Logics for relational geometric structures: distributive mereotopology, extended contact algebras and related quantifier-free logics

In the classical Euclidean geometry the notion of point is taken as one of the basic primitive notions. In contrast the region-based theory of space (RBTS) has as primitives the more realistic notion of region as an abstraction of physical body, together with some basic relations and operations on regions. Some of these relations are mereological - part-of ($x \leq y$), overlap (xOy), its dual underlap ($x\widehat{O}y$). Other relations are topological - contact (xCy), nontangential part-of ($x \ll y$), dual contact ($x\widehat{C}y$). The motivation for taking region as a primary notion is that points, lines and planes do not have separate existence in the reality. RBTS has simpler way of representing of qualitative spatial information.

Contact algebra is one of the main tools in RBTS.

Contact algebra is a Boolean algebra $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, C)$ with an additional binary relation C called *contact*, and satisfying the following axioms:

(C1) If aCb, then $a \neq 0$ and $b \neq 0$,

(C2) If aCb and $a \leq a'$ and $b \leq b'$, then $a'Cb'$,

(C3) If $aC(b + c)$, then aCb or aCc,

(C4) If aCb, then bCa,

(C5) If $a \cdot b \neq 0$, then aCb.

The elements of contact algebra are called regions. Boolean operations are considered as operations for constructing new regions from given ones. The unit element 1 symbolizes the region containing as its parts all regions, and the zero region 0 symbolizes the non-existing region. The contact relation is used also to define non-tangential inclusion, dual contact in the following way: $a \ll b \leftrightarrow a\overline{C}b^*$, $a\widehat{C}b \leftrightarrow a^*Cb^*$.

There is a problem in the motivation of the operation of Boolean complementation. A question arises: if a represents some physical body, what kind of body represents a^*. To avoid this problem, we drop the operation $*$. The topological relations of dual contact and nontangential inclusion cannot be defined without $*$ and because of this we take them as primary in the language. So we consider the language $\mathcal{L}(0, 1; +, \cdot; \leq, C, \widehat{C}, \ll)$ which is an extension of the language of distributive lattice with the predicate symbols for the relations of contact, dual contact and nontangential inclusion. We obtain an axiomatization of the theory consisting of the universal formulas in the language $\mathcal{L}$ true in all contact algebras. The structures in $\mathcal{L}$, satisfying the axioms in question, are called extended distributive contact lattices (EDC-lattices). We develop topological representation theory of EDC-lattices.

Next we consider the predicate c^o - internal connectedness. Let X be a topological space and x be a regular closed set in X. Let $c^o(x)$ means that $Int(x)$ is a connected topological space in the subspace topology. We prove that the predicate internal connectedness cannot be defined in the language of contact algebras. Because of this we add to the language a new ternary predicate symbol $\vdash$. It turns out that the predicate c^o can be defined in the new language. We define *extended contact algebras* - Boolean algebras with added relations $\vdash$, C and c^o, satisfying some axioms, and prove that every extended contact algebra can be isomorphically embedded in the contact algebra of the regular closed subsets of some compact, semiregular, T_0 topological space with added relations $\vdash$ and c^o. So extended contact algebra can be considered an axiomatization of the theory, consisting of the universal formulas true in all topological contact algebras with added relations $\vdash$ and c^o.

It turns out that the logics, related to EDC-lattices and EC-algebras, are decidable.