{12} + t^{16} + t^{20};$$

$$H(\Lambda(S^3 V)^{\mathrm{SL}(5)}, t) = 1 + t^5 + t^{30} + t^{35};$$

Applications: Computing $H(\Lambda(W)^G, t)$

- $H(\Lambda(S^3V)^{\mathrm{SL}(3)}, t) = 1 + t^3 + t^7 + t^{10}$.

Applications: Computing $H(\Lambda(W)^G, t)$

- $H(\Lambda(S^3V)^{\mathrm{SL}(3)}, t) = 1 + t^3 + t^7 + t^{10}$.
- $\Lambda(S^3V)^{\mathrm{SL}(3)}$ is generated by a pair $\{v, *v\}$, where $v \in \Lambda^3(S^3V)$ and $*v$ is the Hodge dual of v, i.e., the unique element in $\Lambda^7(S^3V)$ such that

$$v \wedge *v = \langle v, v \rangle \, \mathrm{vol}.$$

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- Let $\{e_1, e_2, e_3\}$ denote the standard basis for $V = \mathbb{C}^3$. A basis for S^3V is given by:

$$a_1 = e_1^3, \quad a_2 = e_2^3, \quad a_3 = e_3^3, \quad a_4 = e_1^2 e_2, \quad a_5 = e_1^2 e_3,$$
$$a_6 = e_2^2 e_3, \quad a_7 = e_1 e_2^2, \quad a_8 = e_1 e_3^2, \quad a_9 = e_2 e_3^2, \quad a_{10} = e_1 e_2 e_3.$$

Applications: Computing $H(\Lambda(W)^G, t)$

Then $\Lambda(S^3V)^{\mathrm{SL}(3)}$ is generated by

$$\begin{aligned} v =& a_1 \wedge a_2 \wedge a_3 - 3a_3 \wedge a_4 \wedge a_7 - 3a_1 \wedge a_6 \wedge a_9 + 3a_2 \wedge a_5 \wedge a_8 + \\ & 6a_7 \wedge a_8 \wedge a_{10} - 6a_4 \wedge a_9 \wedge a_{10} + 6a_5 \wedge a_6 \wedge a_{10} + \\ & 3a_5 \wedge a_7 \wedge a_9 + 3a_4 \wedge a_6 \wedge a_8. \end{aligned}$$

$$\begin{aligned} * v = & a_4 \wedge a_5 \wedge a_6 \wedge a_7 \wedge a_8 \wedge a_9 \wedge a_{10} - \frac{1}{3} a_1 \wedge a_2 \wedge a_5 \wedge a_6 \wedge a_8 \wedge a_9 \wedge a_{10} - \\ & - \frac{1}{3} a_2 \wedge a_3 \wedge a_4 \wedge a_5 \wedge a_7 \wedge a_8 \wedge a_{10} - \frac{1}{3} a_1 \wedge a_3 \wedge a_4 \wedge a_6 \wedge a_7 \wedge a_9 \wedge a_{10} - \\ & - \frac{1}{9} a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_5 \wedge a_6 \wedge a_9 + \frac{1}{9} a_1 \wedge a_2 \wedge a_3 \wedge a_5 \wedge a_6 \wedge a_7 \wedge a_8 - \\ & - \frac{1}{9} a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_7 \wedge a_8 \wedge a_9 - \frac{1}{9} a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_6 \wedge a_8 \wedge a_{10} + \\ & + \frac{1}{9} a_1 \wedge a_2 \wedge a_3 \wedge a_5 \wedge a_7 \wedge a_9 \wedge a_{10}. \end{aligned}$$

Applications: Hilbert series for some relatively free algebras

- Let $V = \mathbb{C}^n$ and let

$$A = T(V) / \langle [[u, v], w] : \text{ for all } u, v, w \in T(V) \rangle .$$

A is called the relatively free algebra of rank n in the variety generated by the Grassmann algebra.

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- Decomposition of A as a $\mathrm{GL}(n)$-module: Let

$$\mathcal{P} = \{\text{all partitions } \lambda \in \mathbb{N}_0^n : \lambda = (k, \underbrace{1,\ldots,1}_{s}, \underbrace{0,\ldots,0}_{t}), k,s,t \geq 0\}.$$

  Hence, $\mathcal{P}$ contains all partitions λ with Young diagram consisting of one long row and one long column. Then

$$A \cong \bigoplus_{i \geq 0} \bigoplus_{\substack{\lambda \in \mathcal{P} \\ |\lambda| = i}} V_\lambda.$$

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Hence, $\mathcal{P}$ contains all partitions λ with Young diagram consisting of one long row and one long column. Then

$$A \cong \bigoplus_{i\geq 0} \bigoplus_{\substack{\lambda\in\mathcal{P} \\ |\lambda|=i}} V_\lambda .$$

- Hence, $M(A, x_1, \dots, x_n, t) = \sum_{i\geq 0} \Big(\sum_{\substack{\lambda\in\mathcal{P} \\ |\lambda|=i}} x_1^{\lambda_1} \cdots x_n^{\lambda_n} \Big) t^i .$

Applications: Hilbert series for some relatively free algebras

- For the Hilbert series $H(A^{\mathrm{SL}(n)}, t)$ and $H(A^{\mathrm{Sp}(2d)}, t)$ we obtain:

$$H(A^{\mathrm{SL}(n)}, t) = 1 + t^n;$$

$$H(A^{\mathrm{Sp}(2d)}, t) = 1 + t^2 + t^4 + \cdots + t^{2d}, \quad \text{where } n = 2d.$$

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- Let $\{x_1, \ldots, x_n\}$ be a basis for $V = \mathbb{C}^n$.
  The algebra $A^{\mathrm{SL}(n)}$ is generated by the standard polynomial of degree n

$$f = St_n(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

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- The algebra $A^{\mathrm{Sp}(2d)}$ is generated by

$$f = [x_1, x_{d+1}] + [x_2, x_{d+2}] + \cdots + [x_d, x_{2d}].$$

Applications: Hilbert series for some relatively free algebras

- For the Hilbert series $H(A^{\mathrm{O}(n)}, t)$ and $H(A^{\mathrm{SO}(n)}, t)$ we obtain:

$$H(A^{\mathrm{O}(n)}, t) = \frac{1}{1 - t^2};$$

$$H(A^{\mathrm{SO}(n)}, t) = \frac{1 + t^n}{1 - t^2}.$$

Applications: Hilbert series for some relatively free algebras

- For the Hilbert series $H(A^{\mathrm{O}(n)}, t)$ and $H(A^{\mathrm{SO}(n)}, t)$ we obtain:

$$H(A^{\mathrm{O}(n)}, t) = \frac{1}{1 - t^2};$$

$$H(A^{\mathrm{SO}(n)}, t) = \frac{1 + t^n}{1 - t^2}.$$

- The algebra $A^{\mathrm{O}(n)}$ is generated by $f = x_1 \otimes x_1 + \cdots + x_n \otimes x_n$.
- The algebra $A^{\mathrm{SO}(n)}$ is generated by the elements f_1 and f_2, where

$$f_1 = x_1 \otimes x_1 + \cdots + x_n \otimes x_n,$$

$$f_2 = St_n(x_1, \ldots, x_n).$$

Applications: Hilbert series for some relatively free algebras

- Let $V = \mathbb{C}^n$ with basis $\{x_1, \ldots, x_n\}$ and let
$$A = T(V) / \langle [u_1, u_2] \otimes [u_3, u_4] : \text{ for all } u_1, \ldots, u_4 \in T(V) \rangle .$$
A is called the relatively free algebra of rank n in the variety generated by the algebra of 2×2 upper triangular matrices.

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- The Hilbert series of $A^{\mathrm{Sp}(2d)}$ is

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A is called the relatively free algebra of rank n in the variety generated by the algebra of 2×2 upper triangular matrices.
- The Hilbert series of $A^{\mathrm{Sp}(2d)}$ is
$$H(A^{\mathrm{Sp}(2d)}, t) = \frac{1}{1 - t^2}.$$
- $A^{\mathrm{Sp}(2d)}$ is not finitely generated. A set of generators can be defined inductively by
$$f_1 = [x_1, x_{d+1}] + [x_2, x_{d+2}] + \cdots + [x_d, x_{2d}] = \sum_{i=1}^{d} [x_i, x_{d+i}],$$
$$f_{m+1} = \sum_{i=1}^{d} x_i \otimes f_m \otimes x_{d+i} - x_{d+i} \otimes f_m \otimes x_i, \quad m = 1, 2, \ldots$$

Applications: Hilbert series for some relatively free algebras

- The Hilbert series of $A^{\mathrm{O}(n)}$ is

$$H(A^{\mathrm{O}(n)}, t) = \frac{1 - 2t^2 + 2t^4}{(1 - t^2)^3}.$$

- For the Hilbert series of $A^{\mathrm{SO}(n)}$ we obtain
  (i) If $n = 2$, then

$$H(A^{\mathrm{SO}(2)}, t) = \frac{1 - t^2 + 2t^4}{(1 - t^2)^3}.$$

  (ii) If $n = 3$, then

$$H(A^{\mathrm{SO}(3)}, t) = \frac{1 - 2t^2 + t^3 + 2t^4}{(1 - t^2)^3}.$$

  (iii) If $n > 3$, then

$$H(A^{\mathrm{SO}(n)}, t) = H(A^{\mathrm{O}(n)}, t).$$

- The algebras $A^{\mathrm{O}(n)}$ and $A^{\mathrm{SO}(n)}$ are not finitely generated.

The end

Thank you for your attention!