Algorithmic problems in the geometry of polynomials

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The *Geometry of polynomials* study the geometric relations, on the complex plane $\mathcal{C}$, between the zeros of a complex polynomial $p(z)$ and the zeros (called also critical points) of its derivative $p'(z)$. The fundamental fact in this feeld, see [1, p. 69 - 78], is:

Theorem 1 (Gauss-Lucas) *The convex hull $K(p)$ of a polynomial $p(z)$ contain the the zeros of its derivative $p'(z)$.*

To every polynomial

$$p(z) = \sum_{k=0}^{n} a_k z^k; \quad a_n \neq 0$$

correspond a multiaffine, symmetric polynomial in n complex variables:

$$P(z_1, \ldots, z_n) := \sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} S_k(z_1, \ldots,$$

where

$$S_k(z_1, \ldots, z_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} z_{i_1} \cdots z_{i_k}, \quad k = 1, 2, \ldots, n$$

are the elementary symmetric polynomials of degree k, with $S_0(z_1, \ldots, z_n) := 1$. Obviously, $p(z) = P(z, \ldots, z)$. One say that $P(z_1, \ldots, z_n)$ is the *symmetrization of* $p(z)$.
The n-tuple $\{z_1, \ldots, z_n\}$ is called a *solution* of $p(z)$ if $P(z_1, \ldots, z_n) = 0$.
To study extreme problems in the Geometry of polynomials, we introduce, the following notion, see [2], [3], [4], [5].

Definition 1 *A closed subset Ω of $\mathcal{C}^* = \mathcal{C} \bigcup \infty$ is called a* **locus holder** *of $p(z)$ if Ω contains at least one point from every solution of $p(z)$. A minimal by inclusion locus holder Ω is called a* **locus** *of $p(z)$.*

It was shown in [2] that every locus holder contains a locus. If α is a zero of $p(z)$ and Ω is a locus of $p(z)$, then $\alpha \in \Omega$, since $\{\alpha, \alpha, \ldots, \alpha\}$ is a solution of $p(z)$. It is also shown, see [2], that every locus holder of $p(z)$ contains all zeros of all its derivatives $p^{(s)}(z);\ s = 1, 2, \ldots, n-1$. A restatement of the classical theorem of Grace, see [1, p. 107], says that every circular domain containing the zeros of $p(z)$ is a locus holder of $p(z)$. In fact, every locus of $p(z)$ allows one to formulate an extreme version of Grace's theorem, see [3].
A corollary of the Grace theorem, see [1, p. 126], is:

Theorem 2 (Grace-Heawood) *Let $p(z)$ be a polynomial of degree $n \ge 2$. If $z_1 \neq z_2$ and $p(z_1) = p(z_2)$, then the disk $D_n(z_1, z_2)$ with center $c = \frac{z_1+z_2}{2}$ and radius $r = \frac{|z_1-z_2|}{2} \cot \frac{\pi}{n}$ contains at least one zero of $p'(z)$.*

In fact $D_n(z_1, z_2)$ is a locus holder of a polynomial $\kappa_{n-1}(z)$ of degree $n-1$, depending only on the points z_1 and z_2, but not from the poliynomial $p(z)$. To sharpen the theorem of Grace-Heawood, we have to find a locus holder of $\kappa_{n-1}(z)$ with smaller area then this of the disk $D_n(z_1, z_2)$, see [5]. The disk $D_n(z_1, z_2)$ is only a locus holder of $\kappa_{n-1}(z)$, but not a locus. The problem to find the sharpest analogue of the Grace-Heawood theorem was formulated by Lubomir Tchakaloff, see [6], more than 80 years ago.
There are many extreme problems in the Geometry of polynomials, which may be solved by finding a locus of a polynomial.
It is possible to find a locus holder of a given polynomial by analytical methods. But until now, we do not know an analitical method to find the locus with the smallest area of a polynomial of degree $n \geq 4$.
The lecture is devoted to some ideas for constructing numerical algorithms for calculating the locus of a polynomial with smallest area. Observe that every polynomial has a such locus, but the problem of its uniqueness is steel open.

References

[1] RAHMAN, Q. I. AND SCHMEISSER, G., *Analytic Theory of Polynomials*, Oxford Univ. Press Inc., New York, (2002).

[2] SENDOV, BL. AND SENDOV, H.S., Loci of complex polynomials, part I, *Trans. Amer. Math. Soc.*, 10(366) 5155–5184 (2014).

[3] SENDOV, BL. AND SENDOV, H.S., Loci of complex polynomials, part II: polar derivatives, *Math. Proc. Camb. Phil. Soc.*, 159, 253–273 (2015).

[4] SENDOV, BL. AND SENDOV, H.S., Two Walsh-type theorems for the solutions of multi-affine symmetric polynomials, *Progress in Approximation Theory and Applicable Complex Analysis - In the Memory of Q.I. Rahman*, Springer-Verlag series in "Optimization and Its Applications", accepted (2016).

[5] SENDOV, BL. AND SENDOV, H.S., Stronger Rolle's theorem for complex polynomials *Proc. Amer. Math. Soc.*, v. 148, n. 8, 3367–3380 (2018).

[6] TCHAKALOFF, L.: Sur une généralisation du théoreme de Rolle pour polynomiales, C. R. Acad. Sci. Paris, 202 (1936), 1635 - 1637.