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On the asymptotics for $*$-graded Capelli identities

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- $f = f(x_1, \dots, x_n) \in F\langle X\rangle$ is a polynomial identity for A if $f(a_1, \dots, a_n) = 0$, for all $a_1, \dots, a_n \in A$.

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- $Id(A)$ = the set of all identities of A.
- $\mathrm{var}(\Gamma) = \mathrm{var}(A)$ = the variety of algebras having the elements of $\Gamma = Id(A)$ as identities.
- $\Gamma = Id(A)$ is verbally prime if for any T-ideals I_1, I_2 such $I_1 I_2 \subseteq \Gamma$ we must have $I_1 \subseteq \Gamma$ or $I_2 \subseteq \Gamma$.

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  Also A and $\mathrm{var}(A)$ are called verbally prime.

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- $\mathrm{var}(A)$ is verbally prime $\Leftrightarrow$ A is one of the following

$$M_k(F),\ M_k(G),\ M_{k,l}(G)$$

where $G = G_0 \oplus G_1$ is the Grassmann algebra and

$$M_{k,l}(G) = \begin{array}{c} \\ k \\ l \end{array} \begin{array}{c} \begin{array}{cc} k & l \end{array} \\ \begin{pmatrix} G_0 & G_1 \\ G_1 & G_0 \end{pmatrix} \end{array}.$$

Problem. to describe the verbally prime T-ideals of $F\langle X\rangle$

$$Id(M_k(F)),\ Id(M_k(G)),\ Id(M_{k,l}(G))$$

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V_n= the space of multilinear polynomials of degree n in the variables $x_1, \dots, x_n$.

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- In the early 1980's, Amitsur conjectured that the exponential growth of the codimension sequence should be an integer.

- Giambruno and Zaicev (1998, 1999) if A is a PI-algebra, then

$$exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

exists and it is a non negative integer called PI-exponent of A.

- Regev (1984-1987), Berele-Regev (1995), Giambruno and Zaicev (1999)

$$\exp(M_k(F)) = k^2$$

$$\exp(M_k(G)) = 2k^2$$

$$\exp(M_{k,l}(G)) = (k+l)^2$$

- Regev (1984)

$$c_n(M_k(F)) \simeq \alpha k^{\frac{1}{2}(k^2+4)} n^{\frac{1-k^2}{2}} (k^2)^n$$

where $\alpha = \left(\frac{1}{\sqrt{2\pi}}\right)^{k-1} \left(\frac{1}{2}\right)^{\frac{1}{2}(k^2-1)} 1!2!\cdots(k-1)!$

- Berele-Regev (1995),(2020)

$$c_n(M_{k,l}(G)) \simeq \alpha n^{-\frac{1}{2}(k^2+l^2-1)} (k^2+l^2)^n$$

$$c_n(M_k(G)) \simeq \alpha n^{\frac{1-k^2}{2}} (2k^2)^n$$

with $\alpha = ?$

The polynomial (Razmyslov (1973))

$$Cap_m(x_1, \ldots, x_m; y_1, \ldots, y_{m-1}) =$$

$$= \sum_{\sigma \in S_m} (\mathrm{sgn}\sigma) x_{\sigma(1)} y_1 x_{\sigma(2)} \cdots x_{\sigma(m-1)} y_{m-1} x_{\sigma(m)}$$

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The polynomial (Amitsur-Regev (1982))

$$e^*_{d,l}(x_1, \dots, x_n; y_1, \dots, y_{n-1}) =$$

$$\sum_{\sigma \in S_n} \chi_{d,l}(\sigma) x_{\sigma(1)} y_1 x_{\sigma(2)} \cdots y_{n-1} x_{\sigma(n)},$$

is the Amitsur's Capelli-type polinomial,
$n = (d+1)(l+1)$ and $\chi_{d,l} = \chi_\mu$ with $\mu = ((l+1)^{(d+1)}) \vdash n$.

- Regev (1984)

$$\exp(Cap_{k^2+1}) = \exp(E^*_{k^2,0}) = k^2 = \exp(M_k(F))$$

- Berele-Regev (2001)

$$\exp(E^*_{k^2,k^2}) = 2k^2 = \exp(M_k(G))$$

$$\exp(E^*_{k^2+l^2,2kl}) = (k+l)^2 = \exp(M_{k,l}(G))$$

- Giambruno-Zaicev (2003)

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- B.- Sviridova (2006)

$$c_n(E^*_{k^2,k^2}) \simeq c_n(M_k(G))$$

and

$$c_n(E^*_{k^2+l^2,2kl}) \simeq c_n(M_{k,l}(G))$$

- $X = Y \cup Z$ disjoint union of two countable sets,

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the free superalgebra with grading $(\mathcal{F}^{(0)}, \mathcal{F}^{(1)})$,

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Remark $A = A^{(0)} \oplus A^{(1)}$= superalgebra with grading $(A^{(0)}, A^{(1)})$, if $A^{(0)}, A^{(1)}$ are subspaces of A satisfying:

$$A^{(0)}A^{(0)} + A^{(1)}A^{(1)} \subseteq A^{(0)} \text{ and } A^{(0)}A^{(1)} + A^{(1)}A^{(0)} \subseteq A^{(1)}.$$

- Let $A = A^{(0)} \oplus A^{(1)}$ be a superalgebra. Then a polynomial $f = f(y_1, \ldots, y_n, z_1, \ldots, z_m) \in F\langle Y \cup Z\rangle$ is a graded identity or superidentity for A if

$$f(a_1, \ldots, a_n, b_1, \ldots, b_m) = 0,$$

for all $a_1, \ldots, a_n \in A^{(0)}$ and $b_1, \ldots, b_m \in A^{(1)}$.

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- an associative variety is a prime variety if and only if it is generated by the Grassmann envelope of a finite dimensional simple superalgebra.

From Kemer's theory

If F is an algebraically closed field of characteristic zero, then a simple finite dimensional superalgebra is isomorphic to one of the following algebras:

- $M_k(F)$ with trivial grading $(M_k(F), 0)$;
- $$M_{k,l}(F) = \begin{matrix} & k & l \\ k & & \\ l & & \end{matrix} \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \text{ with grading } \begin{pmatrix} F_{11} & 0 \\ 0 & F_{22} \end{pmatrix}$$

  where F_{11}, F_{12}, F_{21}, F_{22} are $k \times k$, $k \times l$, $l \times k$ and $l \times l$ matrices respectively, $k \geq 1$ and $l \geq 1$;
- $M_k(F \oplus cF)$ with grading $(M_k(F), cM_k(F))$, where $c^2 = 1$.

Problem. to describe the T_2-ideals of graded identities of the simple finite dimensional superalgebras,

$$Id^{sup}(M_k(F)),\ Id^{sup}(M_{k,l}(F)),\ Id^{sup}(M_k(F \oplus cF))$$

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- $c_n^{sup}(A) = \dim_F \left(\frac{V_n^{sup}}{V_n^{sup} \cap Id^{sup}(A)} \right)$ is called the *n*-th supercodimension of A.

V_n^{sup}= the space of multilinear polynomials of degree n in the variables $y_1, z_1, \ldots, y_n, z_n$ (i.e., y_i or z_i appears in each monomial at degree 1).

- Giambruno and Regev (1985) $c_n^{sup}(A)$ is exponentially bounded if and only if A satisfies an ordinary polynomial identity.

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- B., Giambruno and Pipitone (2003) if A is a finitely generated superalgebra satisfying a polynomial identity, then

$$supexp(A) = \lim_{n \to \infty} \sqrt[n]{c_n^{sup}(A)}$$

exists and is a non negative integer called superexponent (or $\mathbb{Z}_2$-exponent) of A.

- B., Giambruno and Pipitone (2003)

$$supexp(M_k(F)) = k^2$$

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Partial results were obtained by

- Di Vincenzo and Nardozza (2012) If G is a group of prime order.
- Giambruno and La Mattina (2010) If G is a finite abelian group.
- Aljadeff, Giambruno and La Mattina (2011) If A is a finitely generated algebra and G a finite abelian group.

- Karasik and Shpigelman (2016) If G is a finite group, then the G-graded codimension sequence of a finite dimensional G-simple F-algebra A is

$$c_n^G(A) \simeq \alpha n^{\frac{1-dim_F A_\epsilon}{2}} (dim_F A)^n$$

where α is some positive real number and A_ϵ denotes the identity component of A.
In the special case where A is the algebra of matrices with an arbitrary elementary G-grading the constant α was explicitly calculated.

- The polynomial

$$Cap_m[T,X] = Cap_m[t_1,\dots,t_m;x_1,\dots,x_{m-1}] =$$

$$= \sum_{\sigma\in S_m} (\mathrm{sgn}\sigma) t_{\sigma(1)} x_1 t_{\sigma(2)} \cdots t_{\sigma(m-1)} x_{m-1} t_{\sigma(m)}$$

is the m-th graded Capelli polynomial in the homogeneous variables $t_1,\dots,t_m$ ($x_1,\dots,x_{m-1}$ are arbitrary variables).

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is the *m-th graded Capelli polynomial* in the homogeneous variables $t_1,\dots,t_m$ ($x_1,\dots,x_{m-1}$ are arbitrary variables).

Remark In particular, $Cap_m[Y,X]$ and $Cap_m[Z,X]$ denote the m-th graded Capelli polynomials in the alternanting variables of homogeneous degree zero $y_1,\dots,y_m$ and of homogeneous degree one $z_1,\dots,z_m$, respectively.

- B. (2013)

$$\mathrm{supexp}(\Gamma^{\mathrm{sup}}_{k^2+1,1}) = k^2 = \mathrm{supexp}(M_k(F))$$

$$\mathrm{supexp}(\Gamma^{\mathrm{sup}}_{k^2+l^2+1,2kl+1}) = (k+l)^2 = \mathrm{supexp}(M_{k,l}(F))$$

$$\mathrm{supexp}(\Gamma^{\mathrm{sup}}_{k^2+1,k^2+1}) = 2k^2 = \mathrm{supexp}(M_k(F \oplus cF)).$$

$\Gamma^{\mathrm{sup}}_{M+1,L+1}$ = the T_2-ideal generated by the polynomials Cap^0_{M+1}, Cap^1_{L+1}.

- Giambruno and Zaicev (2003)

$$c_n^{sup}(\Gamma^{\mathrm{sup}}_{k^2+1,1}) \simeq c_n^{sup}(M_k(F)).$$

- B. (2015)

$$c_n^{sup}(\Gamma^{\mathrm{sup}}_{k^2+l^2+1,2kl+1}) \simeq c_n^{sup}(M_{k,l}(F))$$

and

$$c_n^{sup}(\Gamma^{\mathrm{sup}}_{k^2+1,k^2+1}) \simeq c_n^{sup}(M_k(F \oplus cF)).$$

- We consider

$$F\langle X, *\rangle = F\langle Y \cup Z\rangle$$

the free associative algebra over F with involution $*$

Y= set of symmetric variables, $y_i = x_i + x_i^*$,

Z= set of skew variables, $z_i = x_i - x_i^*$.

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Z= set of skew variables, $z_i = x_i - x_i^*$.

Let $(A, *)$ be an algebra with involution.

- $Id^*(A)$= the set of all $*$-identities of A.
- $\mathrm{var}^*(\Gamma) = \mathrm{var}^*(A)$ = the variety of $*$-algebras having the elements of $\Gamma = Id^*(A)$ as $*$-identities.

From Rowen (1988), Giambruno and M. Zaicev (2005)

If F is an algebraically closed field of characteristic zero, then, up to isomorphisms, all $*$-simple finite dimensional algebras are the following ones:

- $(M_k(F), t)$ the algebra of $k \times k$ matrices with the transpose involution;
- $(M_{2m}(F), s)$ the algebra of $2m \times 2m$ matrices with the symplectic involution;
- $(M_h(F) \oplus M_h(F)^{op}, exc)$ the direct sum of the algebra of $h \times h$ matrices and the opposite algebra with the exchange involution.

Problem. to describe the T-$*$-ideals of $*$-polynomial identities of $*$-simple finite dimensional algebras,

$$Id^*((M_k(F),t)), Id^*((M_{2m}(F),s)), Id^*((M_h(F)\oplus M_h(F)^{op}, exc))$$

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- $c_n^*(A) = \dim\left(\frac{V_n^*}{V_n^* \cap Id^*(A)}\right)$ is called the *n-th $*$-codimension* of $(A, *)$.

V_n^*= the space of multilinear polynomials of degree n in the variables $y_1, z_1, \cdots, y_n, z_n$.

- Giambruno and Regev (1985) if A satisfies a non trivial $*$-polynomial identity then $c_n^*(A) \leq ab^n$, for some constants a and b and for all $n \geq 1$.

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- Bahturin, Giambruno and Zaicev (1999) an explicit exponential bound for $c_n^*(A)$ was exhibited.

- Berele, Giambruno and Regev (1996)

$$c_n^*((M_k(F),t)) \simeq$$

$$\left[\sqrt{k}^{\frac{k(k-1)}{2}} \frac{1}{k!} \Gamma\left(\frac{3}{2}\right)^{-k} \prod_{j=1}^{k} \Gamma\left(1+\frac{1}{2}j\right) \left(\frac{1}{2}\right)^{k-1}\right] \left(\frac{1}{\sqrt{2n}}\right)^{\frac{k(k-1)}{2}} k^{2n},$$

$$c_n^*((M_{2m}(F),s)) \simeq$$

$$\left[\left(\frac{1}{\sqrt{2\pi}}\right)^{m} 2^{\frac{m^2+m+2}{4}} m^{\frac{m(7m-1)}{4}} \frac{1}{n!} \prod_{j=1}^{m} \Gamma(2j+1)\right] \left(\frac{1}{2n}\right)^{\frac{m(2m+1)}{2}} (2m)^{2n},$$

where $\Gamma(x)$ is the Euler's gamma function.

- B. and Valenti (2021), Giambruno, La Mattina and Polcino Milies (2020)

$$c_n^*((M_h(F) \oplus M_h(F)^{op}, exc)) \simeq$$

$$\left(\frac{1}{\sqrt{2\pi}}\right)^{h-1} \left(\frac{1}{2}\right)^{\frac{1}{2}(h^2+1)} 1!2!\cdots(h-1)!\,h^{\frac{1}{2}(h^2+4)} n^{\frac{1-h^2}{2}} (2h^2)^n.$$

- Giambruno and Zaicev (1999) if A is a finite dimensional algebra with involution, then

$$\exp^*(A) = \lim_{n\to\infty} \sqrt[n]{c_n^*(A)}$$

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- Giambruno, Polcino Milies and Valenti (2017) for any algebra with involution A, it was showed that the $*$-exponent of A, $\exp^*(A)$, exists and is a non negative integer.

It follows that

$$\exp^*((M_k(F), t)) = k^2,$$

$$\exp^*((M_{2m}(F), 2s)) = 4m^2,$$

$$\exp^*((M_h(F) \oplus M_h(F)^{op}, exc)) = 2h^2.$$

We denote by

$$Cap^*_m[Y,X] = Cap_m(y_1,\dots,y_m;x_1,\dots,x_{m-1})$$

and

$$Cap^*_m[Z,X] = Cap_m(z_1,\dots,z_m;x_1,\dots,x_{m-1})$$

the m-th $*$-Capelli polynomial in the alternating symmetric variables $y_1,\dots,y_m$ and skew variables $z_1,\dots,z_m$, ($x_1,\dots,x_{m-1}$ are arbitrary variables).

- B. and Valenti (2019)

$$\exp^*(\Gamma^*_{k(k+1)/2+1,k(k-1)/2+1}) = k^2 = \exp^*((M_k(F), t))$$

$$\exp^*(\Gamma^*_{m(2m-1)+1,m(2m+1)+1}) = 4m^2 = \exp^*((M_{2m}(F), 2s))$$

$$\exp^*(\Gamma^*_{h^2+1,h^2+1}) = 2h^2 = \exp^*((M_h(F) \oplus M_h(F)^{op}, exc)).$$

$\Gamma^*_{M+1,L+1}$= the T-$*$-ideal generated by the polynomials Cap^+_{M+1}, Cap^-_{L+1}

- B. and Valenti (2019)

$$c_n^*(\Gamma^*_{k(k+1)/2+1,k(k-1)/2+1}) \simeq c_n^*((M_k(F),t))$$

$$c_n^*(\Gamma^*_{m(2m-1)+1,m(2m+1)+1}) \simeq c_n^*((M_{2m}(F),2s))$$

$$c_n^*(\Gamma^*_{h^2+1,h^2+1}) \simeq c_n^*((M_h(F)\oplus M_h(F)^{op}, exc)).$$

- $F =$ field, $\quad \mathrm{char} F = 0$.
- We consider the free $*$-superalgebra over F with graded involution $*$

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Remark Given a superalgebra $A = A_0 \oplus A_1$ endowed with an involution $*$, we say that $*$ is a graded involution if it preserves the homogeneous components of A, i.e. if $A_i^* \subseteq A_i$, $i = 0, 1$. A superalgebra endowed with a graded involution is called $*$-superalgebra.

Let $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$ be a $*$-superalgebra.

- The polynomial

$$f = f(y_1^+, \dots, y_n^+, y_1^-, \dots, y_m^-, z_1^+, \dots, z_p^+, z_1^-, \dots, z_q^-)$$

is a $*$-graded polynomial identity for A if

$$f(a_{1,0}^+, \dots, a_{n,0}^+, a_{1,0}^-, \dots, a_{m,0}^-, a_{1,1}^+, \dots, a_{p,1}^+, a_{1,1}^-, \dots, a_{q,1}^-) = 0$$

for all $a_{1,0}^+, \dots, a_{n,0}^+ \in A_0^+$, $a_{1,0}^-, \dots, a_{m,0}^- \in A_0^-$, $a_{1,1}^+, \dots, a_{p,1}^+ \in A_1^+$, $a_{1,1}^-, \dots, a_{q,1}^- \in A_1^-$.

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- $\mathrm{var}^*_{\mathbb{Z}_2}(\Gamma) = \mathrm{var}^*_{\mathbb{Z}_2}(A)$ = the variety of $*$-superalgebras having the elements of $\Gamma = Id^*_{\mathbb{Z}_2}(A)$ as $*$-graded identities.

From Giambruno, dos Santos and Vieira (2016)

If F is an algebraically closed field of characteristic zero, then, up to isomorphisms, the only finite dimensional simple $*$-superalgebras are the following

- $(M_{h,l}(F), \diamond)$, with $h \geq l \geq 0$, $h \neq 0$;
- $(M_{h,l}(F) \oplus M^{op}_{h,l}(F), exc)$, with $h \geq l \geq 0$, $h \neq 0$, and induced grading;
- $(M_n(F + cF), \star)$, with involution given by $(a + cb)^\star = a^\diamond - cb^\diamond$;
- $(M_n(F + cF), \dagger)$, with involution given by $(a + cb)^\dagger = a^\diamond + cb^\diamond$;
- $((M_n(F + cF)) \oplus (M_n(F + cF))^{op}, exc)$, with grading $(M_n \oplus M_n^{op}, c(M_n \oplus M_n^{op}))$;

where $\diamond = t, s$ denotes the transpose or symplectic involution and exc is the exchange involution.

Problem. to describe the $T^*_{\mathbb{Z}_2}$-ideals of $*$-graded polynomial identities of finite dimensional simple $*$-superalgebras.

Remark In case char$F$=0, it is well known that $Id^{*}_{\mathbb{Z}_2}(A)$ is completely determined by its multilinear $*$-polynomials

Remark In case char$F$=0, it is well known that $Id^*_{\mathbb{Z}_2}(A)$ is completely determined by its multilinear $*$-polynomials

- $c_n^{(\mathbb{Z}_2,*)}(A) = \dim\left(\frac{V_n^{(\mathbb{Z}_2,*)}}{V_n^{(\mathbb{Z}_2,*)} \cap Id^*_{\mathbb{Z}_2}(A)}\right)$ is called the n-th $*$-graded codimension of the $*$-superalgebra A.

$V_n^{(\mathbb{Z}_2,*)}$= the space of multilinear polynomials of degree n in the variables $y_1^+, y_1^-, z_1^+, z_1^-, \ldots, \ y_n^+, y_n^-, z_n^+, z_n^-$.

- Giambruno, dos Santos and Vieira (2016) If A is a PI-algebra, i.e. satisfies an ordinary polynomial identity, then the sequence $\{c_n^{(\mathbb{Z}_2,*)}(A)\}_{n\geq 1}$ is exponentially bounded.

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$$\exp^{*}_{\mathbb{Z}_2}(A) = \lim_{n\to\infty} \sqrt[n]{c_n^{(\mathbb{Z}_2,*)}(A)}$$

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- Giambruno, Ioppolo and La Mattina (2019) The existence of the $*$-graded exponent is extended to any finitely generated PI-$*$-superalgebra.

- Gordienko (2013) If A is a finite dimensional simple $*$-superalgebra, then

$$\exp^{*}_{\mathbb{Z}_2}(A) = \dim_F A.$$

Problem. If A is a finite dimensional simple $*$-superalgebra, then

$$c_n^{(\mathbb{Z}_2,*)}(A) \simeq ?$$

*-Graded Capelli Polynomials and their Asymptotics

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Definition. We denote by

$$Cap_m^{(\mathbb{Z}_2,*)}[Y^+,X],\ Cap_m^{(\mathbb{Z}_2,*)}[Y^-,X],$$

$$Cap_m^{(\mathbb{Z}_2,*)}[Z^+,X] \text{ and } Cap_m^{(\mathbb{Z}_2,*)}[Z^-,X]$$

the *m-th $*$-graded Capelli polynomial* alternating in the symmetric variables of degree zero $y_1^+,\dots,y_m^+$, in the skew variables of degree zero $y_1^-,\dots,y_m^-$, in the symmetric variables of degree one $z_1^+,\dots,z_m^+$ and in the skew variables of degree one $z_1^-,\dots,z_m^-$, respectively ($x_1,\dots,x_{m-1}$ are arbitrary variables).

*-Graded Capelli Polynomials and their Asymptotics

- $\overline{Cap}_m^{(\mathbb{Z}_2,*)}[Y^+,X]$= the set of 2^{m-1} polynomials obtained by deleting any subset of variables x_i from $Cap_m^{(\mathbb{Z}_2,*)}[Y^+,X]$.

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- In a similar way

$$\overline{Cap}_m^{(\mathbb{Z}_2,*)}[Y^-,X],\ \overline{Cap}_m^{(\mathbb{Z}_2,*)}[Z^+,X] \text{ and } \overline{Cap}_m^{(\mathbb{Z}_2,*)}[Z^-,X].$$

*-Graded Capelli Polynomials and their Asymptotics

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- If M^+, M^-, L^+ and L^- are natural numbers, then

$$\Gamma_{M^{\pm},L^{\pm}}^{(\mathbb{Z}_2,*)}$$

denotes the $T^*_{\mathbb{Z}_2}$-ideal generated by $\overline{Cap}_{M^+}^{(\mathbb{Z}_2,*)}[Y^+,X]$, $\overline{Cap}_{M^-}^{(\mathbb{Z}_2,*)}[Y^-,X]$, $\overline{Cap}_{L^+}^{(\mathbb{Z}_2,*)}[Z^+,X]$ and $\overline{Cap}_{L^-}^{(\mathbb{Z}_2,*)}[Z^-,X]$.

*-Graded Capelli Polynomials and their Asymptotics

- B. and Valenti (2021) If $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$ is a finite dimensional simple $*$-superalgebra with $M^\pm = \dim A_0{}^\pm$ and $L^\pm = \dim A_1{}^\pm$, then

$$\exp^*_{\mathbb{Z}_2}(\Gamma^{(\mathbb{Z}_2,*)}_{M^\pm+1,L^\pm+1}) = M^+ + M^- + L^+ + L^- = \dim A = \exp^*_{\mathbb{Z}_2}(A)$$

*-Graded Capelli Polynomials and their Asymptotics

Theorem(B. and Valenti (2021))

For suitable natural numbers M^+, M^-, L^+, L^- there exists a finite dimensional simple $*$-superalgebra A such that

$$\mathrm{var}^*_{\mathbb{Z}_2}(\Gamma^{(\mathbb{Z}_2,*)}_{M^\pm+1,L^\pm+1}) = \mathrm{var}^*_{\mathbb{Z}_2}(A \oplus D),$$

where D is a finite dimensional $*$-superalgebra such that $\exp^*_{\mathbb{Z}_2}(D) < M + L$, with $M = M^+ + M^-$ and $L = L^+ + L^-$.

*-Graded Capelli Polynomials and their Asymptotics

Theorem(B. and Valenti (2021))

In particular

1) $A = (M_{h,l}, t)$, if $M^{\pm} = \frac{h(h\pm 1)}{2} + \frac{l(l\pm 1)}{2}$ and $L^{\pm} = hl$, $h \geq l > 0$;
2) $A = (M_{h,h}, s)$, if $M^{\pm} = h^2$ and $L^{\pm} = h(h \mp 1)$, $h > 0$;
3) $A = (M_{h,l} \oplus M_{h,l}^{op}, exc)$, if $M^{\pm} = h^2 + l^2$ and $L^{\pm} = 2hl$, $h \geq l > 0$;
4) $A = (M_n + cM_n, *)$ where $(a + cb)^* = a^t \pm cb^t$, if $M^+ = L^{\pm} = \frac{n(n+1)}{2}$, $M^- = L^{\mp} = \frac{n(n-1)}{2}$, $n > 0$;
5) $A = (M_n + cM_n, *)$ where $(a + cb)^* = a^s \pm cb^s$, if $M^+ = L^{\pm} = \frac{n(n-1)}{2}$, $M^- = L^{\mp} = \frac{n(n+1)}{2}$, $n > 0$;
6) $A = ((M_n + cM_n) \oplus (M_n + cM_n)^{op}, exc)$, if $M^{\pm} = L^{\pm} = n^2$, $n > 0$.

*-Graded Capelli Polynomials and their Asymptotics

Corollary (B. and Valenti (2021))

$$c_n^{(\mathbb{Z}_2,*)}(\Gamma^{(\mathbb{Z}_2,*)}_{(\frac{h(h\pm 1)}{2}+\frac{l(l\pm 1)}{2})+1,hl+1}) \simeq c_n^{(\mathbb{Z}_2,*)}((M_{h,l}(F),t));$$

$$c_n^{(\mathbb{Z}_2,*)}(\Gamma^{(\mathbb{Z}_2,*)}_{h^2+1,h(h\mp 1)+1}) \simeq c_n^{(\mathbb{Z}_2,*)}((M_{h,h}(F),s));$$

$$c_n^{(\mathbb{Z}_2,*)}(\Gamma^{(\mathbb{Z}_2,*)}_{(h^2+l^2)+1,2hl+1}) \simeq c_n^{(\mathbb{Z}_2,*)}((M_{h,l}(F)\oplus M_{h,l}(F)^{op},exc));$$

$$c_n^{(\mathbb{Z}_2,*)}(\Gamma^{(\mathbb{Z}_2,*)}_{n^2+1,n^2+1}) \simeq c_n^{(\mathbb{Z}_2,*)}(M_n(F+cF)\oplus M_n(F+cF)^{op},exc);$$

*-Graded Capelli Polynomials and their Asymptotics

Corollary (B. and Valenti (2021))

If $M^+ = L^\pm = \frac{n(n+1)}{2}$, $M^- = L^\mp = \frac{n(n-1)}{2}$, with $n > 0$, then

$$c_n^{(\mathbb{Z}_2,*)}(\Gamma^{(\mathbb{Z}_2,*)}_{M^\pm+1,L^\pm+1}) \simeq c_n^{(\mathbb{Z}_2,*)}((M_n(F+cF),*))$$

where $(a+cb)^* = a^t \pm cb^t$;

If $M^+ = L^\pm = \frac{n(n-1)}{2}$, $M^- = L^\mp = \frac{n(n+1)}{2}$, with $n > 0$, then

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where $(a+cb)^* = a^s \pm cb^s$.

Thank You!!