За алгебричните структури на интегрируемите уравнения Отчет на В. С. Герджиков за 2019 г.

V. S. Gerdjikov

асоцииран член на ИМИ при БАН

1 Plan

- Derivation of $\mathbb{Z}_h$ and $\mathbb{D}_h$-reduced nonlinear evolution equations:
  Derivative NLS-type equations (DNLS)
  MKDV-type equations
  2-dimensional Toda field theories (TFT).
- Reductions and Kac-Moody algebras
- Hamiltonian properties and complete integrability

The Equations

- Derivative NLS

$$i\psi_{k,t}+\gamma\operatorname{cotg}\frac{\pi k}{N}\cdot\frac{\partial^2\psi_k}{\partial x^2}+i\gamma\sum_{p=1}^{N-1}\frac{\partial}{\partial x}(\psi_p\psi_{k-p})=0, \qquad k=1,2,\ldots,N-1,$$

$\gamma = \text{const}$; the index $k-p$ should be understood modulo $h=N$; $\psi_0 = \psi_N = 0$. Additional involutions: from $\mathbb{Z}_h \to \mathbb{D}_h$

$$\begin{aligned} \psi_k &= -\psi_k^*, & \gamma &= -\gamma^*, \\ \psi_k &= \psi_{N-k}^*, & \gamma &= \gamma^*, \end{aligned} \tag{1}$$

Particular cases with $N=3$ reported by Fordy and Gibbons (1981) and Mikhailov (1981).

Basic tools:
simple Lie algebras $sl(N) \simeq A_{N-1}$
Kac-Moody algebras $A_{N-1}^{(1)}$ and $A_{N-1}^{(2)}$.
The group of reductions – A. V. Mikhailov (1981)

- 2-dimensional Toda field theory

$$2\frac{\partial q_k}{\partial x \partial t} = e^{q_{k+1}-q_k} - e^{q_k - q_{k-1}}, \tag{2}$$

- modified KdV eqs.

The general construction of Kac - Moody algebras: an example

- Choose a simple Lie algebras $\mathfrak{g}$ and pick up its Coxeter automorphism $C^h = \mathbb{1}$. For $sl(N)$ the Coxeter number $h = N$.

- Introduce a grading in $\mathfrak{g}$ induced by C

$$\mathfrak{g} = \bigoplus_{k=0}^{h-1} \mathfrak{g}^{(k)}, \qquad C\mathfrak{g}^{(k)}C^{-1} = \omega^{-k}\mathfrak{g}^{(k)}, \qquad \omega = \exp(2\pi i/h),$$

$$[\mathfrak{g}^{(k)}, \mathfrak{g}^{(m)}] \in \mathfrak{g}^{(k+m)}, \qquad k+m \mod (h).$$

- Use the grading to construct KM algebras as:

$$X(\lambda) = \sum_p \lambda^p X_p, \qquad X_p \in \mathfrak{g}^{(p)}.$$

Then the potentials of L and M can be viewed as elements of KM algebras.

Example: $\mathfrak{g} \simeq sl(6)$, $h = 6$ and

$$C = \mathrm{diag}\,(1, \omega, \omega^2, \omega^3, \omega^4, \omega^5), \qquad \omega = \exp(2\pi i/6),$$

$$\mathfrak{g}^{(0)} \simeq \begin{pmatrix} * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}, \quad \mathfrak{g}^{(1)} \simeq \begin{pmatrix} 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{g}^{(2)} \simeq \begin{pmatrix} 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathfrak{g}^{(3)} \simeq \begin{pmatrix} 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{g}^{(4)} \simeq \begin{pmatrix} 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 \end{pmatrix}, \quad \mathfrak{g}^{(5)} \simeq \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \end{pmatrix}.$$

An alternative grading:

$$\mathfrak{g} = \bigoplus_{k=0}^{h-1} \tilde{\mathfrak{g}}^{(k)}, \qquad \tilde{C}\tilde{\mathfrak{g}}^{(k)}\tilde{C}^{-1} = \omega^{-k}\tilde{\mathfrak{g}}^{(k)}, \qquad \omega = \exp(2\pi i/h),$$

$$[\tilde{\mathfrak{g}}^{(k)}, \tilde{\mathfrak{g}}^{(m)}] \in \tilde{\mathfrak{g}}^{(k+m)}, \qquad k+m \bmod (h).$$

$$\tilde{C} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

obtained from the previous one by a similarity transformation.

Lax representation and $\mathbb{Z}_h$ reductions

Let us consider a Lax pair of the form:

$$\mathcal{L}\chi(x,t,\lambda) \equiv \left(i\frac{d}{dx} + U(x,t,\lambda) \right) \chi(x,t,\lambda) = 0,$$

$$\mathcal{M}_n\chi(x,t,\lambda) \equiv \left(i\frac{d}{dt} + V^{(n)}(x,t,\lambda) \right) \chi(x,t,\lambda) - \lambda^n \chi(x,t,\lambda) C^n = 0,$$

where

$$U(x,t,\lambda) = U_0(x,t) - \lambda C,$$

$$V^{(2)}(x,t,\lambda) = V_0(x,t) + \lambda V_1(x,t) - \lambda^2 C^2, \qquad \text{DNLS}$$

$$V^{(3)}(x,t,\lambda) = V_0(x,t) + \lambda V_1(x,t) + \lambda^2 V_2(x,t) - \lambda^3 C^3, \qquad \text{mKdV}$$

$$V^{(-1)}(x,t,\lambda) = V_0(x,t) - \frac{1}{\lambda} V_{-1}(x,t), \qquad \text{2-dim TFT}$$

The Lax pair allows one additional involution:

$$U^\dagger(x,t,\lambda) = -S_1 U(x,t,\epsilon\lambda^*) S_1, \tag{3}$$

or

$$U^*(x,t,\lambda) = S_1 U(x,t,\epsilon\lambda^*) S_1, \qquad S_1 = \sum_{j=0}^{N-1} E_{j,N-j+1}$$

and analogously for $V(x,t,\lambda)$.

The direct and inverse scattering problem

The direct and the inverse scattering problems for L reduces to RHP Mikhailov (1981). The continuous spectrum Γ of L fills up N lines in the complex λ-plane:

$$\Gamma : \ \arg\lambda = \frac{2\pi k}{N}, \qquad k = 1, 2, \dots, N.$$

The direct scattering problems for L: introduce the Jost solutions by:

$$\lim_{x\to\infty} \psi(x,t,\lambda) e^{\lambda U_1 x} = \lim_{x\to -\infty} \phi(x,t,\lambda) e^{\lambda U_1 x} = \mathbb{1},$$

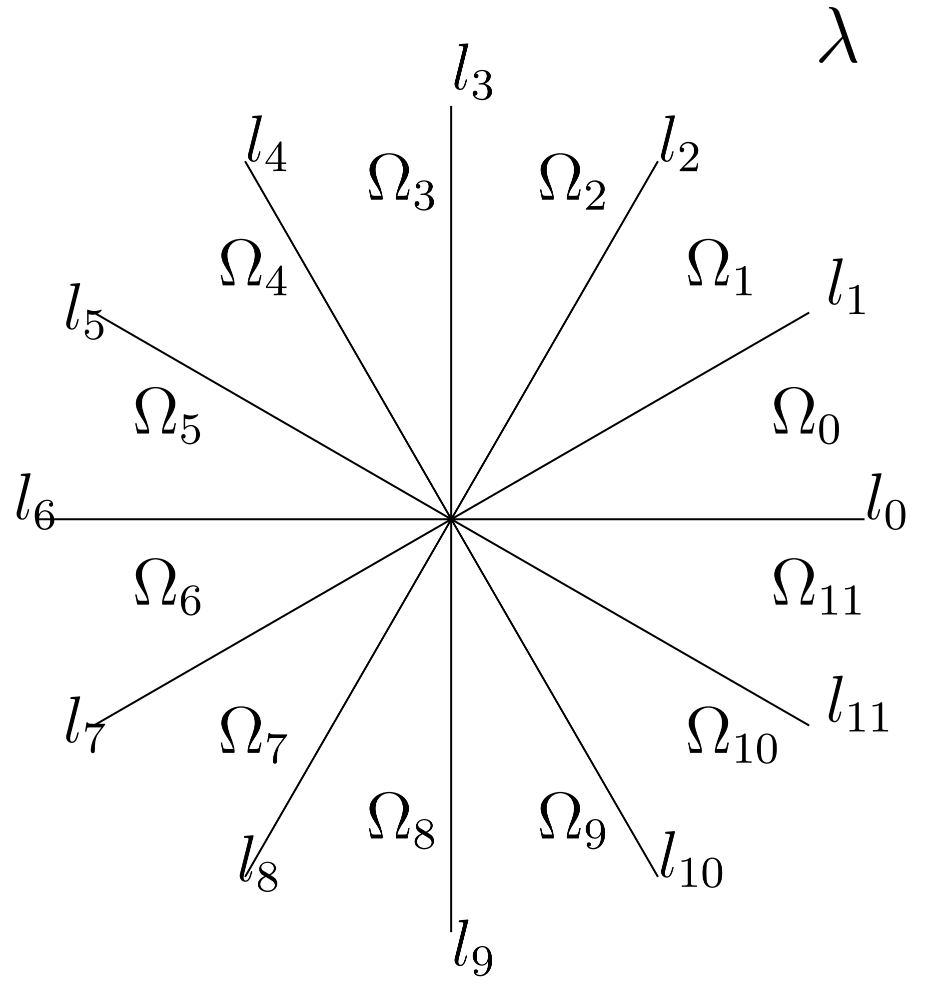
Jost solutions satisfy Volterra-type integral equations. Then determine the transfer matrix

$$T(\lambda, t) = \psi^{-1}(x, t, \lambda)\phi(x, t, \lambda); \qquad \lambda \in \Gamma.$$

The inverse scattering problem for L is equivalent to Riemann-Hilbert problem which allows one to calculate the reflectionless potentials of L and then find the corresponding soliton solutions. It is enough to know that $T(\lambda, t)$ depends on t by:

$$\begin{aligned} &i\frac{dT}{d\lambda} - \lambda^2[C^2, T(\lambda, t)] = 0 && \text{for DNLS} \\ &i\frac{dT}{d\lambda} - \lambda^3[C^3, T(\lambda, t)] = 0 && \text{for MKDV} \\ &i\frac{d\tilde{T}}{d\lambda} - \frac{1}{\lambda}[\tilde{C}, \tilde{T}(\lambda, t)] = 0 && \text{for 2-dim TFT} \end{aligned} \tag{4}$$

The reductions imply also symmetries between the discrete eigenvalues: if λ_1 is a d.e. of L then so are $\lambda_1\omega^j$ for all $j = 1, \ldots, N-1$.



Фигура 1: The contour for the RHP of L with $\mathbb{Z}_6$-symmetry.

ISP and RHP

Fundamental analytic solutions of $L\,\chi_\nu(x,t,\lambda)$ and solutions to the RHP:

$$m_\nu(x,t,\lambda) = \chi(x,t,\lambda)e^{iJ\lambda x}, \qquad \lim_{\lambda\to\infty} \lambda m_\nu(x,t,\lambda) = \mathbb{1}.$$

The rays l_ν are defined by:

$$\operatorname{Im} \lambda\alpha(J) = 0, \qquad \Leftrightarrow \qquad \alpha \in \delta_\nu \qquad \Leftrightarrow \qquad \mathfrak{g}_\nu \subset \mathfrak{g}.$$

Important: all $\mathfrak{g}_\nu sl(2) \oplus sl(2) \oplus \cdots$. The RHP is:

$$m_{\nu+1}(x,\lambda) = m_\nu(x,\lambda)e^{-iJ\lambda x}g_\nu(\lambda)e^{iJ\lambda x}, \qquad \nu = 0,1,\dots,2N$$

$$g_\nu(\lambda) = \hat{S}_\nu^-(\lambda)S_\nu^+(\lambda) = \hat{D}_\nu^-(\lambda)\hat{T}_\nu^+(\lambda)T_\nu^-(\lambda)D_\nu^+(\lambda).$$

Here $S_\nu^\pm(\lambda)$, $T_\nu^\pm(\lambda)$, $D_\nu^\pm(\lambda)$ are defined by the asymptotic of $m_\nu^\pm(x,\lambda)$ when $x\to\pm\infty$:

$$S_\nu^\pm(\lambda) = \lim_{x\to-\infty} \left(e^{i\lambda Jx}m_\nu(x,\lambda e^{\pm i0})e^{-i\lambda Jx}\right), \qquad \lambda \in l_\nu$$

$$T_\nu^\mp(\lambda)D_\nu^\pm(\lambda) = \lim_{x\to\infty} \left(e^{i\lambda Jx}m_\nu(x,\lambda e^{\pm i0})e^{-i\lambda Jx}\right), \qquad \lambda \in l_\nu.$$

One could write $S_\nu^\pm, T_\nu^\pm, D_\nu^\pm$ also into the form

$$S_\nu^\pm(\lambda) = \exp \sum_{\alpha\in\delta_\nu^+} s_{\nu,\alpha}^\pm(\lambda) E_{\pm\alpha}, \quad T_\nu^\pm(\lambda) = \exp \sum_{\alpha\in\delta_\nu^+} t_{\nu,\alpha}^\pm(\lambda) E_{\pm\alpha}$$

$$D_{\nu,\alpha}^\pm(\lambda) = \exp(\pm \sum_{\alpha\in\pi_\nu} d_{\nu,\alpha}^\pm(\lambda) H_\alpha).$$

In other words $S_\nu^\pm, T_\nu^\pm, D_\nu^\pm$ belong to the subgroup G_ν with Lie algebra $\mathfrak{g}_\nu$. The minimal sets of scattering data that determine uniquely $T(\lambda)$ and $U_0(x,t)$ are

$$\mathcal{T}_S = \bigcup_{\nu=0}^{1} \{s_{\nu,\alpha}^\pm(\lambda) : \alpha \in \delta_\nu^+, \lambda \in l_\nu\}, \quad \mathcal{T}_T = \bigcup_{\nu=0}^{1} \{t_{\nu,\alpha}^\pm(\lambda) : \alpha \in \delta_\nu^+, \lambda \in l_\nu\}.$$

Since $m_\nu(x,t,\lambda)$ satisfies

$$i\frac{\partial m_\nu}{\partial x} + U_0(x,t) m_\nu(x,t,\lambda) - \lambda[J_0^{(1)}, m_\nu(x,t,\lambda)] = 0.$$

If we know the solution of RHP then the corresponding potential is recovered from:

$$U_0(x,t) = \lim_{\lambda\to\infty} \lambda \left(J_0^{(1)} - m_\nu(x,t,\lambda) J_0^{(1)} m_\nu^{-1}(x,t,\lambda) \right).$$

Integrals of motion

Since

$$i\frac{dT}{d\lambda} - \lambda^2 [C^2, T(\lambda,t)] = 0.$$

where C^2 is a diagonal matrix, then the diagonal elements of $T(\lambda,t)$ are time-independent and may be considered as generating functionals of the integrals of motion. The first two of them A_1 and A_2, obtained as coefficients in the asymptotic expansion:

$$\ln T_{11}(\lambda) = (\lambda\omega)^{-1} A_1 + (\lambda\omega)^{-2} A_2 + \mathcal{O}(\lambda^{-3})$$

have the form:

$$A_1 = \frac{1}{2\alpha} \int_{-\infty}^{\infty} dx \sum_{p=1}^{N-1} \psi_p \psi_{N-p}(x,t),$$

$$A_2 = \frac{1}{2\alpha^2} \int_{-\infty}^{\infty} dx \left\{ \sum_{p=1}^{N-1} i \operatorname{cotg} \frac{\pi p}{N} \left(\psi_{p,x} \psi_{N-p}(x,t) - \psi_{N-p,x} \psi_p \right)(x,t) \right.$$

$$\left. - \frac{2}{3} \sum_{p+n+m=N} \psi_p \psi_n \psi_m (x,t) \right\},$$

where $p + n + m = 0 (\operatorname{mod} N)$. Note that under the involution (3) both A_1 and A_2 become real. The density of A_1 may be interpreted as the number of particles density, while A_2 provides the Hamiltonian.

The ISM – Generalized Fourier Transform

The analysis based on the Wronskian relations in Section 3 can be generalized to the system L with $\mathbb{Z}_h$ and $\mathbb{D}_h$ reduction groups. The mapping from the phase space $\mathcal{F}_g \ni q$ to any of the two minimal sets of scattering data $\tilde{\mathcal{T}}_k$, $k = 1, 2$ leads to the following form of the adjoint solutions and the skew–scalar product:

$$\mathcal{E}^{\pm}_{\nu;\alpha}(x,t,\lambda) = \chi^{\pm}_{\nu} E_{\alpha} \hat{\chi}^{\pm}_{\nu}(x,t,\lambda), \qquad e^{\pm}_{\nu;\alpha}(x,t,\lambda) = \mathbb{P}_{0J}\mathcal{E}^{\pm}_{\nu;\alpha}(x,t,\lambda),$$

$$[\![X, Y]\!] = \int_{-\infty}^{\infty} dx \, \langle X, [J,Y] \rangle \,,$$

where $\mathbb{P}_{0J}$ is the projector $\mathbb{P}_{0J}X \equiv \mathrm{ad}_J^{-1} \mathrm{ad}_J X$, $\mathrm{ad}_J X \equiv [J,X]$ and $\langle , \rangle$ is the Killing form on g.

The set of squared solutions $e^{\pm}_{\nu;\alpha}(x,t,\lambda)$ form a complete set of solutions in $\mathcal{F}$ for generic potentials (VSG, Yanovski (1994)).

We get the following expansions for $[J_0^{(1)}, U_0]$ and δU_0:

$$[J_0^{(1)}, U_0(x,t)] = \sum_{\nu=0}^{1} \frac{(-1)^{\nu}}{2\pi i} \int_{l_\nu} d\mu \sum_{\alpha\in\delta_\nu^+} \alpha(A) \left(s^{+}_{\nu;\alpha} e^{+}_{\nu;\alpha} + s^{-}_{\nu;\alpha} e^{-}_{\nu;-\alpha} \right) (x,t,\mu),$$

$$\delta U_0(x,t) = \sum_{\nu=0}^{1} \frac{(-1)^{\nu}}{2\pi i} \int_{l_\nu} d\mu \sum_{\alpha\in\delta_\nu^+} \left(\delta s^{+}_{\nu;\alpha} e^{+}_{\nu;\alpha} - \delta s^{-}_{\nu;\alpha} e^{-}_{\nu;-\alpha} \right) (x,t,\mu).$$

The minimal set of scattering data $\tilde{\mathcal{T}}_{1,2}$ consists of the expansion coefficients of the potential $U_0(x,t)$ over the squared solutions. The expansions above are generalized Fourier transforms which linearize the corresponding NLEE.

Introduce the generating operators $\Lambda^{\pm}$ by:

$$(\Lambda^{+,N} - \lambda^N) e^{\pm}_{\nu,\mp\alpha}(x,\lambda) = 0, \qquad (\Lambda^{-,N} - \lambda^N) e^{\pm}_{\nu,\pm\alpha}(x,\lambda) = 0.$$

They are expressed in terms of U_0 by:

$$\Lambda^{\pm} Z = \mathrm{ad}_J^{-1} \left\{ i\frac{dZ}{dx} + \mathbb{P}_{0J}[U_0, Z] + i\sum_{p=1}^{h-1} [U_0, X_0^{(p)}] \int_{\mp\infty}^{x} \left\langle [X_0^{(h-p)}, U_0(y)], Z \right\rangle dy \right\}.$$

The class of NLEE that can be linearized and solved by this generalized Fourier transform has the form:

$$iU_{0,t} + \sum_{k=1}^{h-1} f_k {\Lambda^{\pm}}^k \left[(J_0^{(1)})^k, U_0(x,t) \right] = 0,$$

where $f_k(\lambda)$ are polynomials in λ (or in λ^{-1}). Each NLEE is determined by its dispersion law $f(\lambda) = \sum_{k=1}^{h-1} f_k \lambda^k J_0^{(k)} = \sum_{k=1}^{h-1} \tilde{f}_k(\lambda) E_{kk} \in \mathfrak{h}$.

From the expansions we easily find that the NLEE is equivalent to the following linear evolution of the scattering data:

$$i\frac{dS_\nu^\pm}{dt} + \left[f(\lambda), S_\nu^\pm\right] = 0, \qquad i\frac{dT_\nu^\pm}{dt} + \left[f(\lambda), T_\nu^\pm\right] = 0, \qquad i\frac{dD_\nu^\pm}{dt} = 0.$$

For the $\mathbb{Z}_h$–DNLS the dispersion law is:

$$f_{\mathbb{Z}_h\text{–NLS}}(\lambda) = \lambda^2 J_0^{(2)}.$$

The hierarchy of Hamiltonian structures and action-angle variables

Drinfeld, V. G. Sokolov, V. V. (1984).
Kulish, P. P., Reiman, A. G. (1983).

Poisson brackets

Kulish, Reyman (1984)

Define $\psi_j(x)$ as linear functionals of $U_0(x,t,\lambda)$ by:

$$\psi_j(x) = \frac{1}{N} \operatorname{tr} U(x,t,\lambda) J^{(0)}_{N-j},$$

Then

$$\{\psi_j(x), \psi_k(x)\} = \delta_{k+j-N} \delta'(x-y).$$

Obviously, the Poisson brackets and the Hamiltonian $H = 2\alpha^2\gamma A_2$ lead to $\mathbb{Z}_N$ DNLS.

The symplectic basis

Introduce the symplectic basis

$$\mathcal{P}_{\alpha,\nu}(x,t,\lambda), \qquad \mathcal{Q}_{\alpha,\nu}(x,t,\lambda), \qquad \alpha \in \delta_\nu^+, \lambda \in l_\nu$$

as special linear combinations of $e^{\pm}_{\nu;\alpha}$. Then we can get:

$$[J_0^{(1)}, U_0(x,t)] = \sum_{\nu=0}^{1} \frac{(-1)^\nu}{2\pi i} \int_{l_\nu} d\mu \sum_{\alpha\in\delta_\nu^+} \alpha(A) \left(\kappa_{\nu;\alpha}\mathcal{P}_{\alpha,\nu}(x,t,\lambda)\right)(x,t,\mu),$$

$$\delta U_0(x,t) = \sum_{\nu=0}^{1} \frac{(-1)^\nu}{2\pi i} \int_{l_\nu} d\mu \sum_{\alpha\in\delta_\nu^+} \left(\delta\kappa_{\nu;\alpha}\mathcal{Q}_{\alpha,\nu} + \delta\eta_{\nu;\alpha}\mathcal{P}_{\alpha,\nu}\right)(x,t,\mu).$$

$$\Omega_0 = \left[\!\left[\delta U_0 \mathop{\wedge}_{,} \partial_x^{-1}\delta U_0\right]\!\right] = \sum_{\nu=0}^{1} \frac{(-1)^\nu}{2\pi i} \int_{l_\nu} d\mu \sum_{\alpha\in\delta_\nu^+} \delta\kappa_{\nu;\alpha} \wedge \delta\eta_{\nu;\alpha}$$

$$\Omega_k = \left[\!\left[\delta U_0 \mathop{\wedge}_{,} \Lambda^{kN}\partial_x^{-1}\delta U_0\right]\!\right] = \sum_{\nu=0}^{1} \frac{(-1)^\nu}{2\pi i} \int_{l_\nu} d\mu \sum_{\alpha\in\delta_\nu^+} \lambda^{kN}\delta\kappa_{\nu;\alpha} \wedge \delta\eta_{\nu;\alpha}$$

Conclusions

- There are many other eqs. with $\mathbb{Z}_h$ and $\mathbb{D}_h$ symmetries that can be solved by the inverse scattering method;
- all of them share the same Action - Angle variables;
  so we have a hierarchy of infinite-dimensional completely integrable NLEE.
- The inverse scattering method is a generalization of the Fourier transform method.

Участия в конференции

- **2-6 September 2019, Dijon, France.** International conference Classical and Quantum Integrability. On dressing factors of 2-dimensional Toda field theories and multicomponent MKdV equations.
- **August 5 to 9, Yaroslavl, RUSSIA 2019** The IX-th International Conference: Solitons, collapses and turbulence.
  On dressing factors and soliton solutions of 2-dimensional Toda field theories.
- **11-th AMITANS, June 20-25, 2019, Albena, Bulgaria**
  Kulish-Sklyanin type models: integrability, reductions and soliton solutions.

Участия в проекти с НФНИ

- Интегрируеми системи: нетривиални групи от редукции, преобразования на Дарбу и вълни-убийци. договор НТС – Русия 02/101 от 23.10.2017.

- A. O. Smirnov, M. V. Pavlov, V. B. Matveev, V. S. Gerdjikov. Finite-Gap Solutions of the Mikhalev Equation. AMS collection of papers dedicated to the memory of B. A. Dubrovin (1950–2019). Submitted to Proceedings of Symposiums in Pure Mathematics. “Integrability, Quantization, and Geometry”.

- Две командировки в Петербург, Русия за научни обсъждания по проекта;
  командировка в Ярославл, Русия за участие в конференция;
  командировка в Албена за участие в АМИТАНС 2019.

Публикации

- V. S. Gerdjikov, M. D. Todorov. Manakov model with gain/loss terms and N-soliton Interactions: Effects of Periodic Potentials. Applied Numerical Mathematics **141**, July (2019), Pages 62–80 **arXiv:1801.04897v1 [nlin.SI]**.

- V. S. Gerdjikov, A. A. Stefanov. On an example of derivative nonlinear Schrödinger equation with $\mathbb{D}_2$ reduction. Pliska Stud. Math. **30** (2019) 99-108.

- V. S. Gerdjikov. On the Integrability of Ablowitz-Ladik models with local and nonlocal reductions. To the Proceedings of "Problems of Mathematical Physics and Mathematical Modeling", MEPhi, Moscow, Russia, (2018). Journal of Physics: Conf. Series **1205** (2019) 012015.

- Vladimir S. Gerdjikov, Rossen I. Ivanov and Alexander A. Stefanov. Riemann-Hilbert Problem, Integrability and Reductions. Journal of Geometric Mechanics **11** no. 2, 167–185 (2019);
  Special volume in honor of Darryl Holm's 70-th birthday. **arXive: 1902.10276 [nlin.SI]**

- A. Streche-Pauna, A. D. Florian, V. S. Gerdjikov. On the spectral properties of Lax operators related to BD.I symmetric spaces. Proceedings of BGSIAM18. Advanced Computing in Industrial Mathematics, Eds: Ivan Georgiev, Hristo Kostadinov, Elena Lilkova.
  Studies in Computational Intelligence Series Ed.: Kacprzyk, Janusz (In press)

Thank you for your attention!