PAIRS OF ELEMENTS OF ORDERS 2 AND 3 IN THE REE GROUPS $^2G_2(3^n)$

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Outline

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- A Hurwitz group is any finite nontrivial quotient of the $(2, 3, 7)$- triangle group, that is, the infinite abstract group T defined by the presentation $T = \langle x, y | x^2 = y^3 = (xy)^7 = 1 \rangle$.
- The study of Hurwitz groups goes back to the late XIX century and shows an important connection with the theory of Riemann surfaces. In 1893, Hurwitz proved that the automorphism group of an algebraic curve of genus $g \geq 2$ always has order at most 84(g - 1) and that this upper bound is attained precisely when the group is Hurwitz.

Considered problem

- MALLE [5] proved that the Ree group $G = {}^2G_2(3^n)$ (n odd) is Hurwitz for any $n > 1$, and later prof. Tchakerian found [7] explicit Hurwitz generators x and y of G.

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- MALLE [5] proved that the Ree group $G = {}^2G_2(3^n)$ (n odd) is Hurwitz for any $n > 1$, and later prof. Tchakerian found [7] explicit Hurwitz generators x and y of G.
- From Malle's result and Macbeath's classification [4] of Hurwitz subgoups of $PSL_2(q)$ (q a prime power), and the known subgroup structure of G [3], [9], [2] easily imply that G contains exactly the following Hurwitz groups: $SL_2(8)$, $PSL_2(27)$ if $3|n$, and ${}^2G_2(3^m)$ for each divisor $m > 1$ of n if $n > 1$.

Formulation of the problem

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- Then the second aim is to classify all $(2,3)$-generations of G, that is all pairs of elements x and y with $x^2 = y^3 = 1$ and $G = \langle x, y \rangle$.

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Preliminaries

Let $G = {}^2G_2(q)$ where $q = 3^n$ and $n > 1$ is odd, $F = GF(q)$, and $\theta = 3^{\frac{n-1}{2}}$ (so that $q = 3\theta^2$).

Theorem 1

Theorem

Let x and y be nonidentity elements of G and $x^2 = y^3 = (xy)^7 = 1$. Then up to a conjugation in G exactly one of the following holds:

(i) $x = \omega_0$, $y = \gamma(1)$, *and* $\langle x, y\rangle \cong SL_2(8)$;

(ii) $3|n$, $x = h(z)\omega_0$, $y = \beta(\varepsilon)$ *where* $\varepsilon = \pm 1$, $z \in F$ *and* $z^3 + z^2 = 1$, *and* $\langle x, y\rangle \cong PSL_2(27)$;

(iii) $x = h(z)\omega_0$, $y = \beta(\varepsilon)\gamma(t)$ *where* $\varepsilon = \pm 1$, $z, t \in F^*$ *and* $t^2 = z^{6\theta+3} - z^{3\theta} - 1$, *and* $\langle x, y\rangle \cong {}^2G_2(3^m)$ *where* $m > 1$ *is the least divisor of* n *for which* $z \in GF(3^m)$.

We have $G = UH \cup UHw_0U$. $x \in UH\omega_0 U$, $|x| = 2$, $x = h(z)\omega_0$. The group G has three conjugacy classes of elements of order 3 with representatives $\gamma(1)$, $\beta(1)$, and $\beta(-1) = \beta(1)^{-1}$.

$$x = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -z^{-3\theta-2} \\ 0 & 0 & 0 & 0 & 0 & -z & 0 \\ 0 & 0 & 0 & 0 & -z^{3\theta+1} & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -z^{-3\theta-1} & 0 & 0 & 0 & 0 \\ 0 & -z^{-1} & 0 & 0 & 0 & 0 & 0 \\ -z^{3\theta+2} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof of Theorem 1

$$y = \gamma(1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix}.$$

$xy = z^{-3\theta-2}E_{-3-3} - z^{-3\theta-2}E_{-30} + z^{-3\theta-2}E_{-31} - z^{-3\theta-2}E_{-33} - zE_{-22} - z^{3\theta+1}E_{-11} - E_{0-3} - E_{00} - z^{-3\theta-1}E_{1-3} - z^{-3\theta-1}E_{1-1} + z^{-3\theta-1}E_{12} - z^{-1}E_{2-2} - z^{-1}E_{21} - z^{3\theta+2}E_{3-3}$.

Set $\eta = tr(xy) = z^{-3\theta-2} - 1$. Then the characteristic polynomial of xy is

$$X^7 - \eta X^6 + (\eta^{3\theta} - \eta)X^5 + (\eta^{3\theta} - \eta^2 + \eta)X^4 - (\eta^{3\theta} - \eta^2 + \eta)X^3 - $$

$$-(\eta^{3\theta} - \eta)X^2 + \eta X - 1.$$

G has a single conjugacy class of elements of order 7, with characteristic polynomial $X^7 - 1$. Thus $|xy| = 7$ if and only if $\eta = 0$ which yields $z = 1$.

Now $x = \omega_0$ and $y = \gamma(1)$ both lie in the subgroup $G_1 \cong {}^2G_2(3)$ of G. $\langle x, y\rangle$ is contained in $G_1' \cong SL_2(8)$ and (by the structure of $SL_2(8)$) it follows that $\langle x, y\rangle \cong SL_2(8)$.

Theorem 2

Theorem

Let x and y be elements of G such that $x^2 = y^3 = 1$ and $G = \langle x, y \rangle$. Then up to a conjugation in G exactly one of the following holds:

(i) $x = h(z)\omega_0$ *and* $y = \gamma(1)$ *where* $F = \mathbb{Z}_3(z)$*;*

(ii) $x = h(z)\omega_0$ *and* $y = \beta(\varepsilon)\gamma(t)$ *where* $\varepsilon = \pm 1$, $t \neq 0$, $z \neq -t^{2\theta} - 1$, *and* $F = \mathbb{Z}_3(z, t)$.

The list of maximal subgroups of G determined in [3],[9],[2]. In the notation of [9], this list implies that if M is a maximal subgroup of G then one of the following holds:

1) $M \cong Z_2 \times PSL_2(q)$ is the centralizer of an involution;
2) $M \cong (E_8 Z_{\frac{q+1}{4}})Z_3$ is the normalizer of a four-group;
3) M is conjugate to $P = UH = N_G(U)$, a parabolic subgroup;
4) M is a Frobenius group of order $6(q+1-\sqrt{3q})$ or $6(q+1+\sqrt{3q})$;
5) M is conjugate to the subgroup $G_m \cong {}^2G_2(3^m)$ for some divisor m of n with $\frac{n}{m}$ prime.

Outline

Bibliography

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