5. Möbius Transformations

5.1. The linear transformation and the inversion.

In this section we investigate the Möbius transformation which provides very convenient methods of finding a one-to-one mapping of one domain into another. Let us start with the *a linear transformation*

$$w = \phi(z) := Az + B, \tag{1}$$

where A and B are fixed complex numbers, $A \neq 0$.

We write (1) as

$$w = \phi(z) := |A| e^{i\mathring{A}(a)} z + B.$$

As we see this transformation is a composition of a *rotation about the origin through the angle Arg* (a)

$$w_1 := e^{i\text{Arg }(a)} z,$$

a *magnification*

$$w_2 = |A| w_1$$

and a *translation*

$$w = w_3 = w_2 + B.$$

Each of these transformations are one-to-one mappings of the complex plane onto itself and gap geometric objects onto congruent objects. In particular, the the ranges of lines and of circles are line and circles, respectively.

Now we consider the *inversion* defined by

$$w := \frac{1}{z}. \tag{2}$$

It is easy to see that the inversion is a one-to-one mapping of the extended complex plane $\overline{\mathbb{C}}$ onto itself ($0 \longrightarrow \infty$ and vice versa $\infty \longrightarrow 0$.)

We are going to show that the image of a line is either a line or a circle. Indeed, let first l passes through the origin. The image of a point $\rho e^{i\theta}$ is $\frac{1}{\rho} e^{-i\theta}$. Letting ρ to tend from the negative infinity to the positive one, we see that the image is another line through the origin with an angle of inclination $-\Theta$.

Let now L be given by the equation

$$L : Ax + By = C, \text{ with } C \neq 0, \text{ and } |A| + |B| > 0. \tag{3}$$

On writing $w = u + iv$, we find

$$z = \frac{\bar{w}}{|w|^2} = \frac{u - iv}{u^2 + v^2}$$

and so

$$x = \frac{u}{u^2 + v^2}, \; y = \frac{-u}{u^2 + v^2}. \tag{4}$$

Making these substitutions into (3), we get

$$A\frac{u}{u^2 + v^2} + B\frac{-v}{u^2 + v^2} = C,$$

or, equivalently,

$$u^2 + v^2 - \frac{A}{C}u + \frac{B}{C}v = 0. \tag{5}$$

This is apparently a equation of a circle.

5.2 We are prepared to go define a Möbius transformation in the general sense.

Definition: The transformation

$$w := \frac{az + b}{cz + d}, \; |a| + |c| > 0, \; ad - bc \neq 0$$

is called a **Möbius transformation**. ℵ

If $c = 0$, then the Möbius transformation is linear. If $c \neq 0$, $a = 0$ then the transformation is a inversion. Consider the case $ac \neq 0$. Then w can be written as

$$w(z) = \frac{a}{c}\left(1 + \frac{bc - ad}{a(cz + d)}\right) \tag{6}$$

which is as a matter of fact a decomposition of a linear transformation and and inverse function. We also notice that

$$w'(z) = \frac{ad - bc}{(cz + d)^2} \neq 0$$

Hence, w is conformal at every point $z \neq -d/c$.

Having in mind this observation and the previous deliberations, we can summarize

Theorem 5.1. *Let f be a Möbius transformation. Then*
f can be expressed as a composition of magnifications, rotations, translations and inversions.
f maps the extended complex plane onto itself.
f maps the class of circled and lines to itself.
f is conformal at every point except its pole.

5.3. The group of Möbius transformations. Let

$$w = f(z) = \frac{az+b}{cz+d} \tag{7}$$

be a möbius transformation. The inverse function $z = f^{-1}(w)$ (that is: $f \circ f^{-1} \equiv I, I-$ the identity) can be computed directly:

$$f^{-1}(w) = \frac{dw-b}{-cw+a}.$$

We see that the inverse is again a Möbius transformation. Furthermore, we easily check that the composition of two transformations $f_1 \circ f_2(z)$ (e.g. if

$$f_i(z) = \frac{a_i z + b_i}{c_i z + d_i}, i = 1, 2$$

then

$$f_1 \circ f_2(z) := f_1(f_2(z)) = \frac{a_1 f_2(z) + b_1}{c_1 f_2(z) + d_2})$$

is again a transformation of Möbius. On the other hand, $f \circ I(z) = I \circ f(z) = f(z)$. So, the set of all Möbius transformations is a group with respect to the composition.

Now we shall prove

Theorem 5.2. *A Möbius transformation is uniquely determined by three points $z_i, i = 1, 2, 3, z_i \neq z_j, i, j = 1, 2, 3$.*

Proof: We first introduce the term of a *double point*. We say that z_0 is double point of $f(z)$, if

$$f(z) = z. \tag{8}$$

It It is obvious that a Möbius transformation has not more that two double points unless it coincides with the identity. Indeed, if (8) is fulfilled for three distinct points, then the quadratic equation

$$az + b = cz^2 + dz$$

will have three distinct zeros, which implies $a = d, b = c = 0$. Notice that a linear transformation has only one double point.

Let now $z_i, i = 1, 2, 3$ and $w_i, i = 1, 2, 3$ be given, $z_i \neq z_j$, $w_i \neq w_j, i.j. = 1, 2, 3$. We are looking for a transformation $w = f(z)$ such that

$$f(z_i) = w_i. \tag{9}$$

Consider the cross-ratio (z, z_1, z_2, z_3) of the points $z,\ z_i, i = 1, 2, 3$, that is

$$T(z) = (z, z_1, z_2, z_3) := \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}.$$

The function $T(z), z \in \mathbb{C},\ z_i - fixed$ maps $\mathbb{C}$ in a one-to-one way onto itself[1]. Notice that the order in which the points are listed is crucial in this notation. Then the desired transformation (9) is given by the composition

$$(z, z_1, z_2, z_3) = (w, w_1, w_2, w_3).$$

It remains only to equate w.

The next step is to show that $w = f(z)$ is the only transformation with property (9).

Indeed, suppose to the contrary that there is another Möbius transformation $v = g(z)$, $f \not\equiv g$ which satisfies (9). Then the Möbius transformation $f \circ g^{-1}$ has three distinct double points which is impossible. **Q.E.D.**

Example 5.1: Find a Möbius transformation $T(z)$ that maps the real line $\mathbb{R}$ onto the unit circle $C_0(1)$.

Solution: We select three arbitrary real points, say $-1, 0, 1$ and three arbitrary points on $C_0(1)$, say $-i, 1, i$. The transformation $w = T(z)$, determined by the cross products, namely

$$(z, -1, 0, 1) = (w, -i, 1, i)$$

[1]if some number z_i equals infinity, then the factors containing it are replaced by the unity

maps the real line $\mathbb{R}$ onto the circle C passing through the points $-i, 1, i$. As we know from the elementary geometry, $C \equiv C_0(1)$. $\aleph$

Example 5.2: Let

$$w(z) := \frac{z+1}{z-1}.$$

Find the image of $\mathbb{R}, \mathbb{I}$ and $C_0(1)$ under the mapping $w = T(z)$.

Solution: We first observe that

$$\mathbb{R} \longrightarrow \mathbb{R}.$$

Further, we see that

$$\begin{aligned} &T(-1) = 0, \\ &T(1) = \infty \\ &T(0) = -1, \\ &T(\infty) = 1, \\ &T(i) = -i. \end{aligned}$$

The image of $\mathbb{I}$ is a line or a circle passing through the points $-1, 1, -i$, that is the unit circle. The image of $C_0(1)$ is a line or a circle that passes through $0, \infty$ and is orthogonal to the images of $\mathbb{R}$ and of $\mathbb{I}$, in our case orthogonal to $\mathbb{R}$ and to $C_0(1)$. This turns to be the imaginary axis.

5.4. The left-hand-rule. Consider the unit circle $C_0(1)$. The points $-1, -i, 1$ determine the direction $-1 \longrightarrow -i \longrightarrow 1 \longrightarrow -1$ in traversing $C_0(1)$. The interior of the circle, the unit disk $D_0(1)$ lies to the left of this orientation. We use to say that the disk is *the left region* with respect to the orientation $-1 \longrightarrow -i \longrightarrow 1$. Analogously, the upper half plane is the left region with respect to the direction $-1, 0, \infty$. Since Möbius transformations transformations are conformal mappings, it can be shown that a Möbius transformation that takes the distinct points z_1, z_2, z_3 to the respective points w_1, w_2, w_3 must map the left region of the circle (or line) oriented by z_1, z_2, z_3 onto the left region of the circle (or line) oriented by w_1, w_2, w_3. Using the conformality, we summarize

Theorem 5.3. *Let G be a domain in $\mathbb{C}$, $\partial G := \Gamma$ and assume that G is left oriented with respect to the direction given by the points $z_1 \longrightarrow z_2 \longrightarrow z_3, z_i \in \Gamma, i = 1,2,3 \in \Gamma$. Let $w = T(z)$ be a Möbius transformationthat maps Γ onto γ and $T(z_i) = w_i, i = 1,2,3$. Then T maps G onto that domain which is left oriented to γ with respect to the direction $w_1 \longrightarrow w_2 \longrightarrow w_3$.*

Example 5.3: Let $w = T(z)$ be the Möbius transformationfrom Example 2. Find the images of the upper half plane, of the left half plane and of the unit disk.

Solution: As we have seen,

$$T: \; \mathbb{R} \longrightarrow \mathbb{R}, \quad \mathbb{I} \longrightarrow C_0(1), \quad C_0(1) \longrightarrow \mathbb{I}.$$

The upper plane the left domain with respect to the direction $-1 \longrightarrow 0 \longrightarrow \infty$, hence the range domain will be left oriented with respect to $0, -1, 1$ (the images of $-1, 0, \infty$), e.g., the half plane below the real axis. Similarly, we find that the left half plane is mapped in the unit disk, whereas the unit disk - in the left half plane.

5.5. The symmetry and the Möbius transformation.

Definition: Given the circle $C_a(r)$ of radius r and centered at $z = a$, we say that the points $z, z^* \notin C_a(r)$ are *symmetric with respect to* $C_a(r)$ *if*

$$z^* = \begin{cases} \frac{R^2}{\bar{z}-\bar{a}} + a, & z \neq a, \\ \infty, & z = a. \end{cases}$$

One can easily prove that in this definition each straight line or circle passing through z and z^* intersects t$C_a(r)$ orthogonally. In the case of a line we have the usual symmetry with respect to it.

Theorem 5.3. *Given a circle C and a Möbius transformation$w = T(z)$, assume that z, z^* are symmetric with respect to C. Then their images $T(z), T(z^*)$ are again symmetric with respect to the image $T(C)$ of C.*

Example 5.4: Find all Möbius transformationtransformations that map the disk $C_0(r)$ onto itself.

Solution; We fix in an arbitrary way a point $a \in D_0(r)$. Its symmetric point with respect to the unit circle is

$$a* = \frac{r^2}{\bar{a}}. \tag{10}$$

Any transformation of the form

$$S(z) := K \frac{z - a}{z - r^2 a*} \tag{11}$$

with K being some constant maps the points a and $a*$ at 0 and infinity, respectively. Because of the symmetry, the image of $C_0(r)$ will be a circle C centered at the zero. Now we are going to choose the constant K in such a way that C coincides with $C_0(r)$. Take a point $z_0 := re^{i\phi}$ and calculate $S(z_0)$. We have, by (10),

$$|S(z_0)| = K\frac{re^{i\phi} - a}{a * re^{i\phi} - r^2},$$

or

$$|S(z_0)| = \frac{|K|}{r}\frac{|re^{i\phi} - a|}{|a * -re^{i\phi}|}.$$

Putting

$$|S(z_0)| = r,$$

we arrive at

$$T(z) = r^2 e^{i\Theta}\frac{z - a}{z - r^2 a*}$$

for some real Θ. $\rightarrow$

Exercises:

1. Let

$$w = 1/z.$$

Find the image of the lines $y = kx$, $y = ax + b$, and of the circle $x^2 + y^2 = ax$, $x^2 + y^2$.

2. Let

$$w = \frac{z - i}{z + i}.$$

Find the image of $\{x, y \geq 0\}$.

3. Let $w = \frac{z}{z-1}$. Find the image of the angle $0 \leq \phi \leq \frac{\pi}{4}$

4. Find all Möbius transformations that maps the upper half plane onto itself.

5. Given $T(z) := \frac{z-1}{z+1}$, find the image of the domain $G := \{z, |z| < 1\} \bigcap \{z, |z - 1| < 1\}$.