(2, 3)-GENERATION OF THE GROUPS $PSL_6(q)$

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Outline

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- A group is (2, 3)-generated if and only if it is a homomorphic image of the modular group $PSL_2(\mathbb{Z})$.
- The theorem of Liebeck-Shalev and Lübeck-Malle states that all finite simple groups, except the symplectic groups $PSp_4(2^m)$, $PSp_4(3^m)$, the Suzuki groups $Sz(2^m)$ (m odd), and finitely many other groups, are (2, 3)-generated (see [11])

Considered problem

For the $PSL_n(q)$,
$(2,3)$-generation has been proved in the cases $n = 2, q \neq 9$ [8], $n = 3, q \neq 4$ [4], [1], $n = 4, q \neq 2$ [12], [13], [9], $n = 5$, any q [14], $n \geq 5$, odd $q \neq 9$ [2],[3], and $n \geq 13$, any q [10].

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Main theorem

Theorem

The group $PSL_6(q)$ is $(2,3)$-generated for any q.

Preliminaries

- $G = SL_6(q)$, $\overline{G} = G/Z(G) = PSL_6(q)$, where $q = p^m$ and p is a prime. Set $d = (6, q-1)$, also $Q = q^5 - 1$ if $q \neq 3, 7$ and $Q = (q^5 - 1)/2$ if $q = 3$ or 7.
- The group G acts naturally on a six-dimensional vector space V over the field $F = GF(q)$ and $\overline{G}$ acts on the corresponding projective space $P(V)$.

Lemma 1

Lemma

Let $\overline{M}$ be a maximal subgroup of the group $\overline{G}$. Then either $\overline{M}$ is reducible on the space $P(V)$ or $\overline{M}$ has no element of order $Q/(d, Q)$.

Proof of Lemma 1

The maximal subgroups of $PSL_6(q)$ are determined (up to conjugacy) in [5]. In particular, this implies that one of the following holds:

(i) $\overline{M}$ belongs to the family C_1 of reducible subgroups of $\overline{G}$;

(ii) $\overline{M}$ is a member of one of the remaining families C_2, C_3, C_4, C_5, C_8 of (irreducible) geometric subgroups of $\overline{G}$;

(iii) $\overline{M} \cong PSL_3(q)$ if q is odd or $\overline{M} \cong PSL_2(11),\ A_7,\ M_{12},\ PSL_3(4).\mathbb{Z}_2,\ PSU_4(3)$, or $PSU_4(3).\mathbb{Z}_2$ for specific values of p and q.

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Case 1: $q \neq 2, 4$

Let $\omega \in GF(q^5)^*$, $|\omega| = Q$

$$f(t) = \prod_{i=0}^{4}(t - \omega^{q^i}) = t^5 - \alpha t^4 + \beta t^3 - \gamma t^2 + \delta t - \varepsilon.$$

Then $f(t) \in F[t]$ and the polynomial $f(t)$ is irreducible over F.

Generators

$$x = \begin{pmatrix} -1 & 0 & 0 & \gamma\varepsilon^{-1} & 0 & \gamma \\ 0 & -1 & 0 & \beta\varepsilon^{-1} & 0 & \beta \\ 0 & 0 & 0 & \alpha\varepsilon^{-1} & -1 & \delta \\ 0 & 0 & 0 & 0 & 0 & \varepsilon \\ 0 & 0 & -1 & \delta\varepsilon^{-1} & 0 & \alpha \\ 0 & 0 & 0 & \varepsilon^{-1} & 0 & 0 \end{pmatrix}, \ x \in G, |x| = 2,$$

$$y = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \ y \in G, |y| = 3.$$

$$z = xy = \begin{pmatrix} 0 & 0 & -1 & 0 & \gamma & \gamma\varepsilon^{-1} \\ -1 & 0 & 0 & 0 & \beta & \beta\varepsilon^{-1} \\ 0 & 0 & 0 & -1 & \delta & \alpha\varepsilon^{-1} \\ 0 & 0 & 0 & 0 & \varepsilon & 0 \\ 0 & -1 & 0 & 0 & \alpha & \delta\varepsilon^{-1} \\ 0 & 0 & 0 & 0 & 0 & \varepsilon^{-1} \end{pmatrix}.$$

The characteristic polynomial of z is $f_z(t) = (t - \varepsilon^{-1})f(t)$ and the characteristic roots $\varepsilon^{-1}, \omega, \omega^q, \omega^{q^2}, \omega^{q^3}, \omega^{q^4}$ of z are pairwise distinct. Then, in $GL_6(q^5)$, z is conjugate to $\mathrm{diag}(\varepsilon^{-1},\ \omega,\ \omega^q,\ \omega^{q^2},\ \omega^{q^3},\ \omega^{q^4})$ and hence z is an element of G of order Q.

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Lemma 2

Lemma

Let $H = \langle x, y \rangle,\ H \leq G$*.*
The group H *acts irreducibly on the space* V*.*

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Case 2: $q = 2, 4$

Let now $q = 2$ or 4
The element $y \in G$, |y|=3 is the same like in Case 1 and

$$x = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & \eta & 0 & \eta^2 \\ 0 & 0 & 0 & \eta & 1 & \eta^2 \\ 0 & 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 1 & \eta & 0 & \eta^2 \\ 0 & 0 & 0 & \eta^2 & 0 & 0 \end{pmatrix}, \; x \in G, |x| = 2.$$

Here $\langle \eta \rangle = F^*$.

$$z = xy = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \eta^2 & \eta \\ 0 & 0 & 0 & 1 & \eta^2 & \eta \\ 0 & 0 & 0 & 0 & \eta & 0 \\ 0 & 1 & 0 & 0 & \eta^2 & \eta \\ 0 & 0 & 0 & 0 & 0 & \eta^2 \end{pmatrix}$$

The characteristic polynomial of z is $f_z(t) = (t + \eta^2)g(t)$, where $g(t) = t^5 + \eta^2 t^4 + \eta^2 t^3 + \eta^2 t^2 + (\eta^2 + \eta)t + \eta$.
It follows that both for $q = 2$ and $q = 4$ the element z has order $q^5 - 1 = Q$.

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Conclusion

Now, in $\overline{G}$, the elements $\overline{x}$, $\overline{y}$, and $\overline{z}$ have orders 2, 3, and $Q/(d,Q)$ in Case 1 (Q/d - Case 2), respectively. So the group $\overline{H} = \langle \overline{x}, \overline{y} \rangle$ has an element of order $Q/(d,Q)$ (or Q/d) and $\overline{H}$ is irreducible on $P(V)$ as H is irreducible on V by Lemma 2. Lemma 1 implies that $\overline{H}$ cannot be contained in any maximal subgroup of $\overline{G}$. Thus $\overline{H} = \overline{G}$ and $\overline{G} = \langle \overline{x}, \overline{y} \rangle$ is a $(2,3)$-generated group.

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