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# On the Minimal Commutativity of a General Operator of Differentiation Type

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The operator $My(z)=\sum_{k=q}^{\infty}\frac{1}{k!}\frac{d^k}{dz^k}y(0)b_kz^{k-q}$, $q\in\mathbb{N}$, with $b_k$ being arbitrary complex numbers, is considered in the space $S$ of the polynomials. It generalizes many well-known operators of differentiation type. The commutant of $M$ is described and then the cases when the operator $M$ is minimally commutative are found.

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*Key Words*: commutant, minimal commutativity

## 1. Introduction

Let $S$ be the space of the polynomials of the complex variable $z\in\mathbb{C}$ and $q>0$ be an arbitrarily fixed integer. We want to consider a generalization of the differentiation operator $\frac{d^q}{dz^q}$ allowing arbitrary coefficient of the different powers of $z$ as follows:

If $b_k\neq 0,\ b_k\in\mathbb{C},\ k=q,q+1,q+2,\ldots$, are arbitrary complex numbers, then the operator

$$My(z)=\sum_{k=q}^{\infty}\frac{1}{k!}\frac{d^k}{dz^k}y(0)b_kz^{k-q},\quad y\in S, \tag{1}$$

is a linear operator in $S$.

In fact, if $y(z)=\sum_{k=0}^{\infty}a_kz^k$, $a_k=\frac{1}{k!}\frac{d^k}{dz^k}y(0)$, with finite number of $a_k$ different from zero, then $My(z)=\sum_{k=q}^{\infty}a_kb_kz^{k-q}$. A simple description of $M$ can be given if only the action of $M$ on the arbitrary powers $z^k$, $k = 0, 1, 2, \ldots$, of $z$ is considered:

$$Mz^k = \begin{cases} b_k z^{k-q}, & k \geq q, \\ 0, & 0 \leq k \leq q-1. \end{cases} \tag{2}$$

In the particular case of the differentiation operator $\dfrac{d^q}{dz^q}$, the numbers $b_k$ are

$$\frac{d^q}{dz^q} z^k = \begin{cases} b_k z^{k-q},\ b_k = k(k-1)\ldots(k-q+1) \neq 0, & k \geq q, \\ 0, & 0 \leq k \leq q-1. \end{cases} \tag{3}$$

Another particular case are the Dunkl operators $D_m$ with parameters $m > 0$, defined in [4] and considered in many papers, in particular by the authors in [9] and [10]:

$$D_m y(z) = \frac{dy(z)}{dz} + m\frac{y(z) - y(-z)}{z} \tag{4}$$

Here $q = 1$ and the action on the powers is

$$D_m z^k = \begin{cases} b_k z^{k-1},\ b_k = k + m[1-(-1)^n], & k \geq 1, \\ 0, & k = 0. \end{cases} \tag{5}$$

**Definition 1.** It is said that a continuous linear operator $L$ *commutes with a fixed operator* $M$, if $LM = ML$. The set of all such operators is called the *commutant of* $M$ and will be denoted by $C_M$.

**Definition 2.** It is said that a continuous linear operator $T$ is *generated by an operator* $M$, if $T$ is a polynomial of $M$, i.e. $T = \sum_{n=0}^{\infty} d_n M^n$, $d_n \in \mathbb{C}$.. The set of all operators generated by $M$ will be denoted by $G_M$.

It is clear that every operator $T$, which is generated by $M$, i.e. $T \in G_M$, also commutes with $M$, i.e. $T \in C_M$, and hence $G_M \subset C_M$. The opposite inclusion $G_M \supset C_M$ is, in general, not true. Therefore the following definition is natural:

**Definition 3.** (Raichinov [1]) An operator $M$ is called *minimally commutative* if $G_M \supset C_M$, i.e. if the commutant $C_M$ consists only of operators $T$ generated by $M$ and hence if $C_M = G_M$.

In this paper only the case when the general operator $M$ *decreases* the powers is considered. First a description of the commutant $C_M$ of the operator (1) in the space $S$ of the polynomials is given. Next the question about the minimal commutativity of $M$ in the sense of Rajchinov [1] is considered.

In fact, the commutant of the operator $M$ is considered in [7], but in suitable subspaces of the space $S$ of the polynomials. Here we make a modification considering the whole $S$ with initial zero values of the arbitrary constants $b_k$. The description of the commutant $C_M$ is given in this paper for sake of convenience.

Similar operators, but in the case when they *increase* or *preserve* the powers, are considered in many papers (see e.g. [1], [2], [3], [5], [8], etc.). The most general variant with arbitrary coefficients in this case, i.e. for integration type operators, appears in M. Hristova [6].

## 2. Representation of the commutant

Here we describe the commutant $C_M$ of the general operator $M$, defined by (1), in the case $q > 0$, i.e. when the powers are decreased:

**Theorem 1.** *A continuous linear operator $L : S \to S$ commutes with the operator $M$ defined by (1) with parameter $q > 0$*

$$(6)\ My(z) = \sum_{k=q}^{\infty} \frac{1}{k!}\frac{d^k}{dz^k} y(0) b_k z^{k-q},\ y \in S, \quad \text{or } Mz^k = \begin{cases} b_k z^{k-q}, & k \geq q \\ 0, & 0 \leq k \leq q-1, \end{cases}$$

*if and only if it has the form*

$$\begin{aligned}(7)\ Ly(z) \ &= \sum_{m=0}^{q-1}\sum_{k=0}^{\infty} a_k c_{k,m} z^m \\ &+ \sum_{m=q}^{\infty}\sum_{k=\left[\frac{m}{q}\right]q}^{\infty} a_k \frac{b_k b_{k-q}\dots b_{k-\left(\left[\frac{m}{q}\right]-1\right)q}}{b_m b_{m-q}\dots b_{m-\left(\left[\frac{m}{q}\right]-1\right)q}}\, c_{k-\left[\frac{m}{q}\right]q,\, m-\left[\frac{m}{q}\right]q} \cdot z^m,\end{aligned}$$

*where* $a_k = \dfrac{1}{k!}\dfrac{d^k}{dz^k}y(0)$ *and $c_{k,m}$ are arbitrary complex numbers with indices $k = 0, 1, 2, \dots$ and $0 \leq m \leq q-1$.*

P r o o f. Let the action of an arbitrary operator $L$ of the commutant $C_M$ on an arbitrarily fixed power $z^k$ be

$$(8) \qquad Lz^k = \sum_{m=0}^{\infty} c_{k,m} z^m$$

with unknown complex coefficients $c_{k,m}$, such that only finite number of them are different from zero.

Now let us express $LMz^k$ and $MLz^k$:

$$LMz^k = \begin{cases} Lb_k z^{k-q} = b_k \sum_{m=0}^{\infty} c_{k-q,m} z^m = \sum_{m=0}^{\infty} b_k c_{k-q,m} z^m, & k \geq q \\ 0, & 0 \leq k \leq q-1. \end{cases} \tag{9}$$

$$MLz^k = M \sum_{m=0}^{\infty} c_{k,m} z^m = \sum_{m=q}^{\infty} c_{k,m} b_m z^{m-q} = \sum_{m=0}^{\infty} c_{k,m+q} b_{m+q} z^m. \tag{10}$$

First, equating the coefficients of the powers $z^m$ in the case $0 \leq m \leq q-1$, one has $c_{k,m+q} b_{m+q} = 0$. Since $b_{m+q} \neq 0$, if $m+q$ is replaced by $m$, it follows that

$$c_{k,m} = 0 \qquad \text{for } 0 \leq k \leq q-1,\ m \geq q. \tag{11}$$

Next, in the other case $m \geq q$ it follows that $b_k c_{k-q,m} = c_{k,m+q} b_{m+q}$ and replacing again $m+q$ by $m$, the following important recurrent relation holds:

$$c_{k,m} = \frac{b_k}{b_m}\, c_{k-q,m-q} \qquad \text{for } k \geq q,\ m \geq q. \tag{12}$$

Now this recurrent relation (12) will be used to express arbitrary coefficient $c_{k,m}$, $k \geq q$, $m \geq q$, by some coefficient $c_{\varkappa,\mu}$, where either $0 \leq \varkappa \leq q-1$ or $0 \leq \mu \leq q-1$.

In the sequel, $[A]$ will denote the integer part of a number $A$. In particular, $\left[\frac{k}{q}\right]$ is the quotient and $k - \left[\frac{k}{q}\right] q$ is the remainder when $k$ is divided by $q$.

In the case $\left[\frac{k}{q}\right] < \left[\frac{m}{q}\right]$, one can apply $\left[\frac{k}{q}\right]$ times the recurrent formula (12) and then

$$\begin{aligned} c_{k,m} &= \frac{b_k}{b_m}\, c_{k-q,m-q} = \frac{b_k b_{k-q}}{b_m b_{m-q}}\, c_{k-2q,m-2q} \\ &= \ldots = \frac{b_k b_{k-q} \ldots b_{k-\left(\left[\frac{k}{q}\right]-1\right)q}}{b_m b_{m-q} \ldots b_{m-\left(\left[\frac{k}{q}\right]-1\right)q}}\, c_{k-\left[\frac{k}{q}\right]q,\, m-\left[\frac{k}{q}\right]q}. \end{aligned} \tag{13}$$

In this case $0 \leq k - \left[\frac{k}{q}\right] q \leq q-1$, i.e. the first index is the remainder when $k$ is divided by $q$. Then by (11) the coefficient $c_{k-\left[\frac{k}{q}\right]q,\, m-\left[\frac{k}{q}\right]q}$ must be zero. Therefore, using (13), one has

$$c_{k,m} = 0, \quad \text{for } \left[\frac{k}{q}\right] < \left[\frac{m}{q}\right]. \tag{14}$$

In the other case, when $\left[\dfrac{k}{q}\right] \ge \left[\dfrac{m}{q}\right]$, one can apply $\left[\dfrac{m}{q}\right]$ times the recurrent formula (12) to get

$$\begin{aligned} c_{k,m} &= \frac{b_k}{b_m}\, c_{k-q,m-q} = \frac{b_k b_{k-q}}{b_m b_{m-q}}\, c_{k-2q,m-2q} \\ &= \ldots = \frac{b_k b_{k-q} \ldots b_{k-\left(\left[\frac{m}{q}\right]-1\right)q}}{b_m b_{m-q} \ldots b_{m-\left(\left[\frac{m}{q}\right]-1\right)q}}\, c_{k-\left[\frac{m}{q}\right]q,\, m-\left[\frac{m}{q}\right]q}. \end{aligned} \tag{15}$$

This time the second index $m-\left[\dfrac{m}{q}\right] q$ is the remainder when $m$ is divided by $q$.

Let us combine (14) and (15) as

$$c_{k,m} = \begin{cases} 0 & \text{for } \left[\frac{k}{q}\right] < \left[\frac{m}{q}\right], \\ \dfrac{b_k b_{k-q} \ldots b_{k-\left(\left[\frac{m}{q}\right]-1\right)q}}{b_m b_{m-q} \ldots b_{m-\left(\left[\frac{m}{q}\right]-1\right)q}}\, c_{k-\left[\frac{m}{q}\right]q,\, m-\left[\frac{m}{q}\right]q} & \text{for } \left[\frac{k}{q}\right] \ge \left[\frac{m}{q}\right]. \end{cases} \tag{16}$$

This important formula shows that:

*All coefficients* $c_{k,m}$ *with* $0 \le m \le q-1$ *can be chosen arbitrarily,* and then *all other coefficients* $c_{k,m}$ *with* $m \ge q$ *are either equal to zero or can be expressed by some of the arbitrarily chosen* $c_{\varkappa,\mu}$ *with* $\varkappa = k - \left[\dfrac{m}{q}\right] q \ge 0,\ 0 \le \mu = m - \left[\dfrac{m}{q}\right] q \le q-1.$

The recurrent relation (16) allows a representation of $Mz^k$ as a polynomial of degree at most $\left(\left[\dfrac{k}{q}\right]+1\right) q - 1$:

$$\begin{aligned} Lz^k &= \sum_{m=0}^{q-1} c_{k,m} z^m \\ &+ \sum_{m=q}^{\left(\left[\frac{k}{q}\right]+1\right)q-1} \frac{b_k b_{k-q} \ldots b_{k-\left(\left[\frac{m}{q}\right]-1\right)q}}{b_m b_{m-q} \ldots b_{m-\left(\left[\frac{m}{q}\right]-1\right)q}}\, c_{k-\left[\frac{m}{q}\right]q,\, m-\left[\frac{m}{q}\right]q} \cdot z^m. \end{aligned} \tag{17}$$

Finally, the action of an arbitrary operator $L \in C_M$ on any polynomial $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is

$$(18)\qquad Lf(z) = L\sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{\infty} a_k L z^k$$

$$= \sum_{k=0}^{\infty} a_k \left( \sum_{m=0}^{q-1} c_{k,m} z^m + \sum_{m=q}^{\left(\left[\frac{k}{q}\right]+1\right)q-1} \frac{b_k b_{k-q} \dots b_{k-\left(\left[\frac{m}{q}\right]-1\right)q}}{b_m b_{m-q} \dots b_{m-\left(\left[\frac{m}{q}\right]-1\right)q}} \, c_{k-\left[\frac{m}{q}\right]q, m-\left[\frac{m}{q}\right]q} \cdot z^m \right).$$

interchanging the summations, this can be written also as

$$(19)\qquad Lf(z) = \sum_{m=0}^{q-1} \sum_{k=0}^{\infty} a_k c_{k,m} z^m$$

$$+ \sum_{m=q}^{\infty} \sum_{k=\left[\frac{m}{q}\right]q}^{\infty} a_k \frac{b_k b_{k-q} \dots b_{k-\left(\left[\frac{m}{q}\right]-1\right)q}}{b_m b_{m-q} \dots b_{m-\left(\left[\frac{m}{q}\right]-1\right)q}} \, c_{k-\left[\frac{m}{q}\right]q, m-\left[\frac{m}{q}\right]q} \cdot z^m.$$

This is in fact the desired representation (7) of the commutant $C_M$ of the operator $M$.

Thus, we proved the *necessity*, i.e. if $L \in C_M$, then the operator $L$ must be of the form (7).

Now, let us check the *sufficiency*, i.e. if an operator $L$ has the form (7), then it commutes with the operator $M$, i.e. $LM = ML$. It is enough to verify this for all powers $z^k$, $k = 0, 1, 2, \dots$. In fact, for arbitrarily fixed $k$ we can use the representation (17) instead of the general expression (7).

In the case $0 \le k \le q-1$ the representation (17) reduces to the first sum $Lz^k = \sum_{m=0}^{q-1} c_{k,m} z^m$ and calculating $MLz^k$ and $LMz^k$

$$MLz^k = M\sum_{m=0}^{q-1} c_{k,m} z^m = \sum_{m=0}^{q-1} c_{k,m} M z^m = \sum_{m=0}^{q-1} c_{k,m}.0 = 0,$$

$$LMz^k = L0 = 0,$$

one has $MLz^k = LMz^k = 0$. Here we used the second case in (2).

In the case $k \ge q$, if we apply $M$ to (2), then the first sum in (17) will vanish since (2) gives $Mz^m = 0$ for $0 \le m \le q-1$.

Next, (2) and (17) used for $k \ge q$ give

$$(20)\quad MLz^k = \sum_{m=q}^{\left(\left[\frac{k}{q}\right]+1\right)q-1} \frac{b_k b_{k-q} \dots b_{k-\left[\frac{m}{q}\right]q}}{b_m b_{m-q} \dots b_{m-\left[\frac{m}{q}\right]q}} \, c_{k-\left[\frac{m}{q}\right]q, m-\left[\frac{m}{q}\right]q} \cdot b_m z^{m-q}.$$

It is suitable to separate the sum as $\sum_{m=q}^{2q-1} + \sum_{\mu=2q}^{\left(\left[\frac{k}{q}\right]+1\right)q-1}$, i.e.

$$
\begin{aligned}
(21)\; MLz^k \;&=\; \sum_{m=q}^{2q-1} \frac{b_k}{b_m}\, c_{k-q,m-q} \;.\; b_m z^{m-q} \\
&+\; \sum_{m=2q}^{\left(\left[\frac{k}{q}\right]+1\right)q-1} \frac{b_k b_{k-q}\dots b_{k-\left[\frac{m}{q}\right]q}}{b_m b_{m-q}\dots b_{m-\left[\frac{m}{q}\right]q}}\, c_{k-\left[\frac{m}{q}\right]q,m-\left[\frac{m}{q}\right]q} \;.\; b_m z^{m-q}.
\end{aligned}
$$

In both sums $b_m$ can be canceled:

$$
MLz^k = \sum_{m=q}^{2q-1} b_k c_{k-q,m-q} z^{m-q} + \sum_{m=2q}^{\left(\left[\frac{k}{q}\right]+1\right)q-1} \frac{b_k b_{k-q}\dots b_{k-\left[\frac{m}{q}\right]q}}{b_{m-q}\dots b_{m-\left[\frac{m}{q}\right]q}}\, c_{k-\left[\frac{m}{q}\right]q,m-\left[\frac{m}{q}\right]q} z^{m-q}.
$$

(22)

It is also suitable to replace $m-q$ by only one letter $m$:

$$
MLz^k = \sum_{m=0}^{q-1} b_k c_{k-q,m} z^m + \sum_{m=q}^{\left[\frac{k}{q}\right]q-1} \frac{b_k b_{k-q}\dots b_{k-\left(\left[\frac{m}{q}\right]+1\right)q}}{b_m\dots b_{m-\left[\frac{m}{q}\right]q}}\, c_{k-\left(\left[\frac{m}{q}\right]+1\right)q,m-\left[\frac{m}{q}\right]q} z^m.
$$

(23)

Next, let us represent $LMz^k$ in the same case $k \geq q$:

$$
(24)\quad LMz^k = Lb_k z^{k-q} = b_k L z^{k-q} =
$$

$$
b_k \left( \sum_{m=0}^{q-1} c_{k-q,m} z^m + \sum_{m=q}^{\left(\left[\frac{k-q}{q}\right]+1\right)q-1} \frac{b_{k-q} b_{k-2q}\dots b_{k-q-\left[\frac{m}{q}\right]q}}{b_m b_{m-q}\dots b_{m-\left[\frac{m}{q}\right]q}}\, c_{k-q-\left[\frac{m}{q}\right]q,m-\left[\frac{m}{q}\right]q} \;.\; z^m \right).
$$

In fact, the right hand sides of (23) and (24) are one and the same. Thus $MLz^k = LMz^k$ for arbitrary power $z^k$ and the sufficiency of (7) is also proved. ∎

## 3. Minimal commutativity

In order to investigate the operator $M$, defined by (1), about minimal commutativity in the sense of Rajchinov [1] (Definition 3), it is needed to describe the operators $T$ generated by $M$:

**Lemma 1.** *If* $T = \sum_{n=0}^{\infty} d_n M^n \in G_M$ *is an operator, generated by the operator* $M$ *(defined by* (1) *or* (2)*), then its action on the powers* $z^k$ *has the form*

$$Tz^k = d_0 z^k + \sum_{n=1}^{\left[\frac{k}{q}\right]} d_n b_k b_{k-q} \dots b_{k-(n-1)q} z^{k-nq}. \tag{25}$$

P r o o f. $M^n z^k$ has the following expression:

$$M^n z^k = \begin{cases} z^k, & n = 0 \\ b_k b_{k-q} \dots b_{k-(n-1)q} z^{k-nq}, & 1 \leq n \leq \left[\frac{k}{q}\right] \\ 0, & n > \left[\frac{k}{q}\right]. \end{cases}$$

Therefore,

$$\begin{aligned} Tz^k &= \sum_{n=0}^{\infty} d_n M^n z^k \\ &= d_0 z^k + d_1 b_k z^{k-q} + d_2 b_k b_{k-q} z^{k-2q} + \dots \\ &\quad + d_{\left[\frac{k}{q}\right]} b_k b_{k-q} \dots b_{k-\left(\left[\frac{k}{q}\right]-1\right)q} z^{k-\left[\frac{k}{q}\right]q} \\ &= d_0 z^k + \sum_{n=1}^{\left[\frac{k}{q}\right]} d_n b_k b_{k-q} \dots b_{k-(n-1)q} z^{k-nq}. \end{aligned} \tag{26}$$

■

One can immediately describe the action on any polynomial $y \in S$:

**Theorem 2.** *If* $T = \sum_{n=0}^{\infty} d_n M^n$ *is an operator, generated by the operator* $M$ *and* $y \in S$ *is a polynomial, then*

$$\begin{aligned} Ty(z) &= \sum_{m=0}^{\infty} \left[ \frac{1}{m!} \frac{d^m}{dz^m} y(0) d_0 \right. \\ &\quad \left. + \sum_{s=1}^{\infty} \left( \frac{1}{(m+sq)!} \frac{d^{m+sq}}{dz^{m+sq}} y(0) \prod_{t=1}^{s} b_{m+tq} \right) d_s \right] z^m. \end{aligned} \tag{27}$$

P r o o f. From the lemma

$$
\begin{aligned}
Ty(z) &= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k}{dz^k} y(0) T z^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k}{dz^k} y(0) \left( d_0 z^k + \sum_{n=1}^{\left[\frac{k}{q}\right]} d_n b_k b_{k-q} \dots b_{k-(n-1)q} z^{k-nq} \right). \qquad (28)
\end{aligned}
$$

Let $a_k = \frac{1}{k!} \frac{d^k}{dz^k} y(0)$, $k = 0, 1, 2, \dots$, be a short notation of the coefficients of the polynomial $y$. One can gather the equal powers of $z$ at one place as follows:

$$
Ty(z) = \sum_{m=0}^{\infty} \left[ a_m d_0 + \sum_{s=1}^{\infty} \left( a_{m+sq} \prod_{t=1}^{s} b_{m+tq} \right) d_s \right] z^m. \qquad (29)
$$

This is in fact the desired representation (27). ■

**Theorem 3.** *The general operator $M$, defined by* (1), *is minimally commutative if and only if $q = 1$.*

P r o o f. Instead of considering the action on an arbitrary polynomial, given in Theorem 2, it is enough to fix arbitrarily an integer $k$ and to compare the representation (25) of $Tz^k$ from Lemma 1 with (17), which is an expression of $Lz^k$ from the description of the commutant:

$$
\begin{aligned}
Lz^k &= \sum_{m=0}^{q-1} c_{k,m} z^m \qquad (30)\\
&+ \sum_{m=q}^{\left(\left[\frac{k}{q}\right]+1\right)q-1} \frac{b_k b_{k-q} \dots b_{k-\left(\left[\frac{m}{q}\right]-1\right)q}}{b_m b_{m-q} \dots b_{m-\left(\left[\frac{m}{q}\right]-1\right)q}} \, c_{k-\left[\frac{m}{q}\right]q, m-\left[\frac{m}{q}\right]q} \cdot z^m.
\end{aligned}
$$

If $q \geq 2$, note that only one power of $z$ between 0 and $q-1$ in (26) could be different from zero, namely $z^{k-\left[\frac{k}{q}\right]q}$, while all coefficients $c_{k,m}$ in the first sum $\sum_{m=0}^{q-1} c_{k,m} z^m$ of (30) could be chosen arbitrarily, in particular different from zero. This shows that:

*If $q \geq 2$, then the general operator $M$, defined by* (1), *is not minimally commutative.*

Hence $M$ could be minimally commutative only the case $q = 1$. In this case we want to check whether $Lz^k$ has the form of $Tz^k$ for arbitrary $k$. One has

$$Lz^k = c_{k,0}z^0 + \sum_{m=1}^{k} \frac{b_k b_{k-1}\dots b_{k-m+1}}{b_m b_{m-1}\dots b_1}\, c_{k-m,0}\,.\, z^m, \tag{31}$$

$$\begin{aligned} Tz^k &= \sum_{n=0}^{\infty} d_n M^n z^k \\ &= d_0 z^k + d_1 b_k z^{k-1} + d_2 b_k b_{k-1} z^{k-2} + \dots + d_k b_k b_{k-1}\dots b_1 z^0. \end{aligned} \tag{32}$$

Let us now equate the coefficients of the equal powers in (31) and (32):

$$\begin{aligned} c_{k,0} &= d_k b_k b_{k-1}\dots b_1 \\ \frac{b_k}{b_1}\, c_{k-1,0} &= d_{k-1} b_k b_{k-1}\dots b_2 \\ \frac{b_k b_{k-1}}{b_1 b_2}\, c_{k-2,0} &= d_{k-2} b_k b_{k-1}\dots b_3 \\ \dots &= \dots \\ \frac{b_k b_{k-1}\dots b_{k-m+1}}{b_1 b_2\dots b_m}\, c_{k-m,0} &= d_{k-m} b_k b_{k-1}\dots b_{k-m+1} \\ \dots &= \dots \\ \frac{b_k b_{k-1}\dots b_2}{b_1 b_2\dots b_{k-1}}\, c_{1,0} &= d_1 b_k \\ \frac{b_k b_{k-1}\dots b_1}{b_1 b_2\dots b_k}\, c_{0,0} &= d_0. \end{aligned} \tag{33}$$

In fact, replacing $k-m$ by $n$ and solving with respect to $c_{n,0}$ and $d_n$, $n = 0, 1, \dots, k$, we can write the following formulae which relate the arbitrarily chosen coefficient $c_{n,0}$ in the description of the commutant $C_M$ and the coefficients $d_n$ in the description of the generated by $M$ operators in $G_M$:

$$c_{n,0} = d_n\, b_1 b_2 \dots b_n, \qquad d_n = c_{n,0}\, \frac{1}{b_1 b_2 \dots b_n}, \qquad n = 0, 1, 2, \dots \tag{34}$$

These formulae do not depend on the arbitrarily chosen $k$ and thus:

*In the case $q = 1$, any operator $L$ from the commutant $C_M$ can be treated as an operator generated by $M$, i.e. $L \in G_M$, $C_M \subset G_M$ and the operator $M$ is minimally commutative.* ∎

## 4. Final remarks

This publication completes in some sense the investigation of the minimal commutativity made in [6]. There the general operator increases or preserves the powers, while here it decreases the powers. Combining the results from both papers a general conclusion can be made:

*A general operator which changes the powers by a fixed number $p$, $p = 0, \pm 1, \pm 2, \ldots$, is minimally commutative only if $p = 1$, $p = 0$, or $p = -1$.*

Another remark is that we made the description of the commutant only in the space $S$ of the polynomials and used infinite upper limit of the sums. Thus it was not needed to specify the degrees of the polynomials. But in fact, it is possible to make the same descriptions also in the space $A$ of the functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ analytic in a neighbourhood of the origin. One can give at least sufficient conditions which ensure the convergence of the power series in the descriptions. For instance, a possibility is to suppose that

$$\limsup_{k\to\infty} \sqrt[k]{|b_k|} < \infty,$$

and then

$$\limsup_{k\to\infty} \sqrt[k]{|a_k b_k|} \le \limsup_{k\to\infty} \sqrt[k]{|a_k|}.\limsup_{k\to\infty} \sqrt[k]{|b_k|} < \infty$$

guarantees the convergence in the space $A$.

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