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MATHEMATIQUES
Équations différentielles

MEAN-PERIODIC SOLUTIONS OF EULER DIFFERENTIAL EQUATIONS

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Abstract

The Euler operator $\delta = t\dfrac{d}{dt}$ is considered in the space $C = C(\mathbb{R}_+)$ of the continuous functions on $\mathbb{R}_+ = (0, \infty)$. Nonlocal operational calculi for it are presented and they are used for solving nonlocal Cauchy boundary value problems for Euler differential equations. At last, Euler differential equations are solved in mean-periodic functions for δ with respect to an arbitrary linear functional in the resonance case.

Key words: Euler operator, nonlocal Cauchy problem, convolution, convolution function, mean-periodic function, Euler differential equation, resonance solution, Appell polynomials

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1. Nonlocal operational calculi for the Euler operator. Let Φ be a non-zero linear functional on $C = C(\mathbb{R}_+)$. The solution of the elementary boundary value problem

$$\delta y - \lambda y = f(t), \quad \Phi\{y\} = 0,$$

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has the form

$$y = L_\lambda f(x) = \frac{t^\lambda}{E(\lambda)} \Phi_\tau \left\{ \tau^\lambda \int_\tau^t \frac{f(\sigma) d\sigma}{\sigma^{\lambda+1}} \right\},$$

where $E(\lambda) = \Phi_\tau\{\tau^\lambda\}$ is the Euler indicatrix of the functional Φ. L_λ is said to be the resolvent operator of δ with respect to Φ.

We begin with the following non-classical convolution related to the Euler operator, which is considered in detail in [3–5].

Theorem 1. *Let Φ be a continuous non-zero linear functional on $C(\mathbb{R}_+)$. Then the operation*

$$(f * g)(t) = \Phi_\tau \left\{ \int_\tau^t f\left(\frac{t\tau}{\sigma}\right) g(\sigma) \frac{d\sigma}{\sigma} \right\} \tag{1}$$

is a separately continuous, bilinear, commutative, and associative operation in $C(\mathbb{R}_+)$, such that

$$L_\lambda f(t) = \left\{ \frac{t^\lambda}{E(\lambda)} \right\} * f.$$

Lemma 1. *If $f \in C^1(\mathbb{R}_+)$, then*

$$\delta(f * g) = \delta f * g - \Phi\{f\} g.$$

Here we also pay attention to a very useful property of convolution (1):

Lemma 2. *The convolution given by* (1) *is such that*

$$\Phi\{f * g\} = 0 \tag{2}$$

for arbitrary $f, g \in C(\mathbb{R}_+)$.

Proof. The function

$$h(t, \tau) = \int_\tau^t f\left(\frac{t\tau}{\sigma}\right) g(\sigma) \frac{d\sigma}{\sigma}$$

is antisymmetric with respect to t and τ, i.e. $h(t, \tau) = -h(\tau, t)$ and, hence,

$$\begin{aligned} \Phi\{f * g\} = \Phi_t\{(f * g)(t)\} = \Phi_t \Phi_\tau\{h(t, \tau)\} = \Phi_t \Phi_\tau\{-h(\tau, t)\} \\ = -\Phi_t \Phi_\tau\{h(\tau, t)\} = -\Phi_\tau \Phi_t\{h(\tau, t)\} = -\Phi_t \Phi_\tau\{h(t, \tau)\} = -\Phi\{f * g\}. \end{aligned} \tag{3}$$

Here the Fubini property of the functional Φ is used, i.e. the possibility to interchange Φ_t and Φ_τ. At the end, t and τ are also interchanged, since they are "dumb" variables in the corresponding expressions. Thus the last chain of equalities gives $2\Phi\{f * g\} = 0$ and $\Phi\{f * g\} = 0$ holds. $\square$

Further, if $\Phi\{1\} \neq 0$, i.e. if $E(0) \neq 0$, then without loss of generality we may assume that the functional Φ satisfies $\Phi\{1\} = 1$ and then the following representation for $L := L_0$ holds:

$$Lf = \{1\} * f.$$

Let the space $C = C(\mathbb{R}_+)$ be considered as a commutative and associative algebra with the convolution $*$ as multiplication. Next, the commutative ring $\mathcal{M}$ of convolution fractions $\dfrac{f}{g}$ with g being non-zero non-divisor of zero is defined. Two convolution fractions $\dfrac{f}{g}$ and $\dfrac{f_1}{g_1}$ are considered as equal iff $f * g_1 = g * f_1$. The elements of $C(\mathbb{R}_+)$ may be considered as convolutional fractions due to the embedding

$$f \hookrightarrow \frac{f * \{1\}}{\{1\}}.$$

It embeds the ring $(C(\mathbb{R}_+), *)$ into the ring $\mathcal{M}$ of the convolution fractions.

The reciprocal element

$$S = L^{-1}$$

of L in the ring $\mathcal{M}$ is called the algebraic Euler operator. Its relation to the ordinary Euler operator δ is given by

Lemma 3. *If $f \in C^1(\mathbb{R}_+)$, then*

$$\delta f = Sf - \Phi\{f\},$$

where $\Phi\{f\}$ is the "numerical operator" $\dfrac{\{\Phi\{f\}\}}{\{1\}}$.

By induction it follows that

$$\delta^k f = S^k f - \Phi\{f\}S^{k-1} - \Phi\{\delta f\}S^{k-2} - \cdots - \Phi\{\delta^{k-1} f\}. \tag{4}$$

If λ is such that $E(\lambda) = \Phi_\tau(\tau^\lambda) \neq 0$, then (see [5])

$$\frac{1}{S-\lambda} = \left\{\frac{t^\lambda}{E(\lambda)}\right\} \text{ and } \frac{1}{(S-\lambda)^k} = \left\{\frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial\lambda^{k-1}}\left(\frac{t^\lambda}{E(\lambda)}\right)\right\},\ k > 1.$$

2. Nonlocal Cauchy boundary value problems for Euler equations. The general nonlocal Cauchy boundary value problem for the Euler operator $\delta = t\dfrac{d}{dt}$ has the form

$$P(\delta)y(t) = F(t), \quad \Phi\{\delta^k y\} = \alpha_k, \quad k = 0, 1, 2, \ldots, \deg P - 1, \tag{5}$$

where P is a polynomial, Φ is an arbitrary non-zero linear functional, and α_k are real or complex numbers.

Lemma 4. *Let none of the zeros of the polynomial $P(\lambda)$ be a zero of the indicatrix $E(\lambda)$, i.e. $\{\lambda : P(\lambda) = 0\} \cap \{\lambda : P(\lambda) = 0\} = \emptyset$. Then $P(S)$ is a non-divisor of zero in $\mathcal{M}$.*

Proof. Assume that $P(S)$ is a divisor of zero in $\mathcal{M}$. Then at least for one of the zeros $\lambda = \lambda_1$ of $P(\lambda)$ the element $S - \lambda_1$ of $\mathcal{M}$ should be a divisor of zero. This means that there is a function $u \in C(\mathbb{R}_+)$, $u \neq 0$, such that $(S - \lambda_1)u = 0$. Multiplying by L, we get $u - \lambda_1 Lu = 0$. Hence, $\Phi\{u\} = 0$. Applying the operator δ to $u - \lambda_1 Lu = 0$, we get $\delta u - \lambda_1 u = 0$. All the non-zero solutions of this equation are $u = Ct^{\lambda_1}$ with a constant $C \neq 0$. The condition $\Phi\{u\} = 0$ gives $CE(\lambda_1) = 0$ and hence $E(\lambda_1) = 0$. The contradiction proves the lemma. $\square$

The case, when $P(S)$ is a non-divisor of zero in $\mathcal{M}$, is called the non-resonance case.

In this non-resonance case the operational approach gives the solution simply by substituting (4) in (5) in order to obtain a usual algebraic equation

$$P(S)y = F + Q(S), \tag{6}$$

where $Q(S)$ is a polynomial of S with $\deg Q < \deg P$. It has the formal solution

$$y = \frac{1}{P(S)}F + \frac{Q(S)}{P(S)}.$$

Using the zeros of the polynomial P, the formal quotients $\dfrac{1}{P(S)}$ and $\dfrac{Q(S)}{P(S)}$ can be written as sums of elementary fractions. Representing each such fraction as a function and then using

$$\begin{gathered} \frac{1}{S-\lambda} = \left\{\frac{t^\lambda}{E(\lambda)}\right\} \quad \text{and} \\ \frac{1}{(S-\lambda)^k} = \left\{\frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial\lambda^{k-1}}\left(\frac{t^\lambda}{E(\lambda)}\right)\right\}, \quad \text{for } k > 1, \end{gathered} \tag{7}$$

one obtains the solution of the nonlocal Cauchy boundary problem in the non-resonance case.

3. Mean-periodic solutions of Euler differential equations in the non-resonance case.

Definition 1 (Schwartz [7], §22)**.** *A function $f \in C(\mathbb{R}_+)$ is said to be mean-periodic for the Euler operator with respect to a linear functional* Φ (shortly Φ-*mean-periodic*, *Euler mean-periodic*, or simply *mean-periodic*) *if*

$$\Phi_\tau\{f(t\tau)\} = 0$$

identically on $\mathbb{R}_+$.

Let us denote by C_Φ the subset of the mean-periodic functions in $C(\mathbb{R}_+)$ with respect to the functional Φ.

If we are looking for mean-periodic solutions of the Euler equation $P(\delta)y(t) = F(t)$, it is equivalent to a nonlocal Cauchy problem with homogeneous boundary value conditions, i.e.

$$P(\delta)y(t) = F(t), \quad \Phi\{\delta^k y\} = 0, \quad k = 0, 1, 2, \dots, \deg P - 1. \tag{8}$$

The following two theorems concerning properties of the mean-periodic functions in the convolutional algebra $(C(\mathbb{R}_+), *)$ are proved by the authors in [3] and [4]:

Theorem 2. *The mean-periodic functions for the Euler operator δ with respect to any non-zero functional $\Phi : C(\mathbb{R}_+) \to \mathbb{C}$ form an ideal in the convolutional algebra $(C(\mathbb{R}_+), *)$.*

Proof. Here we propose a simpler proof than the one given in [4]. For the simplicity sake, we consider only the case when $\Phi\{1\} = E(0) \neq 0$. First, we show that $f \in C_\Phi$ implies $Lf \in C_\Phi$. To this end, we consider the function $\varphi(t) = \Phi_\tau\{(Lf)(t\tau)\}$. We have $\delta\varphi(t) = \Phi_\tau\{f(t\tau)\} = 0$ and, hence, $\varphi(t) = c =$ const. But $\varphi(1) = \Phi_\tau\{(Lf)(\tau)\} = 0$. Then $L^n f \in C_\Phi$ for arbitrary $n \in \mathbb{N}$. But $L^n f = A_{n-1}(\ln t) * f$, where A_{n-1} is a polynomial of degree exactly $n-1$. Hence, $Q(\ln t) * f \in C_\Phi$ for arbitrary polynomial Q. Then the separate continuity of the convolution $*$ and the Weierstrass' approximation theorem imply that $f * g \in C_\Phi$ for arbitrary $g \in C(\mathbb{R}_+)$. The case when $\Phi\{1\} = 0$ is considered in [4]. $\square$

Theorem 3. *If $F \in C_\Phi$ and $\{\lambda : P(\lambda) = 0\} \cap \{\lambda : E(\lambda) = 0\} = \emptyset$, then $P(\delta)y = F$ has a unique solution $y \in C_\Phi$ and it has the explicit form*

$$y = G * F,$$

with $G = \{G(t)\} = \dfrac{1}{P(S)}$.

4. Operational method for mean-periodic resonance solutions of Euler differential equations. As we have mentioned in the previous sections, the Φ-mean-periodic resonance solutions of an Euler differential equation have to satisfy the homogeneous Cauchy boundary value conditions

$$P(\delta)y(t) = F(t), \quad \Phi\{\delta^k y\} = 0, \quad k = 0, 1, 2, \dots, \deg P - 1. \tag{9}$$

Let $P(\lambda) = (\lambda - \lambda_1)^{k_1} \dots (\lambda - \lambda_m)^{k_m} Q(\lambda)$, where $\lambda_1, \dots, \lambda_m$ are the resonance zeros of $P(\lambda)$ with corresponding multiplicities $k_1, \dots, k_m$. Here we assume that they are simple zeros of the indicatrix $E(\lambda)$.

Further we extend an idea of S. Grozdev [6] of reducing the resonance case to the non-resonance one.

Denote by $\widetilde{C}_{\lambda_1,\dots,\lambda_m}$ the subalgebra of $(C(\mathbb{R}_+),*)$ consisting of the functions $F \in C(\mathbb{R}_+)$ satisfying the conditions

$$F * t^{\lambda_k} = 0, \quad k = 1,2,\dots,m$$

or in an equivalent form

$$\Phi_\tau\left\{\tau^{\lambda_k}\int_1^\tau \frac{F(\sigma)d\sigma}{\sigma^{\lambda_k+1}}\right\} = 0, \quad k = 1,2,\dots,m.$$

Obviously these conditions are necessary for existing of a mean-periodic solution. Our next task is to show that they are also sufficient.

We want to find the mean-periodic solutions w of the auxiliary equation

$$(\delta-\lambda_1)^{k_1}\dots(\delta-\lambda_1)^{k_m}w(t) = F(t). \tag{10}$$

Let $\widetilde{L}$ be the restriction of L to $\widetilde{C}_{\lambda_1,\dots,\lambda_m}$ and $\widetilde{S} = \dfrac{1}{\widetilde{L}}$. In the ring $\widetilde{M}_{\lambda_1,\dots,\lambda_m}$ of the convolutional fractions in $\widetilde{C}_{\lambda_1,\dots,\lambda_m}$, equation (10) takes the form

$$(\widetilde{S}-\lambda_1)^{k_1}\dots(\widetilde{S}-\lambda_1)^{k_m}w = F. \tag{11}$$

The elements $(\widetilde{S}-\lambda_k)$, $k = 1,2,\dots,m$, are non-divisors of zero in $\widetilde{M}_{\lambda_1,\dots,\lambda_m}$. Indeed, assume $(\widetilde{S}-\lambda_k)v = 0$ for some $v \in \widetilde{C}_{\lambda_1,\dots,\lambda_m}$. It is equivalent to $v - \lambda_k\widetilde{L}v = 0$ and, hence, $\Phi\{v\} = 0$. Applying δ, we get $\delta v - \lambda_k v = 0$, or $v = Ct^{\lambda_k}$ with a non-zero constant C, which is a contradiction with $v \in \widetilde{C}_{\lambda_1,\dots,\lambda_m}$ since $v * t^{\lambda_k} = -Ct^{\lambda_k}E'(\lambda_k) \neq 0$.

The formal solution of (11) is

$$w = \frac{1}{(\widetilde{S}-\lambda_1)^{k_1}\dots(\widetilde{S}-\lambda_m)^{k_m}}F. \tag{12}$$

We will show that in fact the algebraic multiplier $\dfrac{1}{(\widetilde{S}-\lambda_1)^{k_1}\dots(\widetilde{S}-\lambda_m)^{k_m}}$ is a convolution multiplier $\{G(t)\}*$ with $G \in C(\mathbb{R}_+)$ and we can find it explicitly. To this end it is enough to show this for an arbitrary expression of the form $\dfrac{1}{(\widetilde{S}-\lambda_k)^j}$, which will be represented as a convolution operator $g*$ in $\widetilde{C}_{\lambda_1,\dots,\lambda_m}$ defined by $(g*)f = g*f$. For arbitrary $g \in C(\mathbb{R}_+)$ the convolution operator $g*$ is an inner operator in $\widetilde{C}_{\lambda_1,\dots,\lambda_m}$.

Theorem 4. *Let λ_k be a resonance zero of $P(\lambda)$. Then*

$$\frac{1}{(\widetilde{S}-\lambda_k)^j} = \left\{\frac{t^{\lambda_k}A_j(\ln t)}{E'(\lambda_k)}\right\}*, \tag{13}$$

where $A_j(x)$, $j = 1,2,\dots$ are Appell set of polynomials defined recurrently by

$$A_1(x) = x,\ A_j{}'(x) = A_{j-1}(x),\ \Phi_\tau\{\tau^{\lambda_k}A_j(\ln\tau)\} = 0,\ j > 1.$$

Proof. First, let us consider the case $j = 1$. We are to prove that

$$\frac{1}{\widetilde{S} - \lambda_k}\, F = \left\{ \frac{t^{\lambda_k} \ln t}{E'(\lambda_k)} \right\} * F \tag{14}$$

for arbitrary $F \in \widetilde{C}_{\lambda_1,\ldots,\lambda_m}$. This identity written in the form

$$\left[(\widetilde{S} - \lambda_k) \left\{ \frac{t^{\lambda_k} \ln t}{E'(\lambda_k)} \right\} \right] * F = F$$

can be verified easily using the basic formula

$$\widetilde{S} f = \delta f + \Phi\{f\}$$

from Lemma 3.

The case $j > 1$ can be settled by induction.

Since $\dfrac{1}{(\widetilde{S} - \lambda_k)^j} = \varphi_{k,j}(t)$ is a function satisfying the boundary value condition $\Phi\{\varphi_{k,j}\} = 0$, then by Lemma 3

$$\frac{1}{(\widetilde{S} - \lambda_k)^{j-1}} = (\widetilde{S} - \lambda_k) \frac{1}{(\widetilde{S} - \lambda_k)^j} = (\delta - \lambda_k) \frac{1}{(\widetilde{S} - \lambda_k)^j},$$

and we have

$$t^{\lambda_k} A_{j-1}(\ln t) = (\delta - \lambda_k)\{t^{\lambda_k} A_j(\ln t)\},$$

or

$$\delta A_j(\ln t) = A_{j-1}(\ln t), \quad \Phi_\tau\{\tau^{\lambda_k} A_k(\ln t)\} = 0, k > 1.$$

These relations are equivalent to

$$A_j{}'(x) = A_{j-1}(x),\ A_1(x) = x,\ \widetilde{\Phi}_\xi\{e^{\lambda_k \xi} A_j(\xi)\} = 0,$$

where $\widetilde{\Phi} = \Phi \circ \ln^{-1}$. Hence, $\{A_j(x)\}_{j=1}^{\infty}$ are ordinary Appell polynomials with respect to the functional $\widetilde{\Phi} = \Phi \circ \ln^{-1}$ in $C(\mathbb{R})$ (see Bourbaki [1], Chapter 6.2). □

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