On the universal optimality of the 600-cell: the Levenshtein framework lifted

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The potential energy of a spherical code $C \subset \mathbb{S}^{n-1}$ with interaction potential h is defined as $E(C,n,h) := \sum_{x \neq y \in C} h(\langle x, y \rangle)$. In 2016, based on the Delsate-Yudin linear programming approach, we derived a universal lower bound (ULB) on energy for absolutely monotone potentials

$$E(C,n,h) \geq N^2 \sum_{i=1}^{m} \rho_i h(\alpha_i),$$

where the nodes $\{\alpha_i\}$ and weights $\{\rho_i\}$ depend only on the cardinality N and dimension n and not on the interacting potential. The nodes and weights are obtained from a quadrature rule framework introduced by Levenshtein in relation to maximal codes. The ULB is attained for all universally optimal codes discovered by Cohn and Kumar, except the 600-cell (a code with 120 points on $\mathbb{S}^3$).

In this talk we present a method for solving the linear program for polynomials of higher degree, essentially lifting the Levenshtein framework and obtaining a second level ULB. While there are numerous cases for which our method applies, we will emphasize the model examples of 24 points (24-cell) and 120 points (600-cell) on $\mathbb{S}^3$. In particular, we provide a new proof that the 600-cell is universally optimal, and furthermore, we completely characterize the optimal linear programing polynomials of degree at most 17 by finding two new polynomials, which together with the Cohn-Kumar's polynomial form the vertices of the convex hull that consists of all optimal polynomials.

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