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Resonance Mean-Periodic Solutions of Euler Differential Equations

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Abstract. The Euler operator $\delta = t\dfrac{d}{dt}$ is considered in the space $C = C(\mathbb{R}_+)$ of the continuous functions on $\mathbb{R}_+ = (0,\infty)$. Nonlocal operational calculi for it are developed and used for solving nonlocal Cauchy boundary value problems for Euler differential equations of the form $P(\delta)y = f$ with a polynomial P. A function $f \in C(\mathbb{R}_+)$ is said to be mean-periodic for the Euler operator with respect to the linear functional Φ (or simply Φ-*mean-periodic*) if $\Phi_\tau\{f(t\tau)\} = 0$ identically on $\mathbb{R}_+$. The solution of Euler differential equations in mean-periodic functions for δ with respect to an arbitrary linear functional Φ reduces to non-local homogeneous Cauchy problems.

Denoting the algebraic equivalent of the Euler differential operator δ by S, the solution of an Euler differential equation $P(\delta)y = f$ in Φ-mean-periodic functions reduces to the interpretation of $y = \dfrac{1}{P(S)} f$ as a function. This is done both in the non-resonance and in the resonance cases.

Keywords: Euler operator, mean-periodic function, resolvent operator, commutant, Euler convolution.
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RESULTS FOR THE NON-RESONANCE CASE

The non-resonance case of solving Euler differential equations in mean-periodic functions and some preliminary definitions and theorems were considered in previous papers and will be briefly presented in this section.

Nonlocal operational calculi for the Euler operator

Let Φ be a non-zero linear functional on $C = C(\mathbb{R}_+)$. The solution of the elementary boundary value problem

$$\delta y - \lambda y = t\frac{dy}{dt} - \lambda y = f(t), \quad \Phi\{y\} = 0,$$

has the form

$$y = L_\lambda f(x) = \frac{t^\lambda}{E(\lambda)} \Phi_\tau \left\{ \tau^\lambda \int_\tau^t \frac{f(\sigma)d\sigma}{\sigma^{\lambda+1}} \right\},$$

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where $E(\lambda) = \Phi_\tau\{\tau^\lambda\}$ is the Euler indicatrix of the functional Φ. L_λ is said to be the *resolvent operator* of δ with respect to Φ.

We begin with the following non-classical convolution related to the Euler operator, which is considered in details in [3], [4], and [6]:

Theorem 1 *Let Φ be a continuous non-zero linear functional on $C(\mathbb{R}_+)$. Then the operation*

$$(f * g)(t) = \Phi_\tau\left\{\int_\tau^t f\left(\frac{t\tau}{\sigma}\right) g(\sigma)\frac{d\sigma}{\sigma}\right\} \tag{1}$$

is a separately continuous, bilinear, commutative, and associative operation in $C(\mathbb{R}_+)$, such that

$$L_\lambda f(t) = \left\{\frac{t^\lambda}{E(\lambda)}\right\} * f.$$

Lemma 1 *If $f \in C^1(\mathbb{R}_+)$, then*

$$\delta(f * g) = (\delta f) * g - \Phi\{f\}g.$$

Here we pay attention also to a very useful property of the convolution (1):

Lemma 2 *The convolution given by* (1) *is such that*

$$\Phi\{f * g\} = 0 \tag{2}$$

for arbitrary $f, g \in C(\mathbb{R}_+)$.

Further, if $\Phi\{1\} \neq 0$, i.e. if $E(0) \neq 0$, then without loss of generality we may assume that the functional Φ satisfies $\Phi\{1\} = 1$ and then the following representation for $L = L_0$ holds:

$$Lf = \{1\} * f.$$

Let the space $C = C(\mathbb{R}_+)$ be considered as a commutative and associative algebra with the convolution $*$ as multiplication. Next, the commutative ring $\mathscr{M}$ of convolution fractions $\frac{f}{g}$ with g being nonzero non-divisor of zero) is defined. Two convolution fractions $\frac{f}{g}$ and $\frac{f_1}{g_1}$ are considered as equal iff $f * g_1 = g * f_1$. The elements of $C(\mathbb{R}_+)$ may be considered as convolutional fractions due to the embeding

$$f \hookrightarrow \frac{f * \{1\}}{\{1\}}.$$

It embeds the ring $(C(\mathbb{R}_+), *)$ into the ring $\mathscr{M}$ of the convolution fractions.

The reciprocal element

$$S = L^{-1}$$

of L in the ring $\mathscr{M}$ is called the *algebraic Euler operator*. Its relation to the ordinary Euler operator δ is given by

Lemma 3 *If $f \in C^1(\mathbb{R}_+)$, then*

$$\delta f = Sf - \Phi\{f\},$$

where $\Phi\{f\}$ is the "numerical operator" $\dfrac{\{\Phi\{f\}\}}{\{1\}}$.

By induction it follows that

$$\delta^k f = S^k f - \Phi\{f\}S^{k-1} - \Phi\{\delta f\}S^{k-2} - \ldots - \Phi\{\delta^{k-1} f\}. \tag{3}$$

If λ is such that $E(\lambda) = \Phi_\tau(\tau^\lambda) \neq 0$, then (see [8])

$$\frac{1}{S-\lambda} = \left\{\frac{t^\lambda}{E(\lambda)}\right\} \text{ and } \frac{1}{(S-\lambda)^k} = \left\{\frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial\lambda^{k-1}}\left(\frac{t^\lambda}{E(\lambda)}\right)\right\},\ k > 1.$$

Nonlocal Cauchy boundary value problems for Euler equations

The general nonlocal Cauchy boundary value problem for the Euler operator $\delta = t\dfrac{d}{dt}$ has the form

$$P(\delta)y(t) = F(t), \quad \Phi\{\delta^k y\} = \alpha_k, \quad k = 0,1,2,\ldots,\deg P - 1, \tag{4}$$

where P is a polynomial, Φ is an arbitrary non-zero linear functional, and α_k are real or complex numbers.

Lemma 4 *Let none of the zeros of the polynomial $P(\lambda)$ be a zero of the indicatrix $E(\lambda)$, i.e. $\{\lambda : P(\lambda) = 0\} \cap \{\lambda : P(\lambda) = 0\} = \emptyset$. Then $P(S)$ is a non-divisor of zero in $\mathscr{M}$.*

The case, when $P(S)$ is a non-divisor of zero in $\mathscr{M}$, is called the *non-resonance case*.

In this non-resonance case the operational approach gives the solution simply by substituting (3) in (4) in order to obtain an usual algebraic equation

$$P(S)y = F + Q(S), \tag{5}$$

where $Q(S)$ is a polynomial of S with $\deg Q < \deg P$. It has the formal solution

$$y = \frac{1}{P(S)}F + \frac{Q(S)}{P(S)}.$$

Using the zeros of the polynomial P, the formal quotients $\dfrac{1}{P(S)}$ and $\dfrac{Q(S)}{P(S)}$ can be written as sums of elementary fractions. Representing each such fraction as a function and, then using

$$\begin{gathered} \frac{1}{S-\lambda} = \left\{\frac{t^\lambda}{E(\lambda)}\right\} \quad \text{and} \\ \frac{1}{(S-\lambda)^k} = \left\{\frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial\lambda^{k-1}}\left(\frac{t^\lambda}{E(\lambda)}\right)\right\}, \quad \text{for } k > 1, \end{gathered} \tag{6}$$

one obtains the solution of the nonlocal Cauchy boundary problem in the nonresonance case.

Mean-periodic solutions of Euler differential equations in the non-resonance case

Definition 1 (Schwartz [10], §22) *A function $f \in C(\mathbb{R}_+)$ is said to be mean-periodic for the Euler operator with respect to a linear functional* Φ (shortly Φ-*mean-periodic*, *Euler mean-periodic*, or simply *mean-periodic*) *if*

$$\Phi_\tau\{f(t\tau)\} = 0$$

identically on $\mathbb{R}_+$.

Let us denote by MP_Φ the subset of the mean-periodic functions in $C(\mathbb{R}_+)$ with respect to the functional Φ.

If we are looking for mean-periodic solutions of the Euler equation $P(\delta)y(t) = f(t)$, it is equivalent to a nonlocal Cauchy problem with *homogeneous* boundary value conditions, i.e.

$$P(\delta)y(t) = f(t), \quad \Phi\{\delta^k y\} = 0, \quad k = 0, 1, 2, \ldots, \deg P - 1. \tag{7}$$

The following two theorems concerning properties of the mean-periodic functions in the convolutional algebra $(C(\mathbb{R}_+), *)$ are proved by the authors in [3] and [4]:

Theorem 2 *The mean-periodic functions for the Euler operator δ with respect to any non-zero functional $\Phi : C(\mathbb{R}_+) \to \mathbb{C}$ form an ideal in the convolutional algebra $(C(\mathbb{R}_+), *)$.*

Theorem 3 *If $f \in MP_\Phi$ and $\{\lambda : P(\lambda) = 0\} \cap \{\lambda : E(\lambda) = 0\} = \emptyset$, then $P(\delta)y = f$ has a unique solution $y \in MP_\Phi$ and it has the explicit form*

$$y = G * f,$$

with $G = \{G(t)\} = \dfrac{1}{P(S)}$ using the representations (6).

OPERATIONAL METHOD FOR MEAN-PERIODIC RESONANCE SOLUTIONS OF EULER DIFFERENTIAL EQUATIONS

Here we give a result for multiple resonance zeros which generalizes the case of simple resonance zeros considered by the authors in [5].

As it was mentioned above, the Φ-mean-periodic resonance solutions of an Euler differential equation have to satisfy the homogeneous Cauchy boundary value conditions:

$$P(\delta)y(t) = f(t), \quad \Phi\{\delta^k y\} = 0, \quad k = 0, 1, 2, \ldots, \deg P - 1. \tag{8}$$

If λ_0 is one of the resonance zeros, let us denote by k_0 and $\varkappa_0$ its multiplicities in the polynomial P and the Euler indicatrix E, respectively, i.e.

$$P(\lambda) = (\lambda-\lambda_0)^{k_0} Q(\lambda), \quad Q(\lambda_0) \neq 0, \tag{9}$$

and

$$E(\lambda_0) = E'(\lambda_0) = \ldots = E^{(\varkappa-1)}(\lambda_0) = 0, E^{(\varkappa_0)}(\lambda_0) \neq 0. \tag{10}$$

Now an idea of S. Grozdev [9] will be used for reducing the resonance case to the non-resonance one.

Denote by $\widetilde{C}_{\lambda_0}$ the subalgebra of $(C(\mathbb{R}_+),*)$ consisting of the functions $f \in C(\mathbb{R}_+)$ satisfying the conditions

$$f * t^{\lambda_0} = 0, f * t^{\lambda_0} \ln t = 0, \ldots, f * t^{\lambda_0} \ln^{k_0-1} t = 0. \tag{11}$$

Obviously these conditions are necessary for existing of a mean-periodic solution. Our next task is to show that they are also sufficient.

Lemma 5 *Let λ_0 be a resonance zero of the polynomial P. Then, if $\widetilde{S}$ denotes the restriction of S to $\widetilde{C}_{\lambda_0}$,it holds*

$$\frac{1}{(\widetilde{S}-\lambda_0)^n} = \left\{ \frac{t^{\lambda_0} A_n(\ln t)}{E^{(\varkappa)}(\lambda_0)} \right\} * , \tag{12}$$

where $A_n(x)$, $n = 1, 2, \ldots$, are an Appell set of polynomials defined recurrently by

$$A_1(\ln t) = \ln^{\varkappa} t, \;\; \Phi_t \left\{ t^{\lambda_0} A_n(\ln t) \right\} = 0, \;\; A'_{n+1}(x) = A_n(x), \;\; x = \ln t. \tag{13}$$

We have to consider two cases: 1) $k_0 \leq \varkappa_0$ and 2) $k_0 > \varkappa_0$.

Case 1: $k_0 \leq \varkappa_0$

For any λ which is not a zero of the polynomial P we can use the representation

$$R_\lambda \widetilde{f} = \left\{ \frac{t^\lambda}{E(\lambda)} \right\} * \widetilde{f} \quad \text{for } \widetilde{f} \in \widetilde{C}_{\lambda_0} \text{ and } P(\lambda) \neq 0. \tag{14}$$

Then it is natural to let λ tend to λ_0:

$$\lim_{\lambda \to \lambda_0} R_\lambda \widetilde{f} = \lim_{\lambda \to \lambda_0} \left\{ \frac{t^{\lambda_0}.t^{\lambda-\lambda_0}}{E(\lambda)} \right\} * \widetilde{f}. \tag{15}$$

Representing

$$\begin{aligned} t^{\lambda-\lambda_0} &= e^{(\lambda-\lambda_0)\ln t} = \sum_{k=0}^{\varkappa_0-1} \frac{(\lambda-\lambda_0)^k \ln^k t}{k!} + \\ &+ (\lambda-\lambda_0)^{\varkappa_0} \left[\frac{\ln^{\varkappa_0} t}{\varkappa_0!} + (\lambda-\lambda_0) \sum_{k=\varkappa_0+1}^{\infty} \frac{(\lambda-\lambda_0)^{k-\varkappa_0-1} \ln^k t}{k!} \right] \end{aligned} \tag{16}$$

and

$$E(\lambda)=(\lambda-\lambda_0)^{\varkappa}\left[\frac{E^{(\varkappa_0)}(\lambda_0)}{\varkappa_0!}+(\lambda-\lambda_0)\sum_{k=\varkappa_0+1}^{\infty}\frac{(\lambda-\lambda_0)^{k-\varkappa_0-1}E^k(\lambda_0)}{k!}\right], \tag{17}$$

it follows that

$$\lim_{\lambda\to\lambda_0} R_\lambda \widetilde{f}=\left\{\frac{t^{\lambda_0}\ln^{\varkappa_0} t}{E^{(\varkappa_0)}(\lambda_0)}\right\}*\widetilde{f}. \tag{18}$$

Now we can express the elementary fractions needed for the solution as

$$\frac{1}{\widetilde{S}-\lambda_0}=\left\{\frac{t^{\lambda_0}\ln^{\varkappa_0} t}{E^{(\varkappa_0)}(\lambda_0)}\right\}* \tag{19}$$

and by induction

$$\frac{1}{(\widetilde{S}-\lambda_0)^n}=\left\{\frac{t^{\lambda_0}A_n(\ln t)}{E^{(\varkappa_0)}(\lambda_0)}\right\}*, \tag{20}$$

where A_n are polynomials of Appel type in the sense of Bourbaki [1] (Chapter 6.2) related to the Euler operator with the properties

$$A_1(\ln t)=\ln^{\varkappa_0} t,\;\; \Phi_t\left\{t^{\lambda_0}A_n(\ln t)\right\}=0,\;\; A'_{n+1}(x)=A_n(x),\;\; x=\ln t.$$

Later we will give the form of the solution of (8) and the part corresponding to the resonance zero $\lambda=\lambda_0$ depends in this case on k_0 arbitrary constants.

Case 2: $k_0>\varkappa_0$

In this case the representation of $\frac{1}{(\widetilde{S}-\lambda_0)^n}$ is the same. The only difference is that the part of the solution corresponding to the resonance zero $\lambda=\lambda_0$ depends on $\varkappa_0$ arbitrary constants as it will be indicated below. □

General solution

In the case when the polynomial P in (8) has more than one multiple resonance zeros we can proceed as follows:

Let the resonance zeros be $\lambda_1,\ldots,\lambda_m$ with multiplicities $k_1,\ldots,k_m$.

In fact, after removing the non-resonance zeros of P, we want to find the mean-periodic solutions w of the auxiliary equation

$$(\delta-\lambda_1)^{k_1}\ldots(\delta-\lambda_m)^{k_m}w(t)=F(t). \tag{21}$$

Lemma 5 shows how to treat the case of only one zero. In the general case we can take for each zero λ_μ, $\mu=1,\ldots,m$, the subalgebra $\widetilde{C}_{\lambda_\mu}$ of $(C(\mathbb{R}_+),*)$ consisting of the functions $f\in C(\mathbb{R}_+)$ satisfying the conditions

$$f*t^{\lambda_\mu}=0, f*t^{\lambda_\mu}\ln t=0,\ldots,f*t^{\lambda_\mu}\ln^{k_\mu-1} t=0. \tag{22}$$

Next, denote by $\widetilde{L}$ the restriction of L to the intersection $\widetilde{C}_{\lambda_1,\ldots,\lambda_m} = \cap_{\mu=1}^{m} \widetilde{C}_{\lambda_\mu}$. Let $\widetilde{S} = \dfrac{1}{\widetilde{\widetilde{L}}}$. In the ring $\widetilde{M}_{\lambda_1,\ldots,\lambda_m}$ of the convolutional fractions in $\widetilde{C}_{\lambda_1,\ldots,\lambda_m}$, equation (21) takes the form

$$(\widetilde{S}-\lambda_1)^{k_1}\ldots(\widetilde{S}-\lambda_m)^{k_m} w = F. \tag{23}$$

The elements $(\widetilde{S}-\lambda_k)$, $k = 1,2,\ldots,m$, are non-divisors of zero in $\widetilde{M}_{\lambda_1,\ldots,\lambda_m}$ and the formal solution of (23) is

$$w = \frac{1}{(\widetilde{S}-\lambda_1)^{k_1}\ldots(\widetilde{S}-\lambda_m)^{k_m}} F. \tag{24}$$

Next step is to represent the algebraic multiplier $\dfrac{1}{(\widetilde{S}-\lambda_1)^{k_1}\ldots(\widetilde{S}-\lambda_m)^{k_m}}$ in the form

$$\sum_{\mu=1}^{m}\sum_{j=1}^{k_\mu} \frac{B_{\mu,j}}{(\widetilde{S}-\lambda_\mu)^j}\ .$$

The general solution of the auxiliary equation (21), which contains only the resonance zeros of P from the Euler equation (8), is then a sum

$$w = \sum_{\mu=1}^{m}\left(\sum_{j=1}^{k_\mu} B_{\mu,j}\left\{\frac{t^{\lambda_\mu}A_{\mu,j}(\ln t)}{E^{(\varkappa)}(\lambda_\mu)}\right\} * F + \sum_{j=1}^{\min\{k_\mu,\varkappa_\mu\}} C_{\mu,j} t^{\lambda_\mu} \ln^{j-1} t\right)$$

with a uniquely determined constants $B_{\mu,j}$ and arbitrary constants $C_{\mu,j}$, $\mu = 1,\ldots,m$, $j = 1,\ldots,\min\{k_\mu,\varkappa_\mu\}$. Here $A_{\mu,j}$, $\mu = 1,\ldots,m$, $s = 1,2,\ldots$, are m polynomial systems of Appel type in the sense of Bourbaki [1] (Chapter 6.2) related to the Euler operator with the properties

$$A_{\mu,1}(\ln t) = \ln^{\varkappa_\mu} t,\ \ \Phi_t\left\{t^{\lambda_m} A_{\mu,s}(\ln t)\right\} = 0,\ \ A'_{\mu,s+1}(x) = A_{\mu,s}(x),\ \ x = \ln t.$$

In order to have a full solution of (8), the non-resonance zeros have also to be taken into account using for them the representations (6).

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