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Locally nilpotent derivations of polynomial and other free algebras

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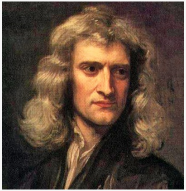
Every student knows the Leibniz rule for the derivative of the product of two functions:

$$\frac{d}{dx}(fg) = \left(\frac{d}{dx}f\right)g + f\frac{d}{dx}g$$



in the notation of Newton

$$(\dot{fg}) = \dot{f}g + f\dot{g}$$

in the notation of Lagrange

$$(fg)' = f'g + fg'$$

In the sequel all algebras will be over a fixed algebraically closed field K of characteristic 0.

Not everyone knows what is a derivation of a ring or an algebra

Let R be an algebra (not necessarily commutative or associative).
The linear map $\delta : R \to R$ is a *derivation* if

$$\delta(ab) = \delta(a)b + a\delta(b) \text{ for all } a, b \in R$$

Example. If R is an associative or a Lie algebra, then the map ad(a), $a \in R$, defined by ad$(a) : b \to [a, b] = ab - ba$, $a, b \in R$, is a derivation called *inner*.

Property. *If R is an algebra, then the vector space $\mathrm{Der}(R)$ of all derivations of R is a Lie algebra with respect to the operation*

$$[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1, \quad \delta_1, \delta_2 \in \mathrm{Der}(R).$$

Short history:

- A. Nowicki, Polynomial Derivations and Their Rings of Constants, Uniwersytet Mikolaja Kopernika, Torun, 1994. http://www-users.mat.uni.torun.pl/∼anow/ps-dvi/pol-der.pdf

The notion of the ring with derivation (=differentiation) is quite old and plays a significant role in the integration of analysis, algebraic geometry and algebra. In the 1940s it was found that the Galois theory of algebraic equations can be transferred to the theory of ordinary linear differential equations (the Picard - Vessiot theory). The field theory also included the derivations in its inventory of tools. The classical operation of differentiation of forms on varieties led to the notion of differentiation of singular chains on varieties, a fundamental notion of the topological and algebraic theory of homology.

In 1950s a new part of algebra called differential algebra was initiated by the works of Ritt and Kolchin. In 1950 Ritt [93] and in 1973 Kolchin [48] wrote the well known books on differential algebra. Kaplansky, too, wrote an interesting book on this subject in 1957 ([42]).

- J.F. Ritt, Differential Algebra, Colloquium Publications 33, New York, AMS, 1950.
- E.R. Kolchin, Differential Algebra and Algebraic Groups, Pure and Applied Mathematics 54, New York-London: Academic Press, 1973.
- I. Kaplansky, An Introduction to Differential Algebra, Actualités Scientifiques et Industrielles 1251, Publ. Inst. Math. Univ. Nancago. V. Paris: Hermann & Cie., 1957.

See also the survey

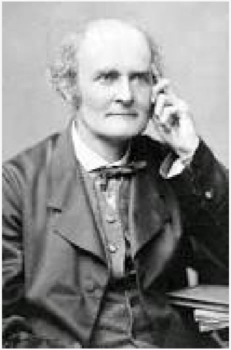
- M. Ashraf, Sh. Ali, C. Haetinger, On derivations in rings and their applications, Aligarh Bull. Math. 25 (2006), No. 2, 79-107. http://tabmaths.com/issues/ABM_25_2_2006.pdf

Derivations of polynomial algebras $K[X_d] = K[x_1, \ldots, x_d]$

$$\mathrm{Der}(K[X_d]) = \left\{ \delta = \sum_{i=1}^{d} f_i(X_d) \frac{\partial}{\partial x_i} \mid f_i(X_d) \in K[X_d] \right\}.$$

Algebras related with Der($K[X_d]$)

It seems that algebras related with the derivations of $K[X_d]$ appeared first in the paper by Cayley in 1857.



A. Cayley, On the theory of analytical forms called trees, Phil. Mag. 13 (1857), 19-30. Collected Math. Papers, University Press, Cambridge, Vol. 3, 1890, 242-246.

A SYMBOL such as $A\partial_x + B\partial_y + \ldots$, where A, B, &c. contain the variables x, y, &c. in respect to which the differentiations are to be performed, partakes of the natures of an operand and an operator, and may be therefore called an Operandator. Let P, Q, R .. be any operandators, and let U be a symbol of the same kind, or to fix the ideas, a mere operand; PU denotes the result of the operation P performed on U, and QPU denotes the result of the operation Q performed on PU; and generally in such combinations of symbols, each operation is considered as affecting the operand denoted by means of all the symbols on the right of the operation in question. Now considering the expression QPU, it is easy to see that we may write

$$QPU = (Q \times P)U + (QP)U,$$

where on the right-hand side $(Q \times P)$ and (QP) signify as follows: viz. $Q \times P$ denotes the mere algebraical product of Q and P, while QP (consistently with the general notation as before explained) denotes the result of the operation Q performed upon P as operand; and the two parts $(Q \times P)U$ and $(QP)U$ denote respectively the results of the operations $(Q \times P)$ and (QP) performed each of them upon U as operand. It is proper to remark that $(Q \times P)$ and $(P \times Q)$ have precisely the same meaning, and the symbol may be written in either form indifferently.

In the modern language this is the *right-symmetric Witt algebra* W_1^{rsym} in one variable

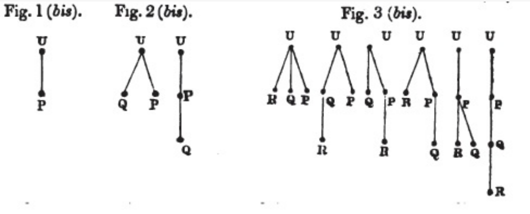
$$W_1^{\mathrm{rsym}} = \left\{ f\frac{d}{dx} \mid f \in K[x] \right\}$$

equipped with the multiplication

$$\left(f_1\frac{d}{dx}\right) * \left(f_2\frac{d}{dx}\right) = \left(f_2\frac{df_1}{dx}\right)\frac{d}{dx}$$

which is *left-commutative*, i.e. a nonassociative algebra satisfying the polynomial identity $x_1(x_2x_3) = x_2(x_1x_3)$.

The algebra W_1^{rsym} is left-commutative and *right-symmetric*.
(Right-symmetric algebras satisfy the polynomial identity
$(x_1, x_2, x_3) = (x_1, x_3, x_2)$, where $(x_1, x_2, x_3) = (x_1x_2)x_3 - x_1(x_2x_3)$ is the associator.)
Cayley also considered the realization of the right-symmetric Witt algebras W_d^{rsym} in terms of rooted trees.



The Weyl algebra

$$A_n = \left\{ \sum_{n_i \geq 0} f_{n_1,\dots,n_d}(X_d) \prod_{n_1,\dots,n_d} \frac{\partial^{n_i}}{\partial x_i} \right\}$$

is the associative algebra of differential operators with polynomial coefficients. Weyl algebras are named after Hermann Weyl, who (according to Wikipedia) introduced them to study the Heisenberg uncertainty principle in quantum mechanics.

The Weyl algebra appears also in many other places of mathematics, e.g. in the Jacobian conjecture.

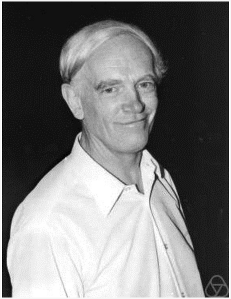
If we denote $y_i = \dfrac{\partial}{\partial x_i}$ then A_n has presentation

$$A_n = K\langle X_d, Y_d \mid [x_i, x_j] = [y_i, y_j] = 0, [y_i, x_j] = \delta_{ij}\rangle,$$

where $K\langle X_d, Y_d\rangle$ is the free associative algebra and δ_{ij} is the Kronecker symbol.

The Witt algebra named after Ernst Witt is the Lie algebra of derivations of the algebra $\mathbb{C}[X_d, X_d^{-1}]$ of the Laurent polynomials. The complex Witt algebra was first defined by Cartan (1909), and its analogues over finite fields were studied by Witt in the 1930s.

Cartan



Witt

Locally nilpotent derivations

The derivation δ of the algebra R is *locally nilpotent* if for every element $a \in R$ there exists a positive integer $n = n(a)$ such that $\delta^n(a) = 0$. The constants in $\mathbb{C}$ are the only differentiable functions $\mathbb{C} \to \mathbb{C}$ which vanish under $\frac{d}{dx}$. By analogy, the elements of the kernel of the locally nilpotent derivation δ of R are called constants and the kernel of δ is an algebra called the algebra of constants: $R^\delta = \ker(\delta)$.

From now on we consider locally nilpotent derivations only.

Why we study locally nilpotent derivations of $K[X_d]$? 1. Relations with Aut$(K[X_d])$

The exponent of δ

$$\exp(\delta) = 1 + \frac{\delta}{1!} + \frac{\delta^2}{2!} + \cdots$$

defines a map $\exp(\delta) : R \to R$ which is an automorphism (because $\delta^n(a) = 0$ for a sufficiently large n, the infinite series $\exp(\delta)(a)$ becomes a finite sum and is well defined).

It is known that *if* $\varphi : (K, +) \to \mathrm{Aut}(K[X_d])$ *is a homomorphism of the additive group* $(K, +)$ *to the automorphism group of* $K[X_d]$*, then there exists a locally nilpotent derivation* δ *of* $K[X_d]$ *such that* $\varphi(\alpha) = \exp(\alpha\delta)$*,* $\alpha \in K$*. The algebra* $K[X_d]^{(K,+)}$ *of the fixed points of the action of* $(K, +)$ *coincides with the algebra* $K[X_d]^{\delta}$ *of the constants of* δ*.*

2. Relations with invariant theory

When $\varphi : (K,+) \to \mathrm{Aut}(K[X_d])$ maps $\alpha \in (K,+)$ to $g_\alpha \in GL_d(K) = GL(V_d)$, $V_d = \mathrm{span}(X_d)$, i.e. we have a linear action of $(K,+)$, *then* $(g_1 - 1)^d = 0$ *and*

$$\delta = \log\varphi(1) = \log(g_1) = \frac{g_1 - 1}{1} - \frac{(g_1-1)^2}{2} + \cdots + (-1)^d \frac{(g_1-1)^{d-1}}{d-1}$$

is a locally nilpotent derivation of $K[X_d]$ *which acts as a nilpotent linear operator in the* d*-dimensional vector space* V_d. Such *linear locally nilpotent derivations* are called *Weitzenböck derivations*.

Weitzenböck derivations

Weitzenböck

In 1932 Weitzenböck published a paper where he gave a (wrong) proof of the theorem that the algebra of invariants $\mathbb{C}[X_d]^G$ is finitely generated for any linear group G.

- R. Weitzenböck, Über die Invarianten von linearen Gruppen, Acta Math. 58 (1932), 231-293.

Hermann Weyl (Zbl 0004.24301) found a gap in the proof. Nowadays the main result of the paper is considered to be the following very nice theorem.

Theorem. (Weitzenböck) *The algebra of constants $K[X_d]^\delta$ is finitely generated for any Weitzenböck derivation.*

The fourteenth problem of Hilbert

The 14th Hilbert problem from his lecture in 1900 states:

Problem. *Let F be a subfield of the field of rational functions $K(X_d)$ and let $K \subset F$. Is it true that $F \cap K[X_d]$ is a finitely generated K-algebra?*

The main motivation for the problem comes from invariant theory:

Problem. *Let G be a subgroup of $GL_d(K)$. Is it true that the algebra $K[X_d]^G$ of G-invariants is finitely?*

References

There is an enormous number of books, surveys and papers devoted to the 14th Hilbert problem. Here are three books and a survey article.

- A. Nowicki, Polynomial Derivations and Their Rings of Constants, Uniwersytet Mikolaja Kopernika, Torun, 1994. http://www-users.mat.uni.torun.pl/∼anow/ps-dvi/pol-der.pdf
- A. van den Essen, Polynomial Automorphisms and the Jacobian Conjecture, Progress in Math. (Boston, Mass.) 190, Birkhäuser, Basel, 2000.
- G. Freudenburg, Algebraic Theory of Locally Nilpotent Derivations, Encyclopaedia of Mathematical Sciences, 136, Invariant Theory and Algebraic Transformation Groups, VII. Springer-Verlag, Berlin, 2006.
- G. Freudenburg, A survey of counterexamples to Hilbert's fourteenth problem, Serdica Math. J. 27 (2001), 171-192. http://www.math.bas.bg/serdica/2001/2001-171-192.pdf

Negative solutions to the 14th Hilbert problem

The negative solution to the Hilbert problem was given in 1958 by Nagata.

- M. Nagata, On the 14-th problem of Hilbert, Amer. J. Math. 81 (1959), 766-772.
- M. Nagata, On the fourteenth problem of Hilbert, Proc. Internat. Congress Math. 1958, Cambridge Univ. Press, Cambridge, 1960, 459-462.

One of the counterexamples of Nagata is the following.

Theorem. *There is an action of the direct sum* $G = (K,+)^{13}$ *of 13 copies of the additive group* $(K,+)$ *acting on the algebra* $K[X_{32}]$ *such that the algebra of invariants* $K[X_{32}]^G$ *is not finitely generated.*

Translation in the language of Weitzenböck derivations.

Theorem. *There are* 13 *Weitzenböck derivations* $\delta_1, \ldots, \delta_{13}$ *of the algebra* $K[X_{32}]$ *of the polynomials in* 32 *variables such that the intersection* $\bigcap_{i=1}^{13} K[X_{32}]^{\delta_i}$ *is not finitely generated.*

Inspired problems

- *What is the minimal number of variables d such that there exists a linear group G such that $K[X_d]^G$ is not finitely generated?*
- *What is the minimal number of Weitzenböck derivations $\delta_1, \ldots, \delta_m$ of $K[X_d]$ such that $\bigcap_{i=1}^{m} K[X_d]^{\delta_i}$ is not finitely generated?*

Lower bounds

Theorem. *The 14th Hilbert problem has a positive solution for $K[X_2]$.*
Corollary. *If δ is a locally nilpotent derivation of $K[X_3]$ then the algebra of constants $K[X_3]^\delta$ is finitely generated.*

- O. Zariski, Interprétations algébrico-géométriques du quatorzième problème de Hilbert, Bull. Sci. Math., II. Sér. 78 (1954), 155-168.

Best results for the upper bounds

Theorem. (Freudenburg) *There is a subgroup G of the unitriangular group $UT_{11}(K)$ such that the algebra $K[X_{11}]^G$ is not finitely generated.*

- G. Freudenburg, A linear counterexample to the fourteenth problem of Hilbert in dimension eleven, Proc. Amer. Math. Soc. 135 (2007), 51-57.
- G. Freudenburg, Foundations of invariant theory for the down operator, J. Symb. Comput. 57 (2013), 19-47.

Theorem. (Mukai) *There are three Weitzenböck derivations $\delta_1, \delta_2, \delta_3$ of $K[X_{18}]$ such that the intersection $K[X_{18}]^{\delta_1} \cap K[X_{18}]^{\delta_2} \cap K[X_{18}]^{\delta_3}$ is not finitely generated.*

- S. Mukai, Geometric realization of T-shaped root systems and counterexamples to Hilbert's fourteenth problem, Algebraic Transformation Groups and Algebraic Varieties, Springer-Verlag, Berlin, 2004, Encyclopaedia Math. Sci. 132, 123-129.

Problem. *Are there two Weitzenböck derivations* δ_1 *and* δ_2 *of* $K[X_d]$ *such that the algebra* $K[X_d]^{\delta_1} \cap K[X_d]^{\delta_2}$ *is not finitely generated?*

Not finitely generated algebras of constants

In 1990 Roberts constructed a counterexample to the 14th Hilbert problem based on another principle. Later A'Campo-Neuen and Deveney and Finston showed that *the counterexample of Roberts is equal to the algebra of constants* $K[X_7]^\delta$ *of a locally nilpotent derivation.*

- P. Roberts, An infinitely generated symbolic blow-up in a power series ring and a new counterexample to Hilbert's fourteenth problem, J. Algebra 132 (1990), 461-473.
- A. A'Campo-Neuen, Note on a counterexample to Hilbert's fourteenth problem given by P. Roberts, Indag. Math., New Ser. 5 (1994), 253-257.
- J.K. Deveney, D.R. Finston, G_a actions on $\mathbb{C}^3$ and $\mathbb{C}^7$, Commun. Algebra 22 (1994), 6295-6302.

In the sequel the idea of Roberts was further developed in a series of papers. The minimal counterexample is for $K[X_5]$.

Theorem. *The algebra of constants $K[X_5]^\delta$ of the locally nilpotent derivation*

$$\delta = x_1^2\frac{\partial}{\partial x_3} + (x_1x_3 + x_2)\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_5}$$

is not finitely generated.

- D. Daigle, G. Freudenburg, A counterexample to Hilbert's fourteenth problem in dimension 5, J. Algebra 221 (1999), 528-535.

Problem. *Is there a locally nilpotent derivation δ of $K[X_4]$ such that the algebra of constants $K[X_4]^\delta$ is not finitely generated?*

Counterexample in dimension $d \geq 3$

We shall close the discussion on the 14th Hilbert problem with the following counterexample for $d = 3$ which holds for all $d \geq 3$.

- S. Kuroda, A counterexample to the fourteenth problem of Hilbert in dimension three, Mich. Math. J. 53 (2005), 123-132.

Theorem. *Let* β_{ij}*,* $i,j=1,2$*, and* γ *be positive integers such that*

$$\frac{\beta_{11}}{\beta_{11}+\beta_{21}}+\frac{\beta_{22}}{\beta_{22}+\beta_{12}}<\frac{1}{2}.$$

Let $K(f_1,f_2,f_3)$ *be the subfield of* $K(X_3)$ *generated by the rational functions*

$$f_1=x_1^{\beta_{21}}x_2^{-\beta_{22}}-x_1^{\beta_{11}}x_2^{-\beta_{12}},$$

$$f_2=x_3^{\gamma}-x_1^{-\beta_{11}}x_2^{\beta_{12}},$$

$$f_3=2x_1^{\beta_{21}-\beta_{11}}x_2^{\beta_{12}-\beta_{22}}-x_1^{-2\beta_{11}}x_2^{2\beta_{12}}.$$

Then the algebra $K(f_1,f_2,f_3)\cap K[X_3]$ *is not finitely generated.*
Since $K(f_1,f_2,f_3)\cap K[X_d]=K(f_1,f_2,f_3)\cap K[X_3]$ for $d\geq 3$, this provides a counterexample also for all $d\geq 3$.

Weitzenböck derivations and invariant theory

The Weitzenböck derivation δ of $K[X_d]$ acts as a nilpotent linear operator on the vector space V_d with basis X_d. Up to a change of the basis δ is determined by its Jordan matrix

$$J(\delta) = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_m \end{pmatrix},$$

which consists of Jordan cell with zero diagonals

$$J_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad i = 1, \ldots, m.$$

Corollary. *For any dimension d there is a finite number essentially different Weitzenböck derivations.*

Problem. *For a given Weitzenböck derivation δ of $K[X_d]$:*

- *Find the generators of $K[X_d]^\delta$.*
- *Compute the Hilbert (or Poincaré) series of the graded vector space $K[X_d]^\delta$.*

Let $J(\delta)$ be the Jordan matrix of the Weitzenböck derivation of $K[X_d]$. If

$$g_\alpha = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \quad \alpha \in K,$$

then the map $g_\alpha \to \exp(\alpha J(\delta))$, $\alpha \in K$, defines a homomorphism of the unitriangular group $UT_2(K)$ and an action of $UT_2(K)$ on V_d. The algebras $K[X_d]^\delta$ and $K[X_d]^{UT_2(K)}$ coincide.
Hence *we can apply methods of invariant theory for the study of the algebra of constants of* δ *and vice versa to apply Weitzenböck derivations to invariant theory.*

Special case: The conjecture of Nowicki

In his book from 1994 Nowicki made the following conjecture.

Conjecture. *If all Jordan cells of the Weitzenböck derivation δ of $K[X_{2d}]$ are 2×2 matrices and*

$$\delta(x_{2j-1})=0,\delta(x_{2j})=x_{2j-1},\quad j=1,\dots,d,$$

then the algebra $K[X_{2d}]^{\delta}$ is generated by

$$x_{2j-1},\quad j=1,\dots,d,$$

$$\begin{vmatrix} x_{2k-1} & x_{2k} \\ x_{2l-1} & x_{2l} \end{vmatrix} = x_{2k-1}x_{2l}-x_{2k}x_{2l-1},\quad 1\le k<l\le d.$$

There are 9 proofs based on different methods confirming the Nowicki conjecture and its generalizations: by Khoury; Bedratyuk; the speaker and Makar-Limanov; Kuroda; Derksen; Panyushev; Goto, Hayasaka, Kurano, Nakamura; Miyazaki; the latest by the speaker in 2020.

Another special case: All Jordan cells are 3×3 matrices

Theorem. *Let δ be a Weitzenböck derivation of $K[X_{3d}]$, where*

$$\delta(x_{3j-2}) = 0, \delta(x_{3j-1}) = x_{3j-2}, \delta(x_{3j}) = x_{3j-1}, \quad j = 1, \ldots, d.$$

Then the algebra $K[X_{3d}]^{\delta}$ of the constants of δ is generated by

$$x_{3j-2}, j = 1, \ldots, d, \quad \begin{vmatrix} x_{3k-2} & x_{3k-1} \\ x_{3l-2} & x_{3l-1} \end{vmatrix}, 1 \leq k < l \leq d,$$

$$x_{3k-2}x_{3l} - x_{3k-1}x_{3l-1} + x_{3k}x_{3l-2}, \quad 1 \leq k < l \leq d,$$

$$\begin{vmatrix} x_{3p-2} & x_{3p-1} & x_{3p} \\ x_{3q-2} & x_{3q-1} & x_{3q} \\ x_{3r-2} & x_{3r-1} & x_{3r} \end{vmatrix}, \quad 1 \leq p < q < r \leq d.$$

- L. Bedratyuk, A note about the Nowicki conjecture on Weitzenböck derivations, Serdica Math. J. 35 (2009), 311-316.

For more information on Weitzenböck derivations one can see the books

- A. Nowicki, Polynomial Derivations and Their Rings of Constants, Uniwersytet Mikolaja Kopernika, Torun, 1994.
- H. Derksen, G. Kemper, Computational Invariant Theory, Encyclopaedia of Mathematical Sciences, Invariant Theory and Algebraic Transformation Groups 130, Springer-Verlag, 2002.
- I. Dolgachev, Lectures on Invariant Theory, London Mathematical Society Lecture Note Series, Cambridge University Press, 2003.

For methods for the computation of the Hilbert series of the algebra of constants see the above books and the paper

- F. Benanti, S. Boumova, V. Drensky, G.K. Genov, P. Koev, Computing with rational symmetric functions and applications to invariant theory and PI-algebras, Serdica Math. J. 38 (2012), Nos 1-3, 137-188.

Locally nilpotent derivations and automorphisms of $K[X_d]$

Main problem. *Describe the group* $\mathrm{Aut}(K[X_d])$, $d > 1$.

The interest to the problem appeared for the needs of algebraic geometry and commutative algebra but the approach is similar to the approach for the description of the automorphism groups of the finitely generated free groups in the 1910s-1920s.

Natural candidates for generators of Aut$(K[X_d])$:

- Affine automorphisms

$$\varphi(x_j)=\sum_{i=1}^{d}\alpha_{ij}x_i+\beta_j, j=1,\dots,d, g=(\alpha_{ij})\in GL_d(K),\beta_j\in K.$$

- Triangular automorphisms

$$\tau(x_j)=\alpha_j x_j+f_j(x_{j+1},\dots,x_d),\alpha_j\in K^*,f_j(x_{j+1},\dots,x_d)\in F_d,$$

  and f_j does not depend on the variables $x_1,\dots,x_j$, $j=1,\dots,d$.

The subgroup of Aut$(K[X_d])$ generated by affine and triangular automorphisms consists of the *tame automorphisms*. The other automorphisms (if they exist) are *wild*.

Problem. *Are all automorphisms of* $K[X_d]$ *tame?*

Answer for $d = 2$

Theorem. (Jung for $\text{char}(K) = 0$, van der Kulk in the general case) *All automorphisms of* $K[X_2]$ *are tame.*

- H.W.E. Jung, Über ganze birationale Transformationen der Ebene, J. Reine und Angew. Math. 184 (1942), 161-174.
- W. van der Kulk, On polynomial rings in two variables, Nieuw Archief voor Wiskunde (3) 1 (1953), 33-41.

There several easy proofs of the theorem of Jung–van der Kulk based on locally nilpotent derivations:

- R. Rentschler, Opérations du groupe additif sur le plan affine, C. R. Acad. Sci., Paris, Sér. A 267 (1968), 384-387.
- L. Makar-Limanov, Locally nilpotent derivations, a new ring invariant and applications, Lecture notes, available at http://www.math.wayne.edu/∼lml/lmlnotes.pdf.
  (see also V. Drensky, Free Algebras and PI-Algebras, Springer-Verlag, Singapore, 2000).

In his paper Rentschler also classified the locally nilpotent derivations of $K[X_2]$.

Theorem. *For every locally nilpotent derivation δ of $K[X_2]$ there exists a generating set $Y_2 = \{y_1, y_2\}$ of $K[X_2]$ such that*

$$\delta(y_1) = 0, \delta(y_2) = f(y_1),$$

for a suitable $f(y_1) \in K[y_1]$.

Locally nilpotent derivations of $K[X_d]$ for $d > 2$

If δ is a locally nilpotent derivation of $K[X_d]$ and $0 \neq w(X_d) \in K[X_d]^{\delta}$ then $\Delta = w\delta$ is a locally nilpotent derivation with the same algebra of constants.

The Nagata automorphism

The most famous automorphism of $K[x, y, z]$ is the Nagata automorphism

$$\nu : (x, y, z) \to (x - 2(y^2 + xz)y - (y^2 + xz)^2 z, y + (y^2 + xz)z, z).$$

Nagata proved that *this automorphism is wild as an automorphism of the polynomial algebra $(K[z])[x, y]$ in x, y over $K[z]$.*
Nagata also conjectured that *his automorphism is wild as an automorphism of $K[x, y, z]$.*

- M. Nagata, On the Automorphism Group of $k[x, y]$, Lect. in Math., Kyoto Univ., Kinokuniya, Tokyo, 1972.

The conjecture of Nagata was answered into affirmative by Shestakov and Umirbaev

- I.P. Shestakov, U.U. Umirbaev, Poisson brackets and two-generated subalgebras of rings of polynomials, J. Am. Math. Soc. 17 (2004), No. 1, 181-196.
- I.P. Shestakov, U.U. Umirbaev, The tame and the wild automorphisms of polynomial rings in three variables, J. Am. Math. Soc. 17 (2004), No. 1, 197-227.

Starting from the Weitzenböck derivation of $K[x,y,z]$ defined by

$$\delta(x) = -2y, \quad \delta(y) = z, \quad \delta(z) = 0,$$

$w = xz + y^2 \in K[x,y,z]^\delta$, and $\Delta = (xz+y^2)\delta$ one obtains the Nagata automorphism as $\exp(\Delta)$.

Stably tame automorphisms of $K[X_d]$

The automorphism φ of $K[X_d]$ is *stably tame* if it becomes tame as an automorphism of $K[X_{d+1}]$ extended as $\delta(x_{d+1}) = x_{d+1}$.

Theorem. *If δ is a triangular derivation, i.e. $\delta(x_j) \in K[X_{j-1}]$, $j = 1, \ldots, d$, and $w \in K[X_d]^\delta$ then $\Delta = w\delta$ is a locally nilpotent derivation and the automorphism $\exp(\Delta)$ is a stably tame automorphism of $K[X_d]$.*

- M.K. Smith, Stably tame automorphisms, J. Pure Appl. Algebra 58 (1989), 209-212.

Many locally nilpotent derivations produce stably tame automorphisms and hence candidates for wild automorphisms of $K[X_d]$, $d > 3$. (The existence of wild automorphisms of $K[X_d]$, $d > 3$, is an open problem!)

Theorem. *Let δ be a locally nilpotent derivation of the polynomial algebra $K[X_n, Y_m]$ in $m+n$ variables and let*

$$\delta(x_j) = \sum_{k=1}^{n} a_{kj}(Y_m)x_k + b_j(Y_m),\ j = 1, \ldots, n,$$

$$\delta(y_i) = 0,\ i = 1, \ldots, m,$$

where $a_{kj}(Y_m)$ and $b_j(Y_m)$ do not depend on X_n. Let $w \in K[X_n, Y_m]^{\delta}$. Then $\varphi = \exp(w\delta)$ is a stably tame automorphism and its extension $\varphi^{[1]}$ becomes tame if we add one variable y_{m+1} for $n \geq 3$ and $\varphi^{[2]}$ is tame adding the variables y_{m+1}, x_3 for $n = 2$.

- V. Drensky, A. van den Essen, D. Stefanov, New stably tame automorphisms of polynomial algebras, J. Algebra 226 (2000), 629-638.

The proof of the theorem is based on the approach of Martha Smith combined with the theorem of Suslin that *for* $n \geq 3$ *every matrix from* $SL_n(K[Y_m])$ *is a product of elementary matrices*.

- A.A. Suslin, On the structure of the special linear group over polynomial rings (Russian), Izv. AN. Nauk SSSR, Ser. Mat. 41 (1977), 235-252. Translation: Math. USSR, Izv. 11 (1977), 221-239.

Theorem. *Let* $\tilde{f}=f(y_1,y_2,x_1x_2-x_3^2,y_2^2x_1+y_1^2x_2-2y_1y_2x_3)$ *for some polynomial* $f(t_1,t_2,t_3,t_4)\in K[t_1,t_2,t_3,t_4]$*. Then the automorphism* φ *of* $K[X_3,Y_2]$ *defined by*

$$\varphi(x_1)=x_1+2y_1(y_2x_1-y_1x_3)\tilde{f}+y_1^2(y_2^2x_1+y_1^2x_2-2y_1y_2x_3)\tilde{f}^2,$$

$$\varphi(x_2)=x_2+2y_2(y_2x_3-y_1x_2)\tilde{f}+y_2^2(y_2^2x_1+y_1^2x_2-2y_1y_2x_3)\tilde{f}^2,$$

$$\varphi(x_3)=x_3+(y_2^2x_1-y_1^2x_2)\tilde{f}+y_1y_2(y_2^2x_1+y_1^2x_2-2y_1y_2x_3)\tilde{f}^2,$$

$$\varphi(y_1)=y_1,\ \varphi(y_2)=y_2$$

is stably tame and becomes tame adding one more variable y_3*.*

These automorphisms appear as automorphisms of the pure trace algebra of two generic 2×2 matrices and were introduced in

- V. Drensky, C.K. Gupta, New automorphisms of generic matrix algebras and polynomial algebras, J. Algebra 194 (1997), 409-414.

Many of the locally nilpotent derivations of $K[X_d]$ fix a variable (maybe after a suitable change of the variables). The following example in $K[x, y, z]$ does not have this property.

$$\delta(f) = \begin{vmatrix} \frac{\partial w}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial f}{\partial x} \\ \frac{\partial w}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial f}{\partial y} \\ \frac{\partial w}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial f}{\partial z} \end{vmatrix},$$

$$w = y^2 + xz, \quad v = zw^2 + 2x^2yw - x^5.$$

For details and notations see the paper

- G. Freudenburg, Actions of G_a on $\mathbb{A}^3$ defined by homogeneous derivations, J. Pure Appl. Algebra 126 (1998), No. 1-3, 169-181.

Locally nilpotent derivations in the noncommutative case

There are many results when δ is a locally nilpotent derivation in a free associative, Lie or nonassociative algebra or a relatively free algebra in a variety of algebras. We shall state few results only.

Locally nilpotent derivations of $K\langle x, y\rangle$

Theorem. *All automorphisms of* $K\langle x, y\rangle$ *are tame and the groups* $\mathrm{Aut}(K[x, y])$ *and* $\mathrm{Aut}(K\langle x, y\rangle)$ *are isomorphic.*

Corollary. *The locally nilpotent derivations of* $K\langle x, y\rangle$ *are of the form* $\delta(x) = 0$, $\delta(y) = f(x)$ *for a suitable* $f(x)$ *depending on* x *only.*

- L.G. Makar-Limanov, On automorphisms of free algebra with two generators (Russian), Funk. Analiz i ego Prilozh. 4 (1970), No. 3, 107-108. Translation: Functional Anal. Appl. 4 (1970), 262-263.
- A.J. Czerniakiewicz, Automorphisms of a free associative algebra of rank 2. I, II, Trans. Amer. Math. Soc. 160 (1971), 393-401; 171 (1972), 309-315.

Differences between the locally nilpotent derivations of $K[X_2]$ and $K\langle X_2\rangle$

If δ is a locally nilpotent derivation of $K[x,y]$ and $0 \neq w \in K[x,y]^{\delta}$ then δ and $\Delta = w\delta$ have the same algebras of constants. The situation is completely different in the case of $K\langle x,y\rangle$.

Theorem. *If δ_1 and δ_2 are two locally nilpotent derivations of $K\langle x,y\rangle$ with the same algebra of constants then $\delta_2 = \alpha\delta_1$ for some nonzero $\alpha \in K$.*

- V. Drensky, L. Makar-Limanov, Locally nilpotent derivations of free algebra of rank two, SIGMA (Symmetry, Integrability and Geometry: Methods and Applications) 15 (2019), 091, 10 pages.

The automorphism of Anick of $K\langle x, y, z\rangle$

The following automorphism ω of $K\langle x, y, z\rangle$ suggested by David Anick was conjectured to be wild:

$$\omega(x) = x + z(xz - zy), \quad \omega(y) = y + (xz - zy)z, \quad \omega(z) = z.$$

The Anick automorphism is of the form $\omega = \exp(\delta)$ where δ is the locally nilpotent derivation defined by

$$\delta : (x, y, z) \to (z(xz - zy), (xz - zy)z, 0).$$

Theorem. *The Anick automorphism is wild.*

- U.U. Umirbaev, The Anick automorphism of free associative algebras, J. Reine Angew. Math. 605 (2007), 165-178.

Various conditions for relatively free algebras

Theorem. *Let $\mathfrak{V}$ be a variety of unitary associative algebras over a field K of characteristic 0. Then the following statements are equivalent:*
(i) *The algebras $F_d(\mathfrak{V})^G$ are finitely generated for all $d \geq 2$ and all linearly reductive algebraic subgroups G of $GL_d(K)$;*
(ii) *The algebra $F_d(\mathfrak{V})$ is finitely presented, i.e., the T-ideal $I_d(\mathfrak{V})$ is finitely generated as an ordinary ideal of $K\langle X_d\rangle$;*
(iii) *The algebra $F_2(\mathfrak{V})$ is finitely presented;*
(iv) *The variety $\mathfrak{V}$ is locally noetherian, i.e., every finitely generated algebra in $\mathfrak{V}$ satisfies the descending chain condition for left ideals (and similarly for right ideals);*
(v) *The variety $\mathfrak{V}$ satisfies for some n the Engel identity*

$$x_1(\mathrm{ad}^n x_2) = [x_1, \underbrace{x_2, \ldots, x_2}_{n \text{ times}}] = 0;$$

(vi) *The variety* $\mathfrak{V}$ *satisfies an identity of the form*

$$x_1x_2^n + \sum_{i=1}^{n} \alpha_i x_2^i x_1 x_2^{n-i} = 0, \quad \alpha_i \in K,$$

called a left Lvov identity (and similarly $\mathfrak{V}$ *satisfies a right Lvov identity);*
(vii) *The variety* $\mathfrak{V}$ *does not contain the algebra* $T_2(K)$ *of* 2×2 *upper triangular matrices;*
(viii) *The variety* $\mathfrak{V}$ *satisfies a polynomial identity* $f(x_1, x_2) = 0$ *in two variables which is linear in* x_1*;*
(ix) *The variety* $\mathfrak{V}$ *is Lie nilpotent, i.e., satisfies for some* m *the identity*

$$[x_1, x_2, \ldots, x_{m+1}] = 0;$$

(x) *The algebras* $F_d(\mathfrak{V})^\delta$ *of the constants of all Weitzenböck derivations* δ *and for all* $d \geq 2$ *are finitely generated;*
(xi) *The algebra* $F_{d_0}(\mathfrak{V})^\delta$ *of the constants of one nonzero Weitzenböck derivation* δ *and for one* $d_0 \geq 2$ *is finitely generated;*

(xii) *The algebra* $F_2(\mathfrak{V})^D$ *of the invariants of the one-dimensional torus*

$$D = \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \mid \xi \in K^* \right\}$$

is finitely generated.

(xiii) *For some* d_0 *and some nontrivial rational action of the special linear group* $SL_2(K)$ *on* KX_{d_0} *the algebra* $F_{d_0}(\mathfrak{V})^{SL_2(K)}$ *is finitely generated;*

(xiv) *Let* C_d *be the commutator ideal of* $F_d(\mathfrak{V})$. *For some* d_0 *and some linearly reductive algebraic subgroup* G_0 *of* $GL_{d_0}(K)$ *the algebra* $F_{d_0}(\mathfrak{V})^{G_0}$ *is finitely generated,* $(C_{d_0}/C_{d_0}^2)^{G_0} \neq 0$, *and* $2\text{transc.deg}K[X_{d_0}]^G < \text{transc.deg}K[X_{d_0}, Y_{d_0}]^G$, *where* G_0 *acts on* KY_{d_0} *in the same way as on* KX_{d_0}.

The equivalence of (iv), (v), and (vi) was established by Latyshev. Lvov showed the equivalence with (vii). Markov added the equivalence with (ii), (iii), and (viii). See the survey by Kharlampovich and Sapir for more comments. The implication (ix) $\Rightarrow$ (v) is obvious. The inverse statement (v) $\Rightarrow$ (ix) is due to Kemer. (The theorem that Lie algebras satisfying the Engel identity are nilpotent was proved by Zelmanov.) Vonessen established that if G is a linearly reductive algebraic group acting rationally on a left noetherian finitely generated PI-algebra R, then R^G is finitely generated which gives (iv) $\Rightarrow$ (i) (and hence also (iv) $\Rightarrow$ (xii)). Using an example of Kharchenko, Domokos and Drensky showed (xii) $\Rightarrow$ (vii) (which also gives (i) $\Rightarrow$ (vii)). Finally, the implications (xi) $\Rightarrow$ (vii) and (ix) $\Rightarrow$ (x) were proved, respectively, by Drensky and Gupta and Drensky. Since (x) $\Rightarrow$ (xi) is trivial, this closes the cyrcle of equivalences in the condition (i)-(xii). The coditions (xiii) and (xiv) are added in a paper in preparation of Bedratyuk and Drensky. A survey on the special case of invariant theory of finite groups acting on relatively free algebras can be found in the surveys of Formanek, Drensky, Kharlampovich and Sapir.

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THANK YOU VERY MUCH
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