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COMMUTANTS OF THE EULER OPERATOR

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Abstract

Here the Euler operator $\delta = t\dfrac{d}{dt}$ is considered in the space $C = C(\mathbb{R}_+), \mathbb{R}_+ = (0, \infty)$, and the operators $M\colon C \to C$ such that $M\delta = \delta M$ in $C^1(\mathbb{R}_+)$ are characterized. Next, for a non-zero linear functional $\Phi : C(\mathbb{R}_+) \to \mathbb{C}$ the continuous linear operators M with the invariant hyperplane $\Phi\{f\} = 0$ and commuting with δ in it are also characterized. Further, mean-periodic functions for δ with respect to the functional Φ are introduced and it is proved that they form an ideal in a corresponding convolutional algebra $(C(\mathbb{R}_+), *)$. As an application the mean-periodic solutions of Euler differential equations are characterized.

Key words and phrases: commutant, invariant hyperplane, convolutional algebra, multiplier, cyclic element, mean-periodic function

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Introduction. Compared with the case of the differentiation operator $D = \dfrac{d}{dt}$ in a space C of continuous functions, the problem of characterizing the continuous linear operators $M : C \to C$ commuting with the Euler operator $\delta = t\dfrac{d}{dt}$, i.e. such that $M\delta = \delta M$ in C^1, had not been so intensively treated as the corresponding problem for D. Here we can mention only the classical book of Levin [12] (Ch. 8 and 9, Theorem 20, pp. 379–380), where δ is considered not in C, but in spaces of entire functions. In the operational calculus developed by Elizarraraz and Verde-Star [9] in fact some operators commuting with the Euler operator can also be found.

Due to the analogy of the considerations for δ and D, a short survey of the results for the differentiation operator will be made.

Bourbaki [1], Chapter 6, seems to be the first to characterize the linear continuous operators $M : C(\mathbb{R}) \to C(\mathbb{R})$ with $MD = DM$ in $C^1(\mathbb{R})$.

One of the authors (Dimovski [3]) had found the linear continuous operators $M : C(\mathbb{R}) \to C(\mathbb{R})$, such that the subspace $C_\Phi = \{f \in C(\mathbb{R}), \Phi(f) = 0\}$ is an invariant subspace of M and M commutes with D in C^1_Φ.

Let us mention that DELSARTE [2] introduced the space of the mean-periodic functions determined by the functional Φ as the kernel space of M. For details see also SCHWARTZ [13].

Rather natural is the question about the relationship between the two types of commutants. A partial answer for the differentiation operator is given by the following theorem (DIMOVSKI and SKÓRNIK [6,7]): The space of the mean-periodic functions determined by the functional Φ forms an ideal in the convolutional algebra $(C(\mathbb{R}), *)$.

Similar results for the Pommiez operator $\Delta f(z) = [(f(z) - f(0)]/z$ are presented by the authors in [5].

Finally, an interesting historical survey about commutants of the differentiation operator and related operators like the Euler one can be found in the book of KOROBEINIK [11].

General commutant. The main theorem in the general case is:

Theorem 1. A linear continuous operator $M : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ with $M : C^1(\mathbb{R}_+) \to C^1(\mathbb{R}_+)$ commutes with $\delta = t\dfrac{d}{dt}$ in $C^1(\mathbb{R}_+)$ iff it admits a representation of the form

$$(Mf)(t) = \Phi_\tau\{f(t\tau)\} \tag{1}$$

with a continuous linear functional $\Phi : C(\mathbb{R}_+) \to \mathbb{C}$.

The proof uses the one-parameter family $T^\tau, 0 < \tau < \infty$, of the shift operators defined by

$$(T^\tau f)(t) := f(t\tau), \qquad 0 < \tau < \infty. \tag{2}$$

Each of them commutes with $\delta = t\dfrac{d}{dt}$ in $C^1(\mathbb{R}_+)$ and the following lemma shows their importance:

Lemma 1. A linear operator $M : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ with $M : C^1(\mathbb{R}_+) \to C^1(\mathbb{R}_+)$ commutes with $\delta = t\dfrac{d}{dt}$ in $C^1(\mathbb{R}_+)$ iff M commutes with T^τ for all $\tau, 0 < \tau < \infty$.

In the proof of the lemma the following "multiplicative" version of the Taylor formula is needed:

$$(T^\tau f)(t) = f(t\tau) = \sum_{n=0}^{\infty} (\delta^n f)(t) \frac{(\ln \tau)^n}{n!}. \tag{3}$$

It is true for arbitrary polynomial $f(t)$. Then the possibility to approximate any function in $C(0, \infty)$ by polynomials is used.

The abundance of operators commuting with δ in $C(\mathbb{R}_+)$ given by Theorem 1 is in sharp contrast to the set of linear operators commuting with δ in $C(\Delta)$, where Δ is a segment $[a, b] \subset \mathbb{R}_+$:

Theorem 2. Let $[a, b] \subset \mathbb{R}_+$. Then a continuous linear operator $M : C[a, b] \to C[a, b]$, such that $M : C^1[a, b] \to C^1[a, b]$, commutes with the Euler operator δ in $C^1[a, b]$ if and only if it is an operator of the form

$$Mf(t) = cf(t),$$

with a constant c.

The proof goes by transforming δ into the differentiation operator $D = \dfrac{d}{dx}$ and using the corresponding result for D due to KAHANE [10].

A general convolution related to the Euler operator. Basic for the theory of the differentiation operator $\dfrac{d}{dt}$ considered in a space $C(\Delta)$ of continuous functions on an interval Δ is the operation

$$(f * g)(t) = \Phi_\tau \left\{ \int_\tau^t f(t + \tau - \sigma) g(\sigma) d\sigma \right\}, \tag{4}$$

where Φ is a linear functional on $C(\Delta)$. Its properties are studied in details in [4]. The operation (4) is bilinear, commutative, and associative in $C(\Delta)$. It generalizes the classical Duhamel convolution when the functional Φ in (4) is $\Phi(f) = f(0)$.

In order to extend this result to the Euler operator an analogue of operation (4) is needed.

Definition 1. The analytic function

$$E(\lambda) = \Phi_\tau(\tau^\lambda) \tag{5}$$

is said to be the Euler indicatrix of the functional Φ.

It is also convenient to denote for the rest of the paper

$$\varphi_\lambda(t) = \frac{t^\lambda}{E(\lambda)} = \frac{t^\lambda}{\Phi_\tau(\tau^\lambda)}. \tag{6}$$

Here a "multiplicative" variant of (4) is proposed:

Theorem 3. Let Φ be a continuous non-zero linear functional on $C(\mathbb{R}_+)$. Then the operation

$$(f * g)(t) = \Phi_\tau \left\{ \int_\tau^t f\left(\frac{t\tau}{\sigma}\right) g(\sigma) \frac{d\sigma}{\sigma} \right\} \tag{7}$$

is a separately continuous, bilinear, commutative, and associative operation in $C(\mathbb{R}_+)$, such that

$$\Phi(f * g) = 0. \tag{8}$$

The commutant of δ in an invariant hyperplane. In this section another commutant of δ will be described. Here it is supposed that the operators $M : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ preserve $C^1(\mathbb{R}_+)$, i.e. $M : C^1(\mathbb{R}_+) \to C^1(\mathbb{R}_+)$, and additionally they also preserve an invariant hyperplane

$$C_\Phi := \{f \in C(\mathbb{R}_+) : \Phi\{f\} = 0\}, \tag{9}$$

i.e., $M : C_\Phi \to C_\Phi$, where $\Phi : C(\mathbb{R}_+) \to \mathbb{C}$ is an arbitrary non-zero linear functional.

The main result of this section is the explicit representation $Mf = \mu f + m * f$ of any linear continuous operator $M : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ with $M : C_\Phi \to C_\Phi$ and commuting with $\delta = t\dfrac{d}{dt}$ in $C^1_\Phi := C_\Phi \cap C^1(\mathbb{R}_+)$.

With that end in view some auxilliary results will be considered.

Lemma 2. A linear operator $M : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ with $M : C^1(\mathbb{R}_+) \to C^1(\mathbb{R}_+)$ and $M : C_\Phi(\mathbb{R}_+) \to C_\Phi(\mathbb{R}_+)$ commutes with the Euler operator δ in $C^1_\Phi(\mathbb{R}_+)$ iff M commutes with L_λ in $C(\mathbb{R}_+)$, where

$$L_\lambda f(t) = \int_1^t \left(\frac{t}{\tau}\right)^\lambda f(\tau)\frac{d\tau}{\tau} - \frac{t^\lambda}{E(\lambda)}\Phi_\tau\left\{\int_1^\tau \left(\frac{\tau}{\sigma}\right)^\lambda f(\sigma)\frac{d\sigma}{\sigma}\right\} \tag{10}$$

is the right inverse in $C(\mathbb{R}_+)$ of the perturbed Euler operator $\delta_\lambda = \delta - \lambda$ satisfying the boundary condition $\Phi(L_\lambda f) = 0$.

Lemma 3. The operator L_λ, given by (10), is a convolution operator of the form

$$L_\lambda f = \varphi_\lambda * f = \left\{\frac{t^\lambda}{E(\lambda)}\right\} * f. \tag{11}$$

Theorem 4. The commutant of δ in the invariant hyperplane C_Φ coincides with the commutant of any of the operators L_λ in $C(\mathbb{R}_+)$.

Definition 2. A linear operator $M : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ is said to be a *multiplier* of the convolutional algebra $C(\mathbb{R}_+), *)$ when for arbitrary $f, g \in C(\mathbb{R}_+)$ it holds

$$M(f * g) = (Mf) * g.$$

Theorem 5. A linear operator $M : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ with $M : C^1(\mathbb{R}_+) \to C^1(\mathbb{R}_+)$ is a multiplier of the convolution algebra $(C(\mathbb{R}_+), *)$ iff it has a representation of the form

$$Mf(t) = \mu f(t) + (m * f)(t), \tag{12}$$

where $\mu = \text{const}$ and $m \in C(\mathbb{R}_+)$.

Theorem 6. The function $\varphi_\lambda(t) = \dfrac{t^\lambda}{E(\lambda)}$ is a cyclic element of the operator L_λ.

As it is shown in the book [4] (Theorem 1.3.11, p.33) , the commutant of L_λ coincides with the ring of the multipliers of the convolution algebra $(C(\mathbb{R}_+), *)$ since L_λ has a cyclic element. By Theorem 6 such a cyclic element exists and the following characterization holds:

Theorem 7. A linear operator $M : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ with $M : C^1(\mathbb{R}_+) \to C^1(\mathbb{R}_+)$ which has an invariant hyperplane $C_\Phi = \{f \in C(\mathbb{R}_+) : \Phi\{f\} = 0\}$, commutes with δ in C^1_Φ if and only if it has a representation of the form

$$(Mf)(t) = \mu f(t) + (m * f)(t) \tag{13}$$

with a constant $\mu \in \mathbb{C}$ and $m \in C(\mathbb{R}_+)$.

Remark. The constant μ and the function $m \in C(\mathbb{R}_+)$ in (12) are uniquely determined.

Mean-periodic functions for the Euler operator.

Definition 3. A function $f \in C(\mathbb{R}_+)$ is said to be mean-periodic for the Euler operator with respect to the linear functional Φ if

$$\Phi_\tau\{f(t\tau)\} = 0$$

identically in $\mathbb{R}_+$.

It is clear that the mean-periodic functions with respect to Φ form the kernel space of the operator

$$Mf(t) = \Phi_\tau\{f(t\tau)\}$$

commuting with the Euler operator δ in $C^1(\mathbb{R}_+)$.

Now a connection between the mean-periodic functions for δ with respect to Φ and the convolutional algebra $(C(\mathbb{R}_+), *)$ will be given:

Theorem 8. The mean-periodic functions for the Euler operator δ with respect to any non-zero functional $\Phi : C(\mathbb{R}_+) \to \mathbb{C}$ form an ideal in the convolutional algebra $(C(\mathbb{R}_+), *)$.

Application to the Euler differential equation. Now Theorem 8 will be applied to find necessary and sufficient conditions in order an Euler differential equation

$$P(\delta)y(t) = f(t), \qquad 0 < t < \infty, \tag{14}$$

to have a unique mean-periodic solution with respect to a non-zero linear functional Φ in $C(\mathbb{R}_+)$. Here $\delta = t\dfrac{d}{dt}$ is the Euler operator and $P(\mu) = a(\mu-\mu_1)(\mu-\mu_2)\dots(\mu-\mu_k)$ is a polynomial.

Theorem 9. In order the Euler differential equation (14) to have a unique mean-periodic solution with respect to a non-zero linear functional Φ in $C(\mathbb{R}_+)$, it is necessary and sufficient no one of the roots of the equation $P(\lambda) = 0$ to be a root of the Euler indicatrix $E(\lambda) = \Phi_\tau(\tau^\lambda)$.

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