Clifford Semigroups with Injective Structure Homomorphisms

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- **Clifford semigroup** (strong semilattice of groups)

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- g is called an **inverse** of f
- $End(S)$ is regular if each $f \in End(S)$ has an inverse
- $End(S)$ is **completely regular** if regular and for each $f \in End(S)$ there is an inverse g with $fg = gf$.

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Theorem (Samman & Meldrum, 2002, AC)

Let the semilattice Y have a unique least element δ and let all the structure homomorphisms be injective. Then every group homomorphism f_α associated with a homomorphism f is determined by f_δ and is given by $f_\alpha = \varphi_{\alpha,\delta} f_\delta \varphi_{\alpha,\delta}^{-1}$.

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Lemma (Worawiset & K, 2019)

If $End(S)$ is regular then $End(G_\delta)$ is regular.

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- The restriction $\underline{f}$ of f to E_S induces an endomorphism on Y
- $(\alpha)\underline{f} = \beta$, whenever $(e_\alpha)f = e_\beta$ (**induced index mapping**)
- $End(Y) = \{\underline{f} : f \in End(S)\}$ is the monoid of all endomorphisms on Y.

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Theorem (Worawiset & K, 2019)

$End(S)$ is regular iff for aech $f \in End(S)$ with $s = \underline{f}$ there is $t \in End(Y)$ with $sts = s$ and there is $g \in End(G_\delta)$ with $\mathrm{Im}(\varphi_{a,\delta} g) \subseteq \mathrm{Im}(\varphi_{(\alpha)t,\delta})$ for all $\alpha \in Y$ such that $f_\delta \varphi_{a,\delta} g f_\delta = f_\delta$.

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Corollary (Worawiset & K, 2019)

Let Y be a semilattice which has a unique least element $\nu = \bigwedge Y$ and let $S = \bigcup_{\xi \in Y} G_\xi$ be a Clifford semigroup with bijective structure homomorphisms. Then $End(S)$ is regular if and only if both $End(Y)$ and $End(G_\nu)$ are regular.

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Theorem (Worawiset & K, 2019)

Let $Y = \{\alpha \geq \beta\}$ be a two-element chain and let $S = G_\alpha \cup G_\beta$ be a Clifford semigroup with an injective structure homomorphism. $End(S)$ is a band iff $End(G_\beta)$ is a band.

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Lemma

Let $S = G_\alpha \cup G_\beta \in CL^*$. Then $End(S)$ is completely regular if $End(G_\beta)$ completely regular.

Merci