Images of non-commutative polynomials evaluated on algebras

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Let p be a polynomial in several non-commuting variables with coefficients in a field K of arbitrary characteristic. It has been conjectured that for any n, for p multilinear, the image of p evaluated on the set $M_n(K)$ of n by n matrices is either zero, or the set of scalar matrices, or the set $\mathrm{sl}_n(K)$ of matrices of trace 0, or all of $M_n(K)$. This expository paper describes research on this problem and related areas. We discuss the solution of this conjecture for $n = 2$ in Section 2, some decisive results for $n = 3$ in Section 3, and partial information for $n \geq 3$ in Section 4, also for non-multilinear polynomials. In addition we consider the case of K not algebraically closed, and polynomials evaluated on other finite dimensional simple algebras (in particular the algebra of the quaternions). This review recollects results and technical material of our previous papers, as well as new results of other researches, and applies them in a new context. This article also explains the role of the Deligne trick, which is related to some nonassociative cases in new situations, underlying our earlier, more straightforward approach. We pose some problems for future generalizations and point out possible generalizations in the present state of art, and in the other hand providing counterexamples showing the boundaries of generalizations.