Recent advances in numerical methods for fractional differential equations with non-smooth data: a concise overview

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Introduction. We shall survey some resent results in numerical treatment of initial and boundary value problems for time fractional differential equations involving both Riemann-Liouville and Caputo fractional derivatives. Examples of such problems include fractional time dependent diffusion equation, convection-diffusion problems, and multi-term transient diffusion equation.

Problem formulation. We shall survey the latest development in numerical methods for solving initial and boundary valued problems for fractional differential equations. Our focus will be the following time fractional diffusion and diffusion-wave equations:

$$\partial_t^\alpha u(x,t) - \Delta u(x,t) = f(x,t) \quad x \in \Omega,\ t \in (0,T). \tag{1}$$

Here $\partial_t^\alpha u$ denotes the Caputo fractional derivative with respect to t and $\Omega \subset \mathcal{R}^d$ $(d = 1, 2, 3)$ is a bounded convex polygonal domain with a boundary $\partial\Omega$. We assume that problem (1) is subject to the following initial and boundary value conditions

$$\begin{aligned} &u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ &u(x,0) = v(x), \quad x \in \Omega, \quad (\text{and } \partial_t u(x,0) = w(x) \quad x \in \Omega,\ \text{if } 1 < \alpha < 2). \end{aligned}$$

Here $f(x,t),\ v(x)$, and $w(x)$ are given functions.
We shall also briefly discuss the steady-state sub-diffusion convection-reaction problem

$$-\partial_x^\alpha u(x) + b(x)u'(x) + q(x)u(x) = f(x), \quad x \in D = (0,1), u(0) = u(1) = 0, \tag{2}$$

where the source term f belongs to $L^2(D)$ or suitable subspace, and $\partial_x^\alpha u$ denotes either the left-sided Riemann-Liouville or Caputo fractional derivative of order $\alpha \in (3/2, 2)$. We assume a convection coefficient $b \in W^{1,\infty}(0,1)$ and a potential coefficient $q \in L^\infty(0,1)$. When $\alpha = 2$, the problem recovers the canonical one-dimensional steady-state convection diffusion-reaction equation.

Main topics to be discussed. It is impossible to survey all important and relevant works in a short talk. Instead, we aim at only reviewing relevant works on the numerical methods for the sub-diffusion model (1) with non-smooth problem data. This means that the initial data v belongs only to $L^2(\Omega)$ or the source term f is not compatible with the initial data or/and boundary condition. First, this choice allows us to highlight some distinct features common to many nonlocal models, especially how the smoothness of the problem data influences the solution and the corresponding numerical methods. These are the features that pose substantial new mathematical and computational challenges when compared with standard parabolic problems – and extra care has to be taken when developing and analyzing relevant numerical methods. In particular, since the solution operators of the fractional model have limited smoothing property, a numerical method that requires high regularity of the exact solution will impose severe restrictions (compatibility conditions) on the data and generally does not work well. Schemes that are constructed and analyzed under high regularity assumptions on the solution, substantially limit their potential applications. For example, non-smooth data analysis is fundamental to the rigorous error analysis of various applications in optimal control, inverse problems, and stochastic fractional diffusion. For relevant discussion on this important topic we refer to recent paper [1]. We shall review briefly the following four topics and give some representative results:

(i) Importance of the regularity theory in Sobolev spaces;

(ii) Spatial discretization via Galerkin finite element and finite volume element methods;

(iii) Temporal discretization via time-stepping schemes;

(iv) Space-time formulations (Galerkin or Petrov-Galerkin type).

The talk is based on our resent works in this area [2, 3, 6, 4, 5, 7].

References

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