An inner-product free iteration method for an equation of motion and for a bound-constrained optimal control PDE problem

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Inner product free iteration methods can outperform more commonly used iteration methods based on Krylov subspaces, such as the classical conjugate gradient (CG) method and the generalized minimum residual (GMRES) method, at least when implemented on parallel computer platforms. This is due to the need to compute inner products in CG and GMRES, mainly for the orthogonalization of a new search direction with respect to previous search directions. Thereby each processor must send its local part of the inner product to all other processors or to a master processor, to enable the computation of the global inner product. This requires start up times for synchronization and global communication times. Since the computer chips on modern computers now have reached nearly their physical limit of speed, this type of overhead will be even more dominating for future generations of parallel computer processors, where the only way to decrease the computer time is to use more parallel processors.

There exists however iteration methods which do not require any computation of inner products except for the computation of the norm of the current residual, to see if it is sufficiently small for the required tolerance accuracy to be reached. But this check need to take place rarely, only at the end of the iteration process.

The classical such method is the Chebyshev iteration method, but as shown in a recent paper by O. Axelsson and D.K. Salkuyeh [1], there exists also other methods such as the transformed matrix iteration (TMIT) method. In [1] it has been shown that the Chebyshev and TMIT methods can outperform the GMRES and other methods and can be competitive even on a single processor machine, in elapsed computer times. This is due to that there is no computer time needed for the arithmetic computation involved in the inner products.

However, to be efficient these methods require that accurate, or even sharp bounds of the eigenvalues of the preconditioned matrix are available and that they do not correspond to a huge condition number.

In the present paper we consider an extension of methods to solve two-by-two block matrices arising in optimal control of PDEs where this is possible and which have appeared in a series of papers by O. Axelsson and M. Neytcheva, and coworkers, see e.g. [2]. This is one of the first papers where a very efficient preconditioner, later called PRESB (preconditioned square block) has been used. This method is fully parameter free and gives eigenvalue bounds $1/2 \leq \lambda \leq 1$ which hold uniformly with respect to problem, regularization and step-size parameters. The origin of the method comes from the elementary problem of solving a complex valued equation,

$$(W + iZ)(x + iy) = f + ig$$

avoiding complex arithmetics, which can be done via the real valued form

$$\begin{bmatrix} W & -Z \\ Z & W \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix},$$

see e.g. [3] and references therein to earlier papers. Thereby the PRESB preconditioner

$$\begin{bmatrix} W & -Z \\ Z & W+2Z \end{bmatrix}$$

is used, the action of which involves solving two systems with matrix $W + Z$, in addition a matrix vector multiplication with W and vector operations.

The present paper is concerned with extensions of this preconditioning method to two important types of applications. The first deals with a direct frequency domain analysis of an equation of motion, $M\ddot{q} + C\dot{q} + Kq = p$, where M is the inertia, K the stiffness matrix and C a viscous damping matrix. An early presentation of this problem is found in a paper by Feriani, Perotti, Simoncini [4], but the problem has later been taken up by many other researchers, such as Z.Z. Bai [5]. It has also been tested in [1]. In these studies only the frequency $\omega = \pi$ has been studied. In this talk an extension to more general values is presented, which leads to an indefinite matrix W. For other applications of time-harmonic approaches of optimal control problems, see [6] and the references therein.

The present talk deals also with an extension to an optimal control problem with bound constraints on the solution to the state equation. This is based on the method used in O. Axelsson, M. Neytcheva, A. Ström [7]. We show how this problem can be solved very efficiently by use of a polynomially preconditioned Chebyshev iteration method.

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3. O. Axelsson, M. Neytcheva, B. Ahmad, A comparison of iterative methods to solve complex valued linear algebraic systems. *Numerical Algorithms*, 66(2014), 811-841.

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6. O. Axelsson, Z.-Z. Liang, A note on preconditioning methods for time-periodic eddy current optimal control problems, *Journal of Computational and Applied Mathematics*, accepted, 2018.

7. O. Axelsson, M. Neytcheva, A. Ström. An efficient preconditioning method for state box-constrained optimal control problems. *J. Numerical Mathematics*, accepted, 2018.