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On the approximation by convolution operators in homogeneous Banach spaces on $\mathbb{R}^d$

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The paper presents a description of the optimal rate of approximation as well as of a broad class of functions that possess it for convolution operators acting in the so-called homogeneous Banach spaces of functions on $\mathbb{R}^d$. The description is the same in any such space and uses the Fourier transform. Simple criteria for establishing upper estimates of the approximation error via a K-functional are given. The differential operator in the K-functional is defined similarly to the infinitesimal generator by means of the convolution operator.

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1. Introduction

We discuss a simple method for establishing upper estimates of the approximation error of a class of convolution operators by means of appropriately defined K-functionals. Moreover, we consider convolutions on the quite broad class of Banach spaces called *homogeneous Banach spaces (HBS) on* $\mathbb{R}^d$. They include the Lebesgue spaces $L_p(\mathbb{R}^d)$ for $1 \leq p < \infty$, the space of uniformly continuous and bounded functions on $\mathbb{R}^d$, subspaces of the Lipschitz (Hölder) spaces, some Besov spaces, the analogues of all these spaces for periodic functions, and others. Also, we establish that, under certain quite general and

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natural hypotheses, the optimal rate of approximation of such an operator as well as a wide set of functions that possess it can be described in the same way in any HBS by means of the Fourier transform for tempered distributions.

We shall formulate simpler criteria for establishing upper error estimates than those given in [9], discuss the interconnection between the hypotheses of those criteria and show that the hypotheses are actually quite natural.

Here we shall treat only HBS's on $\mathbb{R}^d$. The HBS's on $\mathbb{T}^d$ – the d-dimensional torus, can be considered as their special case or simultaneously (see [12, Definition I.2.10] for the definition of a HBS on $\mathbb{T}^d$). However, with regard to the matter we shall present here it is more appropriate and easy to consider them separately. It should be mentioned that all the results concerning $L(\mathbb{R}^d)$, can be directly extended to the HBS's on $\mathbb{T}^d$, as these spaces are continuously embedded in $L(\mathbb{T}^d)$ by definition. In [10] we considered the univariate case of HBS's on $\mathbb{T}$.

In the next section we recall the definition of the HBS's on $\mathbb{R}^d$ and some of their basic properties we shall need. There we also state the criterion given in [9] for establishing direct estimates of the error of convolution operators acting in HBS's on $\mathbb{R}^d$. The estimate is in terms of K-functionals. The main purpose of the paper is to show that if the differential operator of the K-functional is defined similarly to the infinitesimal generator of a semi-group of operators, then many of the hypotheses of the above mentioned criterion are satisfied. In Section 3 we discuss the relation between the optimal rate of convergence of the convolution operator and the measure through which it is defined. The optimal rate of convergence is to be used in the definition of the differential operator of the K-functional. Further, in Section 4 we investigate the form of the differential operator and show that it can be equivalently given by means of the Fourier transform. This extends classical results on saturation of convolution operators in $L_p(\mathbb{R})$. We strengthen the above mentioned direct criterion in Section 5. Finally, we illustrate the main results by means of two multivariate forms of the generalized Picard singular integral.

2. Basic notions

We shall consider a rather wide class of Banach spaces of real or complex-valued functions of generally several real variables. As we mentioned, it includes the Lebesgue spaces $L_p(\mathbb{R}^d)$, $1 \leq p < \infty$, $d \in \mathbb{N}$, the space of uniformly continuous and bounded functions on $\mathbb{R}^d$, the Lipschitz (Hölder) and the Besov spaces on $\mathbb{R}^d$.

First, let us introduce a number of basic notations. We denote the elements of $\mathbb{R}^d$ by $x = (x_1, \dots, x_d)$, the multiplication of a vector $x \in \mathbb{R}^d$ with a scalar $\rho \in \mathbb{R}$ by $\rho x = (\rho x_1, \ldots, \rho x_d)$ and the dot product of $x, y \in \mathbb{R}^d$ by $x \cdot y = x_1 y_1 + \cdots + x_d y_d$. Let $|x| = \sqrt{x_1^2 + \cdots + x_d^2}$. We denote the Banach space of all functions summable in the Lebesgue sense on the measurable set $D \subseteq \mathbb{R}^d$ by $L(D)$ with the norm

$$\|f\|_{L(D)} = \int_D |f(x)|\, dx.$$

Also, as usual, $L_\infty(D)$ denotes the space of the essentially bounded measurable functions on D, equipped with the sup-norm, $C^n(D)$ the space of the functions with continuous partial derivatives up to order n, $C^\infty(D)$ the space of infinitely differentiable functions and $CB(D)$ the space of all bounded functions on D. If a norm is taken on the whole space $\mathbb{R}^d$, we shall skip the function domain in the subscript of the norm.

Definition 2.1 (Shapiro [18, Definition 9.3.1.1]) A *homogeneous Banach space* (abbreviated *HBS*) B *on* $\mathbb{R}^d$ is a Banach space of Lebesgue measurable functions on $\mathbb{R}^d$ with norm $\| \circ \|_B$, satisfying the conditions:

(a) The translation is an isometry of B onto itself, i.e. for $f_t(x) = f(x - t)$, where $f \in B$ and $t \in \mathbb{R}^d$, there hold $f_t \in B$ and $\|f_t\|_B = \|f\|_B$;

(b) The translation is continuous on B, i.e. for all $f \in B$ and $t, t_0 \in \mathbb{R}^d$ there holds $\lim_{t \to t_0} \|f_t - f_{t_0}\|_B = 0$;

(c) The functions of B are uniformly locally integrable, as there exists a constant c_B such that for all $f \in B$ there holds

$$\|f\|_{L([0,1]^d)} \le c_B \|f\|_B.$$

Two functions in B are considered equivalent if they coincide almost everywhere in the Lebesgue sense.

The concept of HBS's is due to Shilov [19], but also earlier Bochner and Neumann [2, Definition 1] followed a similar abstract approach to define the almost periodic functions (see also the references cited there and [18, p. 200]).

Let B be a HBS on $\mathbb{R}^d$ and $M(\mathbb{R}^d)$ denote the space of all finite Borel measures μ on $\mathbb{R}^d$ with the norm

$$\|\mu\|_M = \int_{\mathbb{R}^d} |d\mu|.$$

The convolution of a function $f \in B$ and a measure $\mu \in M(\mathbb{R}^d)$ is defined by

$$f * d\mu(x) = \int_{\mathbb{R}^d} f(x - y)\, d\mu(y),$$

as the integral is the Lebesgue-Stieltjes one, or, *equivalently*, Bochner's generalization of the Lebesgue-Stieltjes integral of vector-valued functions (cf. [18, Lemma 9.3.2.2].

As is known (cf. [18, Theorem 9.3.2.3]), $f * d\mu(x)$ exists almost everywhere, belongs to B and

$$\|f * d\mu\|_B \leq \|\mu\|_M \, \|f\|_B. \tag{1}$$

In particular, for an absolutely continuous measure $d\mu(y) = k(y)\,dy$ with $k \in L(\mathbb{R}^d)$ we have

$$k * f(x) = \int_{\mathbb{R}^d} k(y) f(x-y)\,dy$$

and

$$\|k * f\|_B \leq \|k\|_L \, \|f\|_B. \tag{2}$$

Also, for $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$ we shall denote by $\mu(a+x)$ and $\mu(bx)$ respectively the measures $\mu(a+E)$ and $\mu(bE)$, where E is a Borel set on $\mathbb{R}^d$. In particular, we set $\tilde{\mu}(x) = \mu(-x)$.

Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ denote the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$ and $\mathcal{S}'$ be its dual – the space of tempered distributions. Conditions (a) and (c) of Definition 2.1 imply that any HBS on $\mathbb{R}^d$ is continuously embedded into $\mathcal{S}'$ (cf. [18, p. 207]). Indeed, let B be a HBS on $\mathbb{R}^d$ and $f \in B$. Then

$$T_f(\eta) = \int_{\mathbb{R}^d} f(x)\eta(x)\,dx, \quad \eta \in \mathcal{S}, \tag{3}$$

defines a continuous linear functional on $\mathcal{S}$. To show this, one can observe that

$$\left| \int_{\mathbb{R}^d} f(x)\eta(x)\,dx \right| \leq \int_{\mathbb{R}^d} \frac{|f(x)|}{(1+|x|^2)^n}\,dx \, \sup_{x \in \mathbb{R}^d} |(1+|x|^2)^n \eta(x)|.$$

It is easy to show that the first term on the right above is finite for $n \geq n_d = [d/2]+1$. Indeed, on $\mathbb{R}^d_+ = [0,\infty)^d$ we have for any $N \in \mathbb{N}$

$$\begin{aligned}
&\int_0^N \cdots \int_0^N \frac{|f(x)|}{(1+x_1^2+\cdots+x_d^2)^n}\,dx_1 \ldots dx_d \\
&\qquad = \sum_{\substack{s_j=0 \\ j=1,\ldots,d}}^{N-1} \int_{s_1}^{s_1+1} \cdots \int_{s_d}^{s_d+1} \frac{|f(x)|}{(1+x_1^2+\cdots+x_d^2)^n}\,dx_1 \ldots dx_d \\
&\qquad \leq \sum_{\substack{s_j=0 \\ j=1,\ldots,d}}^{N-1} \frac{1}{(1+s_1^2+\cdots+s_d^2)^n} \int_{s_1}^{s_1+1} \cdots \int_{s_d}^{s_d+1} |f(x)|\,dx_1 \ldots dx_d.
\end{aligned}$$

Due to (a) and (c) of Definition 2.1 each integral is bounded by $c_B\|f\|_B$, whereas

$$\sum_{\substack{s_j=0\\ j=1,\dots,d}}^{N-1} \frac{1}{(1+s_1^2+\cdots+s_d^2)^n} < c_d \quad \forall N$$

with some positive constant c_d because $2n > d$.

Thus we have established that there exists a positive constant c_B' such that

$$|T_f(\eta)| \le c_B'\|f\|_B \sup_{x\in\mathbb{R}^d} |(1+|x|^2)^{n_d}\eta(x)|, \quad \eta\in\mathcal{S}. \tag{4}$$

Consequently, $T_f \in \mathcal{S}'$. Henceforth, we shall write simply $f(\eta)$ instead of $T_f(\eta)$ for $\eta\in\mathcal{S}$. Actually, $f(\eta)$ is well-defined by (3) for any locally summable η such that $(1+|x|^2)^{n_d}\eta(x)$ is bounded.

Given a function $\psi\in C^\infty(\mathbb{R}^d)$, which is polynomially bounded, and $T\in\mathcal{S}'$, the tempered distribution ψT is defined by

$$\psi T(\eta) = T(\psi\eta), \quad \eta\in\mathcal{S}.$$

In particular, if $f\in B$, then

$$\psi f(\eta) = \int_{\mathbb{R}^d} f(x)\psi(x)\eta(x)\,dx, \quad \eta\in\mathcal{S},$$

moreover, this relation is meaningful for any locally summable and polynomially bounded function ψ, not necessarily belonging to $C^\infty(\mathbb{R}^d)$.

We denote the Fourier-Stieltjes transform of a measure $\mu\in M(\mathbb{R}^d)$ by

$$\widehat{d\mu}(u) = \int_{\mathbb{R}^d} e^{-i\,u\cdot x}\,d\mu(x), \quad u\in\mathbb{R}^d.$$

In particular, the Fourier transform $\hat f$ of a function $f\in L(\mathbb{R}^d)$ is given by

$$\hat f(u) = \int_{\mathbb{R}^d} f(x)\,e^{-i\,u\cdot x}\,dx, \quad u\in\mathbb{R}^d.$$

Next, to recall, the Fourier transform $\widehat{T}$ of a tempered distribution T is the tempered distribution defined by

$$\widehat{T}(\eta) = T(\hat\eta), \quad \eta\in\mathcal{S}.$$

In particular, we have for $f\in B$

$$\hat f(\eta) = \int_{\mathbb{R}^d} f(x)\hat\eta(x)\,dx, \quad \eta\in\mathcal{S}.$$

Now, observe that given $f \in B$ and $\mu \in M(\mathbb{R}^d)$, the convolution $f * d\mu$ is in B and hence it is a tempered distribution. For its Fourier transform in the sense of distributions, we get by Fubini's theorem and basic properties of the convolution and the Fourier transform that

$$\begin{aligned} f * d\mu(\hat{\eta}) &= \int_{\mathbb{R}^d} f * d\mu(x)\, \hat{\eta}(x)\, dx = \int_{\mathbb{R}^d} f(x)\, \hat{\eta} * d\tilde{\mu}(x)\, dx \\ &= f(\hat{\eta} * d\tilde{\mu}) = f\big((\eta\, \widehat{d\mu})\hat{}\,\big) = \hat{f}\big(\eta\, \widehat{d\mu}\big) \\ &= \widehat{d\mu}\hat{f}(\eta), \quad \eta \in \mathcal{S}. \end{aligned}$$

Thus, we get

$$\widehat{f * d\mu} = \widehat{d\mu}\, \hat{f}, \quad f \in B,\ \mu \in M(\mathbb{R}^d), \tag{5}$$

in the sense of distributions.

We shall also make use of the following function sets

$$\begin{aligned} \mathfrak{D} &= \{\eta \in \mathcal{S} :\ \operatorname{supp} \eta \text{ is compact}\}, \\ \widehat{\mathfrak{D}} &= \{\eta \in L(\mathbb{R}^d) :\ \operatorname{supp} \hat{\eta} \text{ is compact}\}. \end{aligned}$$

Both $\mathfrak{D}$ and $\widehat{\mathfrak{D}}$ are dense in $L(\mathbb{R}^d)$ as well as in any HBS on $\mathbb{R}^d$. We shall shortly denote the support of a function or measure η by S_η.

We shall consider convolution operators $J_{\mu,\rho} : B \to B$, defined by

$$J_{\mu,\rho} f(x) = \int_{\mathbb{R}^d} f(x-y) d\mu_\rho(y), \quad x \in \mathbb{R}^d, \tag{6}$$

where $\rho > 0$, $\mu_\rho(y) = \mu(\rho y)$ and $\mu \in M(\mathbb{R}^d)$ is such that

$$\int_{\mathbb{R}^d} d\mu = 1. \tag{7}$$

Young's inequality (1) implies that $J_{\mu,\rho}$ is bounded, as moreover, the family $\{J_{\mu,\rho}\}_\rho$ is uniformly bounded:

$$\|J_{\mu,\rho} f\|_B \le \|\mu_\rho\|_M \, \|f\|_B = \|\mu\|_M \, \|f\|_B \quad \forall f \in B, \quad \forall \rho > 0.$$

As is known (see [6, Theorem 3.1.6, Problem 3.1.16] and [18, Lemma 9.2.2.1]),

$$\lim_{\rho \to \infty} \|f - J_{\mu,\rho} f\|_B = 0 \quad \forall f \in B. \tag{8}$$

We are interested in the rate of the convergence in (8). A helpful tool for error estimates is the so-called K-functional. The one we shall use is defined as follows. Let X be a Banach space and $\mathcal{D}$ an operator acting on a subspace thereof. Set $\mathcal{D}^{-1}(X) = \{g \in X : \mathcal{D}g \in X\}$. Then the K-functional is defined for $f \in X$ and $\tau > 0$ by

$$K(f, \tau; X, \mathcal{D}) = \inf\{\|f - g\|_X + \tau \, \|\mathcal{D}g\|_X \ : \ g \in \mathcal{D}^{-1}(X)\}.$$

In [9, Theorems 3.6 and 3.8] we presented certain Fourier transform based sufficient conditions for establishing direct and strong converse estimates of the error of convolution operators by means of such K-functionals. Though there we considered only convolution operators constructed by means of absolutely continuous measures, the proof of [9, Theorem 3.6] combined with the fact that $\widehat{d\mu_\rho}(u) = \widehat{d\mu}(\rho^{-1}u)$ actually gives the following criterion.

Theorem 2.2. *Let B be a HBS on $\mathbb{R}^d$ and $J_{\mu,\rho}$ be defined by* (6) *with $\mu \in M(\mathbb{R}^d)$, which satisfies* (7)*. Let also the following conditions be satisfied:*

(i) $\mathcal{D}(\eta * g) = \mathcal{D}\eta * g = \eta * \mathcal{D}g, \quad \eta \in \mathcal{D}^{-1}(L(\mathbb{R}^d)), \quad g \in \mathcal{D}^{-1}(B)$;

(ii) $\mathcal{D}^{-1}(L(\mathbb{R}^d))$ *is dense in* $L(\mathbb{R}^d)$;

(iii) $\widehat{\mathcal{D}\eta} = \psi \, \hat{\eta}, \quad \eta \in \mathcal{D}^{-1}(L(\mathbb{R}^d))$;

(iv) $1 - \widehat{d\mu} = \psi \, \widehat{d\lambda}$*, where* $\lambda \in M(\mathbb{R}^d)$;

(v) ψ *is positive-homogeneous of order* $\kappa > 0$*, i.e.* $\psi(\rho u) = \rho^\kappa \psi(u)$ *for all* $\rho > 0$ *and* $u \in \mathbb{R}^d$*.*

Then for all $f \in B$ and $\rho > 0$ we have

$$\|f - J_{\mu,\rho} f\|_B \le c \, K(f, \rho^{-\kappa}; B, \mathcal{D})$$

with some absolute constant c.

This theorem is based on the fact that (i)–(v) imply the functional equality

$$g - J_{\mu,\rho} g = \rho^{-\kappa} \mathcal{D}g * d\lambda_\rho \quad \forall g \in \mathcal{D}^{-1}(B) \quad \forall \rho > 0.$$

For details we refer the reader to the proof of [9, Theorem 3.6].

It was Butzer [3] (cf. also [6, Chapters 12 and 13]) who introduced Fourier transform methods in approximation theory to establish saturation results for convolution operators on $L_p(\mathbb{R})$, $1 \le p \le 2$, and extend them for $p > 2$ by duality arguments. Also, Shapiro [17] (or [18, Section 9.4] and [6, Section 13.3]) used such an approach to relate the errors of two convolution operators. In [9] we essentially followed the same ideas in establishing direct and strong converse estimates of the error for convolution operators by K-functionals in any HBS. Let us also point out that Jan Boman applied distribution theory to treat saturation of convolution operators in $L_p(\mathbb{R}^d)$, $1 \le p \le \infty$, in [18, Appendix I]. More references are given in [9].

In the present paper we continue this research and consider a natural definition of the differential operator $\mathcal{D}$ via $J_{\mu,\rho}$, which readily yields most of the hypotheses of Theorem 2.2. It is similar to that of the infinitesimal generator of a semi-group of operators. Given a measure $\mu \in M(\mathbb{R}^d)$ with (7) and a function $\phi : (0,\infty) \to \mathbb{C}\backslash\{0\}$ with $\lim_{\rho\to\infty} \phi(\rho) = 0$ we set for $g \in B$ (cf. [6, Definition 13.4.3 and (13.4.1)])

$$\mathcal{D}_{\mu,\phi} g = \text{s-}\lim_{\rho\to\infty} \frac{J_{\mu,\rho} g - g}{\phi(\rho)}, \tag{9}$$

where the limit is taken in the B-norm. Note that thus defined $\mathcal{D}_{\mu,\phi}$ commutes with the convolution, i.e. satisfies (i) of Theorem 2.2. We shall show in Section 4 that it possesses property (iii) too, as ψ has property (v) if ϕ is of the optimal magnitude. There we also give a criterion for establishing (ii), which is closely related to (iv) and (v). Before this, in Section 3 we consider how fast ϕ can vanish. Each time we study first the HBS $L(\mathbb{R}^d)$ and then try to extend the results to any HBS. For one thing, in $L(\mathbb{R}^d)$ it is possible to establish the strongest results. For another, this is the most important case as far as properties (ii) and (iii) in Theorem 2.2 are concerned.

Let us point out that the error of an approximation process, which is generated by a semi-group of operators, was characterized in any Banach space by means of its infinitesimal generator and the related K-functional by Butzer and Berens [4] (see also [6, Section 13.4] and the references cited there), Ditzian [7] and Ditzian and Ivanov [8, Section 5]. But the technique used by these authors is different. Also, in [8, Theorem 9.6] the error of an approximation operator, which does not possess the semi-group property, is characterized precisely by a K-functional in a class of Banach spaces that includes the HBS's. Let us mention that the results we establish here can be extended to this broader class of spaces (see Remark 4.5 below).

3. The optimal rate of convergence

Results concerning the saturation of convolution operators (see [6, Chapters 12 and 13] and e.g. [9]) show that generally their approximation rate cannot be arbitrary high. As a matter of fact, the sufficient conditions considered in the literature cited above reveal that the saturation (optimal) order is closely connected with the quantity $1 - \widehat{d\mu}(u)$ $(u \to 0)$ (cf. especially [6, (12.3.4)]). Note that due to (7) we have $\widehat{d\mu}(0) = 1$. In this section we shall formulate assertions, which show that generally the optimal order of ϕ is $|1-\widehat{d\mu}(\rho^{-1}a)|$ with an appropriate $a \in \mathbb{R}^d$.

First, we consider the simplest and the prototype of all HBS's – $L(\mathbb{R}^d)$, and then proceed to the general case. As we mentioned, stronger results are valid for $L(\mathbb{R}^d)$.

Proposition 3.1. *Let $J_{\mu,\rho}$ be defined by* (6) *with $\mu \in M(\mathbb{R}^d)$, which satisfies* (7). *Let $a \in \mathbb{R}^d$ be such that there exists $g_0 \in \mathcal{D}^{-1}_{\mu,\phi}(L(\mathbb{R}^d))$ with $\hat{g}_0(a) \neq 0$. Then there exists a positive constant c_0 such that*

$$|\phi(\rho)| \geq c_0 |1-\widehat{d\mu}(\rho^{-1}a)|, \quad \rho \geq 1.$$

P r o o f. Since $\{\phi(\rho)^{-1}\|g_0 - J_{\mu,\rho}g_0\|_L\}_\rho$ is convergent as $\rho \to \infty$, then there exists a constant c_1 such that for all $\rho \geq 1$

$$\|g_0 - J_{\mu,\rho}g_0\|_L \leq c_1|\phi(\rho)|. \tag{1}$$

Further, using basic properties of the Fourier-Stieltjes transform, we get

$$|\hat{g}_0(u) - \widehat{d\mu}(\rho^{-1}u)\hat{g}_0(u)| \leq \|g_0 - J_{\mu,\rho}g_0\|_L, \quad u \in \mathbb{R}^d, \tag{2}$$

which, in particular, implies the estimate

$$|1-\widehat{d\mu}(\rho^{-1}a)| \leq \frac{1}{|\hat{g}_0(a)|}\,\|g_0 - J_{\mu,\rho}g_0\|_L. \tag{3}$$

Now, combining (1) and (3) we deduce that

$$|1-\widehat{d\mu}(\rho^{-1}a)| \leq \frac{c_1}{|\hat{g}_0(a)|}\,|\phi(\rho)|, \quad \rho \geq 1;$$

hence the assertion of the proposition. ∎

Remark 3.2. This proposition implies that if $|\phi(\rho)| = o(|1-\widehat{d\mu}(\rho^{-1}a)|)$ for each $a \in \mathbb{R}^d$, $a \neq 0$, then $\mathcal{D}^{-1}_{\mu,\phi}(L(\mathbb{R}^d)) \subseteq \mathbb{C}$ and $\mathcal{D}_{\mu,\phi} = 0$.

Remark 3.3. Let us note that results similar to those above hold in the case of the HBS $L_p(\mathbb{R}^d)$ with $1 < p \leq 2$ as well. Then we have to use the Titchmarsh inequality (see e.g. [6, Theorem 5.2.9]) in the place of (2). This observation expands to the Lipschitz and Besov spaces built on the basis of $L_p(\mathbb{R}^d)$ for $1 \leq p \leq 2$ as they are continuously embedded in the corresponding $L_p(\mathbb{R}^d)$.

We can establish similar results in the general case of an arbitrary HBS under certain stronger hypotheses. Below we present two such assertions.

Theorem 3.4. *Let B be a HBS on $\mathbb{R}^d$ and $J_{\mu,\rho}$ be defined by* (6) *with $\mu \in M(\mathbb{R}^d)$, which satisfies* (7) *and has compact support. Let $a \in \mathbb{R}^d$ be such that there exists $g_0 \in \mathcal{D}^{-1}_{\mu,\phi}(B)$ of compact support with $\hat{g}_0(a) \neq 0$. Then there exists a positive constant c_0 such that*

$$|\phi(\rho)| \geq c_0 |1 - \widehat{d\mu}(\rho^{-1}a)|, \quad \rho \geq 1.$$

P r o o f. First, $g_0 \in B$ yields that g_0 is locally integrable and since it has compact support, we get that $g_0 \in L(\mathbb{R}^d)$; hence $\hat{g}_0 \in CB(\mathbb{R}^d)$. Also, let us add that $J_{\mu,\rho} g_0 \in L(\mathbb{R}^d)$ because $g_0 \in L(\mathbb{R}^d)$, as moreover, $J_{\mu,\rho} g_0$ is of compact support because both μ and g_0 are. Further, observe that conditions (a) and (c) of Definition 2.1 imply that for each bounded measurable subset $\mathbb{S}$ of $\mathbb{R}^d$ there exists a constant $c_{\mathbb{S}}$ such that there holds

$$\|f\|_{L(\mathbb{S})} \leq c_{\mathbb{S}} \|f\|_B, \quad f \in B. \tag{4}$$

Therefore, we have for all $\rho \geq 1$

$$\|g_0 - J_{\mu,\rho} g_0\|_L = \|g_0 - J_{\mu,\rho} g_0\|_{L(S)} \leq c_S \|g_0 - J_{\mu,\rho} g_0\|_B, \tag{5}$$

where

$$S = S_{g_0} + \cup_{\varrho \in [0,1]} \varrho \, S_\mu.$$

Next, just as in the proof of Proposition 3.1 it is established that there exists a constant c_1 such that for all $\rho \geq 1$ we have

$$\|g_0 - J_{\mu,\rho} g_0\|_B \leq c_1 |\phi(\rho)|. \tag{6}$$

Now, by (3), (5) and (6) imply the assertion of the theorem. ∎

Remark 3.5. In passing, let us mention that a certain converse inequality about the rate of approximation of the operator $J_{\mu,\rho}$ can also give the optimality result of the preceding assertion. More precisely, suppose that g_0 satisfies the hypotheses of Theorem 3.4 and the measure $\mu \in M(\mathbb{R}^d)$ of total mass one satisfies the relation

$$\int_{\mathbb{R}^d \setminus \rho \mathbb{B}} |d\mu| \leq \|g_0 - J_{\mu,\rho} g_0\|_{L(\mathbb{B})}, \quad \rho \geq \rho_0, \tag{7}$$

where $\mathbb{B}$ is a ball in $\mathbb{R}^d$ that contains the support of g_0 and a neighbourhood of the origin.[2] Then we get by Fubini's theorem and (7) that for all $\rho \geq \rho_0$

$$\begin{aligned}\|g_0 - J_{\mu,\rho} g_0\|_{L(\mathbb{R}^d \setminus 2\mathbb{B})} &= \int_{\mathbb{R}^d \setminus 2\mathbb{B}} \left| \int_{\mathbb{B}} g_0(y)\, d\mu(\rho(x-y)) \right| dx \\ &\leq \|g_0\|_{L_\infty} \int_{\mathbb{B}} \left(\int_{\mathbb{R}^d \setminus 2\mathbb{B}} |d\mu(\rho(x-y))| \right) dy \\ &\leq \|g_0\|_{L_\infty} \int_{\mathbb{B}} dy \,.\, \int_{\mathbb{R}^d \setminus \rho\mathbb{B}} |d\mu| \\ &\leq \|g_0\|_{L_\infty} \int_{\mathbb{B}} dy \,.\, \|g_0 - J_{\mu,\rho} g_0\|_{L(\mathbb{B})},\end{aligned}$$

as we have used that $x \in \mathbb{R}^d \setminus 2\mathbb{B}$ and $y \in \mathbb{B}$ imply $\rho(x-y) \in \mathbb{R}^d \setminus \rho\mathbb{B}$. Therefore by (4) we infer that there exists a constant c such that for all $\rho \geq \rho_0$

$$\begin{aligned}\|g_0 - J_{\mu,\rho} g_0\|_L &= \|g_0 - J_{\mu,\rho} g_0\|_{L(2\mathbb{B})} + \|g_0 - J_{\mu,\rho} g_0\|_{L(\mathbb{R}^d \setminus 2\mathbb{B})} \\ &\leq c \|g_0 - J_{\mu,\rho} g_0\|_B.\end{aligned}$$

Now, the inequality

$$|\phi(\rho)| \geq c' |1 - \widehat{d\mu}(\rho^{-1} a)|$$

follows for ρ large enough as in the proof of Theorem 3.4.

4. The form of the differential operator $\mathcal{D}_{\mu,\phi}$

It turns out that $\mathcal{D}_{\mu,\phi}$ similarly to ϕ has a simple description in terms of the Fourier transform of the measure μ. Returning to the sufficient conditions for saturation and direct estimates of the error of $J_{\mu,\rho}$ (see e.g. [6, (12.3.5)] and Theorem 2.2 or [9, Theorem 3.6]), we see that it is reasonable to expect that the Fourier transform of $\mathcal{D}_{\mu,\phi} g$ is of the form $\psi \hat{g}$, where the function $\psi(u)$ is closely related to the quantity

$$\frac{\widehat{d\mu}(\rho^{-1} u) - 1}{\phi(\rho)}.$$

Moreover, as we shall see below, ψ actually belongs to a very narrow class of functions when we consider ϕ of the generally optimal order of $|\widehat{d\mu}(\rho^{-1} a) - 1|$ with an appropriate a.

Again we shall first consider the simplest case of $B = L(\mathbb{R}^d)$.

Proposition 4.1. *Let $J_{\mu,\rho}$ be defined by* (6) *with $\mu \in M(\mathbb{R}^d)$, which satisfies* (7). *Set $\mathbb{U} = \{u \in \mathbb{R}^d :\ \hat{g}(u) = 0\ \forall g \in \mathcal{D}_{\mu,\phi}^{-1}(L(\mathbb{R}^d))\}$. Then*

$$\lim_{\rho \to \infty} \frac{\widehat{d\mu}(\rho^{-1} u) - 1}{\phi(\rho)} \tag{1}$$

[2] Note that every measure of compact support satisfies (7).

exists as a finite number for each $u \in \mathbb{R}^d \backslash \mathbb{U}$*. Set for* $u \in \mathbb{R}^d$

$$\psi(u) = \begin{cases} \lim\limits_{\rho\to\infty} \dfrac{\widehat{d\mu}(\rho^{-1}u) - 1}{\phi(\rho)} & \textit{if the limit exists,} \\ 0, & \textit{otherwise.} \end{cases}$$

Then $\widehat{\mathcal{D}_{\mu,\phi} g}(u) = \psi(u)\hat{g}(u)$, $u \in \mathbb{R}^d$, *for each* $g \in \mathcal{D}^{-1}_{\mu,\phi}(L(\mathbb{R}^d))$.

If $\mathcal{D}^{-1}_{\mu,\phi}(L(\mathbb{R}^d))$ *is dense in* $L(\mathbb{R}^d)$*, then the limit* (1) *exists for each* $u \in \mathbb{R}^d$ *as the convergence is uniform on every compact set; hence* $\psi(u)$ *is continuous on* $\mathbb{R}^d$.

Proof. The proof is based on the standard arguments used to establish Proposition 3.1. For all $g \in \mathcal{D}^{-1}_{\mu,\phi}(L(\mathbb{R}^d))$ and $u \in \mathbb{R}^d$ we have

$$\left| \frac{\widehat{d\mu}(\rho^{-1}u) - 1}{\phi(\rho)}\, \hat{g}(u) - \widehat{\mathcal{D}_{\mu,\phi} g}(u) \right| \le \left\| \frac{J_{\mu,\rho} g - g}{\phi(\rho)} - \mathcal{D}_{\mu,\phi} g \right\|_L \to 0, \quad \rho \to \infty. \quad (2)$$

For each $u_0 \in \mathbb{R}^d \backslash \mathbb{U}$ we fix $g \in \mathcal{D}^{-1}_{\mu,\phi}(L(\mathbb{R}^d))$ such that $\hat{g}(u_0) \ne 0$ and derive from (2) that the limit (1) exists for $u = u_0$. Further, for every $u \in \mathbb{R}^d \backslash \mathbb{U}$ we derive again by (2) that

$$\widehat{\mathcal{D}_{\mu,\phi} g}(u) = \psi(u)\hat{g}(u) \quad \forall g \in \mathcal{D}^{-1}_{\mu,\phi}(L(\mathbb{R}^d)). \quad (3)$$

Similarly, (2) implies for $u \in \mathbb{U}$ that $\widehat{\mathcal{D}_{\mu,\phi} g}(u) = 0$ for all $g \in \mathcal{D}^{-1}_{\mu,\phi}(L(\mathbb{R}^d))$ and (3) again holds.

To establish the last assertion of the proposition we just need to fix in (2) a function g such that $|\hat{g}| \ge 1$ on the given compact set. ∎

Conversely, if the function $\psi\hat{g}$ is the Fourier transform of a function in $L(\mathbb{R}^d)$, then g belongs to the domain of a differential operator of the type $\mathcal{D}_{\mu,\phi}$. More precisely, we have the following.

Proposition 4.2. *Let* $\psi \in C(\mathbb{R}^d)$ *be positive-homogeneous of order* $\kappa > 0$ *and* $g \in L(\mathbb{R}^d)$ *be such that* $\psi\hat{g} = \widehat{G}$ *for some* $G \in L(\mathbb{R}^d)$*. Let also* $\mu, \lambda \in M(\mathbb{R}^d)$ *be such that* $\widehat{d\lambda}(0) \ne 0$ *and*

$$1 - \widehat{d\mu} = \psi\, \widehat{d\lambda}. \quad (4)$$

Finally, let $\phi : (0, \infty) \to \mathbb{C}\backslash\{0\}$ *be such that* $\lim_{\rho\to\infty} \rho^\kappa \phi(\rho) = -\widehat{d\lambda}(0)$*. Then* $g \in \mathcal{D}^{-1}_{\mu,\phi}(L(\mathbb{R}^d))$*, as moreover,* $\mathcal{D}_{\mu,\phi} g = G$.

P r o o f. Using (4) and the homogeneity of ψ, we get

$$\left(\frac{J_{\mu,\rho}g-g}{\phi(\rho)}\right)^{\widehat{}}(u) = -\frac{\widehat{d\lambda}(\rho^{-1}u)}{\rho^{\kappa}\phi(\rho)}\,\psi(u)\hat{g}(u)$$
$$= -\frac{\widehat{d\lambda}(\rho^{-1}u)}{\rho^{\kappa}\phi(\rho)}\,\widehat{G}(u) = \ell(\rho)\,\widehat{G * d\lambda'_\rho}(u),$$

where we have set $\ell(\rho) = -\widehat{d\lambda}(0)/(\rho^{\kappa}\phi(\rho))$ and $\lambda'_\rho(y) = \widehat{d\lambda}(0)^{-1}\lambda(\rho y)$. By the uniqueness of the Fourier transform this implies

$$\frac{J_{\mu,\rho}g-g}{\phi(\rho)} = \ell(\rho)\,G * d\lambda'_\rho.$$

Now, we have

$$\left\|\frac{J_{\mu,\rho}g-g}{\phi(\rho)} - G\right\|_L \le \|G - G * d\lambda'_\rho\|_L + |\ell(\rho)-1|\,\|G * d\lambda'_\rho\|_L.$$

If $\rho \to \infty$, the first term on the right above tends to 0 because the measure $\widehat{d\lambda}(0)^{-1}\lambda$ satisfies (7), and the second term tends to 0 because $\ell(\rho) \to 1$ and Young's inequality implies

$$\|G * d\lambda'_\rho\|_L \le \|\lambda'_\rho\|_M\,\|G\|_L = |\widehat{d\lambda}(0)|^{-1}\|\lambda\|_M\,\|G\|_L \quad \forall\rho.$$

Thus $g \in \mathcal{D}^{-1}_{\mu,\phi}(L(\mathbb{R}^d))$ and $\mathcal{D}_{\mu,\phi}g = G$. ■

Remark 4.3. This proposition provides a technically elementary tool for verifying condition (ii) of Theorem 2.2. In particular, if $\psi(u) = |u|^{\kappa}$ or $\psi(u) = |u_1|^{\kappa} + \cdots + |u_d|^{\kappa}$ with $\kappa > 0$, then routine considerations show that $\mathcal{S} \subset \mathcal{D}^{-1}_{\mu,\phi}(L(\mathbb{R}^d))$, and hence condition (ii) of Theorem 2.2 is satisfied. Further, Proposition 4.11 below implies that if ϕ is of the optimal order, then ψ is positive-homogeneous.

The results above can be extended in a certain sense for $L_p(\mathbb{R}^d)$, $1 < p \le 2$. We shall not go into details but rather go on to the general case.

In an arbitrary HBS, if the measure μ has compact support and $\mathfrak{D} \subset \mathcal{D}^{-1}_{\mu,\phi}(B)$, then just as in the proof of Theorem 3.4 and Proposition 4.1 it is established that the limit

$$\psi(u) = \lim_{\rho\to\infty}\frac{\widehat{d\mu}(\rho^{-1}u)-1}{\phi(\rho)},$$

exists for each $u \in \mathbb{R}^d$, as the convergence is uniform on every compact set, and for each $g \in \mathfrak{D}$, we have $\mathcal{D}_{\mu,\phi}g \in L(\mathbb{R}^d)$ and $\widehat{\mathcal{D}_{\mu,\phi}g}(u) = \psi(u)\hat{g}(u)$, $u \in \mathbb{R}^d$.

To extend this result to the general case we can avail ourselves of the fact that the elements of any HBS are tempered distributions and use the Fourier transform in the distributional sense.

Theorem 4.4. *Let B be a HBS on $\mathbb{R}^d$ and $J_{\mu,\rho}$ be defined by* (6) *with $\mu \in M(\mathbb{R}^d)$, which satisfies* (7) *and $\widehat{d\mu} \in C^{2n_d}(\mathbb{R}^d)$. Let us set*

$$\psi_\rho(u) = \frac{\widehat{d\mu}(\rho^{-1}u) - 1}{\phi(\rho)}, \quad u \in \mathbb{R}^d,$$

and assume that the the limit $\psi(u) = \lim_{\rho\to\infty} \psi_\rho(u)$ exists for each $u \in \mathbb{R}^d$ as, moreover, $\psi \in C^{2n_d}(\mathbb{R}^d)$ and there exist naturals $n_{k,j}$, $k = 0, \ldots, 2n_d$ and $j = 1, \ldots, d$, such that

$$\sup_{u\in\mathbb{R}^d} \left| (1 + |u|^2)^{-n_{k,j}} \frac{\partial^k}{\partial u_j^k} (\psi_\rho - \psi)(u) \right| \to 0 \quad as \quad \rho \to \infty. \tag{5}$$

Then ψ is polynomially bounded and there holds

$$\widehat{\mathcal{D}_{\mu,\phi} g} = \psi \hat{g}, \quad g \in \mathcal{D}^{-1}_{\mu,\phi}(B), \tag{6}$$

in the sense of distributions.

P r o o f. It is clear that ψ is polynomially bounded. Let us proceed to verifying (6). Relation (4) shows that strong convergence of a family of elements of the HBS B implies its weak convergence as tempered distributions. Thus we have for each $g \in \mathcal{D}^{-1}_{\mu,\phi}(B)$ and $\eta \in \mathcal{S}$ that

$$\widehat{\mathcal{D}_{\mu,\phi} g}(\eta) = \mathcal{D}_{\mu,\phi} g(\hat{\eta}) = \lim_{\rho\to\infty} \frac{g * d\mu_\rho(\hat{\eta}) - g(\hat{\eta})}{\phi(\rho)}. \tag{7}$$

Thus by (5) we arrive at

$$\widehat{\mathcal{D}_{\mu,\phi} g}(\eta) = \lim_{\rho\to\infty} g\big(\widehat{\psi_\rho \eta}\big). \tag{8}$$

Next, let us observe that for $x \in \mathbb{R}^d$

$$
\begin{aligned}
&(1+|x|^2)^{n_d}\big|((\psi_\rho-\psi)\eta)\widehat{\ }(x)\big| \\
&\quad \le (d+1)^{n_d-1}\big|(1+x_1^{2n_d}+\cdots+x_d^{2n_d})((\psi_\rho-\psi)\eta)\widehat{\ }(x)\big| \\
&\quad = (d+1)^{n_d-1}\left|((\psi_\rho-\psi)\eta)\widehat{\ }(x)+(-1)^{n_d}\sum_{j=1}^{d}\left(\frac{\partial^{2n_d}}{\partial u_j^{2n_d}}((\psi_\rho-\psi)\eta)\right)^{\widehat{}}(x)\right| \\
&\quad \le (d+1)^{n_d-1}\left(\|(\psi_\rho-\psi)\eta\|_L+\sum_{j=1}^{d}\left\|\frac{\partial^{2n_d}}{\partial u_j^{2n_d}}((\psi_\rho-\psi)\eta)\right\|_L\right) \\
&\quad \le c\sum_{j=1}^{d}\sum_{k=0}^{2n_d}\sup_{u\in\mathbb{R}^d}\left|(1+|u|^2)^{-n_{k,j}}\frac{\partial^k}{\partial u_j^k}(\psi_\rho-\psi)(u)\right|,
\end{aligned}
$$

where the constant c is independent of ρ. Consequently, by condition (5) we get

$$\sup_{x\in\mathbb{R}^d}(1+|x|^2)^{n_d}\big|((\psi_\rho-\psi)\eta)\widehat{\ }(x)\big| \to 0 \quad \text{as} \quad \rho\to\infty. \tag{9}$$

Now, (4), (8) and (9) imply (6). ■

Remark 4.5. The last theorem remains valid for every Banach space B of functions on $\mathbb{R}^d$, which is continuously embedded into $\mathcal{S}'$ and satisfies (b) of Definition 2.1 (but not necessarily (a) and (c)). Let us recall that for a given distribution f its translate f_t, $t \in \mathbb{R}^d$, is the distribution defined by

$$f_t(\eta) = f(\eta_{-t}), \quad \eta \in \mathcal{S}.$$

Let for the sake of simplicity consider only absolutely continuous measures $d\mu(x) = \ell(x)\,dx$ with $\ell \in \mathcal{S}$. Then $f * \ell$ is the distribution given by

$$f*\ell(\eta) = f(\tilde{\ell}*\eta), \quad \eta\in\mathcal{S}.$$

Actually, $f*\ell \in C^\infty(\mathbb{R}^d)$ and it as well as all its partial derivatives are polynomially bounded. Condition (b) of Definition 2.1 implies that $f*\ell$ is the strong limit of Riemann sums in B; hence $f*\ell\in B$.

For a multi-index $\alpha = (\alpha_1,\dots,\alpha_d)\in\mathbb{N}_0^d$ we set $|\alpha| = \alpha_1+\cdots+\alpha_d$ and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}}.$$

Let us assume that for each multi-index α there exists a non-negative integer n_α such that $\{(1+|u|^2)^{-n_\alpha}D^\alpha\psi_\rho(u)\}_\rho$ converges to a bounded function uniformly on

$\mathbb{R}^d$ as $\rho \to \infty$. Then $\psi(u) = \lim_{\rho\to\infty} \psi_\rho(u)$ exists for each $u \in \mathbb{R}^d$, $\psi \in C^\infty(\mathbb{R}^d)$, as $D^\alpha\psi$ is polynomially bounded for each multi-index α. Further, for each $n \in \mathbb{N}_0$ and each multi-index α we have by the Leibniz formula that

$$\begin{aligned}
&\sup_{u\in\mathbb{R}^d} |(1+|u|^2)^n D^\alpha(\psi\eta - \psi_\rho\eta)(u)| \\
&\qquad = \sup_{u\in\mathbb{R}^d} \left| (1+|u|^2)^n \sum_{\beta\le\alpha} c_{\alpha\beta} D^\beta(\psi-\psi_\rho)(u) D^{\alpha-\beta}\eta(u) \right| \\
&\qquad \le \sup_{\beta\le\alpha} \sup_{u\in\mathbb{R}^d} |(1+|u|^2)^{n+n_\beta} D^{\alpha-\beta}\eta(u)| \\
&\qquad\qquad \times \sum_{\beta\le\alpha} c_{\alpha\beta} \sup_{u\in\mathbb{R}^d} |(1+|u|^2)^{-n_\beta} D^\beta(\psi-\psi_\rho)(u)|
\end{aligned}$$

with some positive integers $c_{\alpha\beta}$, and consequently,

$$\sup_{u\in\mathbb{R}^d} |(1+|u|^2)^n D^\alpha(\psi\eta - \psi_\rho\eta)(u)| \to 0, \quad \rho \to \infty,$$

for each $\eta \in \mathcal{S}$. Thus for each $\eta \in \mathcal{S}$ the family $\{\psi_\rho\eta\}_\rho$ converges as $\rho \to \infty$ to $\psi\eta$ in the topology of $\mathcal{S}$. Next, since B is continuously embedded into $\mathcal{S}'$, we again have (7) for $g \in \mathcal{D}^{-1}_{\mu,\phi}(B)$ and hence

$$\widehat{\mathcal{D}_{\mu,\phi}g}(\eta) = \lim_{\rho\to\infty} \hat{g}(\psi_\rho\eta) = \hat{g}(\psi\eta) = \psi\hat{g}(\eta), \quad \eta \in \mathcal{S}.$$

Thus finally

$$\widehat{\mathcal{D}_{\mu,\phi}g} = \psi\hat{g}.$$

Approximation in similar spaces was considered by Ditzian and Ivanov [8, Theorem 9.6] by means of a different argument based on duality (cf. [9]).

Quite similarly to Proposition 4.2, using (5), we establish the converse of Theorem 4.4.

Theorem 4.6. *Let B be a HBS on $\mathbb{R}^d$. Let $\psi \in C(\mathbb{R}^d)$ be positive-homogeneous of order $\kappa > 0$. Let $g \in B$ be such that $\psi\hat{g} = \widehat{G}$ for some $G \in B$. Let also $\mu, \lambda \in M(\mathbb{R}^d)$ be such that $\widehat{d\lambda}(0) \neq 0$ and*

$$1 - \widehat{d\mu} = \psi\,\widehat{d\lambda}.$$

Finally, let $\phi : (0,\infty) \to \mathbb{C}\backslash\{0\}$ be such that $\lim_{\rho\to\infty} \rho^\kappa\phi(\rho) = -\widehat{d\lambda}(0)$. Then $g \in \mathcal{D}^{-1}_{\mu,\phi}(B)$ as, moreover, $\mathcal{D}_{\mu,\phi}g = G$.

Remark 4.7. Note that any positive-homogeneous function is polynomially bounded.

In the conditions of Theorem 4.4, if, for example, $g \in L(\mathbb{R}^d)$, relation (6) implies that actually $\widehat{\mathcal{D}_{\mu,\phi} g}$ is a function. As a matter of fact, this is so even under a little bit milder restrictions on ψ.

Theorem 4.8. *Let B be a HBS on $\mathbb{R}^d$ and $J_{\mu,\rho}$ be defined by* (6) *with $\mu \in M(\mathbb{R}^d)$, which satisfies* (7). *Let the limit*

$$\psi(u) = \lim_{\rho\to\infty} \frac{\widehat{d\mu}(\rho^{-1}u) - 1}{\phi(\rho)}$$

exist for each $u \in \mathbb{R}^d$ as the convergence is uniform on the compact sets. Then for each $g \in \mathcal{D}^{-1}_{\mu,\phi}(B) \cap L(\mathbb{R}^d)$ we have that $\widehat{\mathcal{D}_{\mu,\phi} g} \in L_{\infty,loc}(\mathbb{R}^d)$ as

$$\widehat{\mathcal{D}_{\mu,\phi} g}(u) = \psi(u)\hat{g}(u) \quad a.e.$$

Hence, $\widehat{\mathcal{D}_{\mu,\phi} g}$ can be corrected on a set of measure 0 *to a continuous function.*

P r o o f. As we have noted in (7), for $g \in \mathcal{D}^{-1}_{\mu,\phi}(B)$ and $\eta \in \mathcal{S}$ we have

$$\lim_{\rho\to\infty} \int_{\mathbb{R}^d} \frac{J_{\mu,\rho} g(x) - g(x)}{\phi(\rho)} \hat{\eta}(x)\, dx = \int_{\mathbb{R}^d} \mathcal{D}_{\mu,\phi} g(x)\hat{\eta}(x)\, dx, \tag{10}$$

which, in view of the relation

$$\int_{\mathbb{R}^d} \frac{\widehat{d\mu}(\rho^{-1}u) - 1}{\phi(\rho)}\, \hat{g}(u)\eta(u)\, du = \int_{\mathbb{R}^d} \frac{J_{\mu,\rho} g(u) - g(u)}{\phi(\rho)}\, \hat{\eta}(u)\, du,$$

with $g \in L(\mathbb{R}^d)$, yields that

$$\lim_{\rho\to\infty} \int_{\mathbb{R}^d} \frac{\widehat{d\mu}(\rho^{-1}u) - 1}{\phi(\rho)}\, \hat{g}(u)\eta(u)\, du = \int_{\mathbb{R}^d} \mathcal{D}_{\mu,\phi} g(u)\hat{\eta}(u)\, du \tag{11}$$

for any $g \in \mathcal{D}^{-1}_{\mu,\phi}(B) \cap L(\mathbb{R}^d)$ and $\eta \in \mathcal{S}$. Let $g \in \mathcal{D}^{-1}_{\mu,\phi}(B) \cap \widehat{\mathfrak{D}}$. Since

$$\psi(u) = \lim_{\rho\to\infty} \frac{\widehat{d\mu}(\rho^{-1}u) - 1}{\phi(\rho)}$$

uniformly on the compact sets, we get for each $g \in \mathcal{D}^{-1}_{\mu,\phi}(B) \cap \widehat{\mathfrak{D}}$ and $\eta \in \mathcal{S}$

$$\int_{\mathbb{R}^d} \psi(u)\hat{g}(u)\eta(u)\, du = \int_{\mathbb{R}^d} \mathcal{D}_{\mu,\phi} g(u)\hat{\eta}(u)\, du. \tag{12}$$

Next, since $\psi\hat{g}$ is continuous and bounded, it is a tempered distribution and (12) says that the continuous linear functionals $\psi\hat{g}$ and $\widehat{\mathcal{D}_{\mu,\phi} g}$ on $\mathcal{S}$ are equal.

Consequently, for $g \in \mathcal{D}^{-1}_{\mu,\phi}(B)$ such that $g \in \widehat{\mathfrak{D}}$ we have $\widehat{\mathcal{D}_{\mu,\phi} g} = \psi \hat{g}$. To extend this relation to any $g \in \mathcal{D}^{-1}_{\mu,\phi}(B)$, we shall make use of a locally finite partition of unity in $\mathbb{R}^d$ $\{\xi_j\}$. Let $\xi_j = \hat{\zeta}_j$, $\zeta_j \in \mathcal{S} \cap \widehat{\mathfrak{D}}$. Then for any $g \in \mathcal{D}^{-1}_{\mu,\phi}(B) \cap L(\mathbb{R}^d)$

$$\hat{g} = \sum_j \xi_j \hat{g} = \sum_j \hat{\zeta}_j \hat{g} = \sum_j \widehat{\zeta_j * g}.$$

Clearly, $\zeta_j * g \in \widehat{\mathfrak{D}}$. Also, it is easy to establish that $\zeta_j * g \in \mathcal{D}^{-1}_{\mu,\phi}(B)$, as moreover, $\mathcal{D}_{\mu,\phi}(\zeta_j * g) = \zeta_j * \mathcal{D}_{\mu,\phi} g$. Indeed, we have by Young's inequality

$$\begin{aligned} \left\| \frac{J_{\mu,\rho}(\zeta_j * g) - \zeta_j * g}{\phi(\rho)} - \zeta_j * \mathcal{D}_{\mu,\phi} g \right\|_B &= \left\| \zeta_j * \left(\frac{J_{\mu,\rho} g - g}{\phi(\rho)} - \mathcal{D}_{\mu,\phi} g \right) \right\|_B \\ &\le \|\zeta_j\|_L \left\| \frac{J_{\mu,\rho} g - g}{\phi(\rho)} - \mathcal{D}_{\mu,\phi} g \right\|_B \to 0, \quad \rho \to \infty. \end{aligned}$$

Thus, $\zeta_j * g \in \mathcal{D}^{-1}_{\mu,\phi}(B) \cap \widehat{\mathfrak{D}}$ and by what we have already established above, we get

$$\xi_j \widehat{D_{\mu,\phi} g} = (\zeta_j * D_{\mu,\phi} g)\widehat{\ } = (D_{\mu,\phi}(\zeta_j * g))\widehat{\ } = \psi \, \widehat{\zeta_j * g} = \xi_j \, \psi \hat{g}.$$

Summing the last relation in j and taking into account that $\psi \hat{g} \in C(\mathbb{R}^d)$, we complete the proof of the theorem. ■

In the applications it is easy to check that

$$\frac{\widehat{d\mu}(\rho^{-1} u) - 1}{\phi(\rho)}$$

is uniformly convergent on the compact sets when $\rho \to \infty$, or even that condition (5) holds. However, in an arbitrary HBS, it can be shown that we necessarily have a weak* limit and convergence a.e. on bounded sets.

Theorem 4.9. *Let B be a HBS on $\mathbb{R}^d$ and $J_{\mu,\rho}$ be defined by* (6) *with $\mu \in M(\mathbb{R}^d)$, which satisfies* (7)*. Let there exist a ball $\mathbb{B}_0 \subset \mathbb{R}^d$ with center at* 0 *and constants c_0 and ρ_0 such that*

$$|\widehat{d\mu}(\rho^{-1} u) - 1| \le c_0 |\phi(\rho)|, \quad u \in \mathbb{B}_0, \quad \rho \ge \rho_0. \tag{13}$$

Then for each $g \in \mathcal{D}^{-1}_{\mu,\phi}(B) \cap \widehat{\mathfrak{D}}$ we have that $\widehat{\mathcal{D}_{\mu,\phi} g} \in L_\infty(\mathbb{R}^d)$ as

$$\widehat{\mathcal{D}_{\mu,\phi} g} = \mathrm{w}^* \text{-} \lim_{\rho \to \infty} \frac{\widehat{d\mu}(\rho^{-1} \circ) - 1}{\phi(\rho)} \, \hat{g};$$

hence there exists a sequence $\{\rho_n\}_{n=1}^{\infty}$ *with* $\lim_{n\to\infty}\rho_n=\infty$ *such that*

$$\widehat{\mathcal{D}_{\mu,\phi}g}(u)=\lim_{n\to\infty}\frac{\widehat{d\mu}(\rho_n^{-1}u)-1}{\phi(\rho_n)}\,\hat{g}(u)\quad a.e.$$

Moreover, if $\widehat{\mathfrak{D}}\subset\mathcal{D}_{\mu,\phi}^{-1}(B)$, *then for each bounded set* $\mathbb{S}\subset\mathbb{R}^d$ *there exists a sequence* $\{\rho_n\}_{n=1}^{\infty}$ *with* $\lim_{n\to\infty}\rho_n=\infty$ *such that the limit*

$$\lim_{n\to\infty}\frac{\widehat{d\mu}(\rho_n^{-1}u)-1}{\phi(\rho_n)}$$

exist a.e. in $\mathbb{S}$.

Proof. First, let us observe that the hypotheses of the theorem imply that for any compact set $\mathbb{K}$ there exists a positive number $\rho_{\mathbb{K}}$ such that

$$\left|\frac{\widehat{d\mu}(\rho^{-1}u)-1}{\phi(\rho)}\right|\le c_0\quad u\in\mathbb{K},\quad \rho\ge\rho_{\mathbb{K}}.$$

Let $g\in\mathcal{D}_{\mu,\phi}^{-1}(B)\cap\widehat{\mathfrak{D}}$. Then we have

$$\left\|\frac{\widehat{d\mu}(\rho^{-1}\circ)-1}{\phi(\rho)}\,\hat{g}\right\|_{L_\infty}\le c_0\|\hat{g}\|_{L_\infty},\quad \rho\ge\rho_{S_{\hat{g}}}.$$

Now, the weak* compactness theorem for L_∞ (see e.g. [11, Theorems 4.12.3 and 4.14.6]) implies that there exists a sequence $\{\rho_n\}_{n=1}^{\infty}$, which tends to infinity, and a function $G\in L_\infty(\mathbb{R}^d)$ such that

$$G=\mathrm{w}^*\text{-}\lim_{n\to\infty}\frac{\widehat{d\mu}(\rho_n^{-1}\circ)-1}{\phi(\rho_n)}\,\hat{g}. \tag{14}$$

Next, as in the proof of Theorem 4.8 we deduce from (10), (11), (14) and the uniqueness of the weak* limit that $\widehat{\mathcal{D}_{\mu,\phi}g}=G\in L_\infty(\mathbb{R}^d)$.

The second part of the theorem follows from basic properties of convergence of weak type (see e.g. [13, Chapter 5, § 4, Theorem 8]). ∎

Let us turn our attention to the explicit form of $\psi(u)$ for $\phi(\rho)=\widehat{d\mu}(\rho^{-1}a)-1$. As we have seen in the previous section, the optimal order of convergence is of such a form. We shall use the following lemma.

Lemma 4.10. *Let* $\tau:[0,\infty)\to\mathbb{C}$ *be continuous,* $\tau(0)=0$ *and* $\tau(y)\neq0$ *on some interval* $(0,\varepsilon)$. *Set*

$$\Psi(u)=\lim_{y\to0+0}\frac{\tau(uy)}{\tau(y)},\quad u\ge0,$$

as the convergence is uniform on the finite closed intervals. Then $\Psi(u) = u^\kappa$ *for some real* $\kappa > 0$.

Proof. Ψ is continuous on $[0, \infty)$. By the definition of Ψ we get for $a, b > 0$

$$\begin{aligned}\Psi(ab) &= \lim_{y\to 0+0} \frac{\tau(aby)}{\tau(y)} \\ &= \lim_{y\to 0+0} \frac{\tau(aby)}{\tau(by)} \lim_{y\to 0+0} \frac{\tau(by)}{\tau(y)} \\ &= \Psi(a)\Psi(b).\end{aligned}$$

Also, $\Psi(x) \not\equiv 0$ as $\Psi(1) = 1$. Consequently, there exists a complex number κ such that $\Psi(u) = u^\kappa$, $u > 0$. Finally, since $\Psi(x)$ is continuous at 0 and $\Psi(0) = 0$, we get $\kappa > 0$. ∎

The next proposition shows that

$$\psi(u) = \lim_{\rho\to\infty} \frac{\widehat{d\mu}(\rho^{-1}u) - 1}{\phi(\rho)}$$

satisfies property (v) of Theorem 2.2 if ϕ is of the optimal order.

Proposition 4.11. *Let* $\mu \in M(\mathbb{R}^d)$. *Let also* $\widehat{d\mu}(0) = 1$ *and there exists* $a \in \mathbb{R}^d$ *such that* $\widehat{d\mu}(ya) \neq 1$ *for* $y \in (0, 1)$. *Set*

$$\psi(u) = \lim_{\rho\to\infty} \frac{1 - \widehat{d\mu}(\rho^{-1}u)}{1 - \widehat{d\mu}(\rho^{-1}a)}, \quad u \in \mathbb{R}^d,$$

as the convergence is uniform on every compact set. Then ψ *is continuous and positive-homogeneous of positive order.*

Proof. For a positive real σ and any $u \in \mathbb{R}^d$ we just have

$$\begin{aligned}\psi(\sigma u) &= \lim_{\rho\to\infty} \frac{1 - \widehat{d\mu}(\sigma\rho^{-1}u)}{1 - \widehat{d\mu}(\rho^{-1}a)} \\ &= \lim_{\rho\to\infty} \frac{1 - \widehat{d\mu}(\sigma\rho^{-1}u)}{1 - \widehat{d\mu}(\sigma\rho^{-1}a)} \lim_{\rho\to\infty} \frac{1 - \widehat{d\mu}(\sigma\rho^{-1}a)}{1 - \widehat{d\mu}(\rho^{-1}a)} \\ &= \sigma^\kappa \psi(u)\end{aligned}$$

with some $\kappa > 0$. At the last step we have applied Lemma 4.10 with $\tau(y) = 1 - \widehat{d\mu}(ya)$. ∎

5. Direct estimates for convolution operators

The result concerning the form of the Fourier transform of $\mathcal{D}_{\mu,\phi}$ as well as certain facts about the representation of a function as the Fourier transform of another allow us to strengthen Theorem 2.2 in two cases, which are important for the applications.

Theorem 5.1. *Let B be a HBS on $\mathbb{R}^d$ and $J_{\mu,\rho}$ be defined by* (6) *with $\mu \in M(\mathbb{R}^d)$, which satisfies* (7). *Let there hold the representation*

$$1 - \widehat{d\mu}(u) = \psi(u)\,\widehat{d\lambda_0}(u), \quad u \in \mathbb{B},$$

where $\mathbb{B} \subseteq \mathbb{R}^d$ is a neighbourhood of the origin, $\psi \in C(\mathbb{R}^d) \cap C^{n_d}(\mathbb{R}^d \backslash \{0\})$ is positive-homogeneous of order $\kappa > 0$, $\psi(u) \neq 0$ for $u \neq 0$, $\lambda_0 \in M(\mathbb{R}^d)$ and $\widehat{d\lambda_0}(0) \neq 0$. Finally, let $\phi : (0,\infty) \to \mathbb{C}\backslash\{0\}$ be such that $\lim_{\rho\to\infty} \rho^\kappa \phi(\rho) = -\widehat{d\lambda_0}(0)$. Then there exists a constant c such that for all $f \in B$ and $\rho > 0$ we have

$$\|f - J_{\mu,\rho} f\|_B \leq c\, K(f, \rho^{-\kappa}; B, \mathcal{D}_{\mu,\phi}).$$

P r o o f. The assertion follows from Theorem 2.2 with $\mathcal{D} = \mathcal{D}_{\mu,\phi}$. As we have already observed in Section 2.3, $\mathcal{D}_{\mu,\phi}$ satisfies condition (i) of Theorem 2.2. Property (iii) follows from Proposition 4.1 as we take into consideration that

$$\begin{aligned}\lim_{\rho\to\infty} \frac{\widehat{d\mu}(\rho^{-1}u) - 1}{\phi(\rho)} &= -\lim_{\rho\to\infty} \frac{\psi(\rho^{-1}u)\widehat{d\lambda_0}(\rho^{-1}u)}{\phi(\rho)} \\ &= -\psi(u) \lim_{\rho\to\infty} \frac{\widehat{d\lambda_0}(\rho^{-1}u)}{\rho^\kappa \phi(\rho)} = \psi(u).\end{aligned}$$

Further, using λ_0, Jan Boman constructed in [18, Appendix I, p. 259] a measure λ that satisfies (iv). Then by Proposition 4.2 and [18, Appendix I, Lemma 1(b)] we get that $\widehat{\mathfrak{D}} \subset \mathcal{D}_{\mu,\phi}^{-1}(L(\mathbb{R}^d))$, which verifies (ii).

Finally, (v) is included in the hypotheses of the present theorem. ■

Remark 5.2. The condition $\widehat{d\lambda_0}(0) \neq 0$ is quite natural. It provides the optimality of the exponent κ. The results in Section 3 yield that the best approximation order we can generally get is $|\widehat{d\mu}(\rho^{-1}a) - 1| = |\psi(\rho^{-1}a)\widehat{d\lambda_0}(\rho^{-1}a)|$, $a \neq 0$, and hence

$$\lim_{\rho\to\infty} \frac{|\widehat{d\mu}(\rho^{-1}a) - 1|}{\rho^{-\kappa}} = |\psi(a)\,\widehat{d\lambda_0}(0)|.$$

So, if $\widehat{d\lambda_0}(0) \neq 0$, then κ is the largest possible.

Combining Theorem 5.1 with $\phi(\rho) = -\widehat{d\lambda_0}(0)\rho^{-\kappa}$, Boman's result, cited in its proof, and Theorem 4.6 we arrive at the following optimal upper estimate of the error of the convolution operator.

Corollary 5.3. *Let B be a HBS on $\mathbb{R}^d$ and $J_{\mu,\rho}$ be defined by* (6) *with $\mu \in M(\mathbb{R}^d)$, which satisfies* (7)*. Let there hold the representation*

$$1 - \widehat{d\mu}(u) = \psi(u)\,\widehat{d\lambda_0}(u), \quad u \in \mathbb{B},$$

where $\mathbb{B} \subseteq \mathbb{R}^d$ is a neighbourhood of the origin, $\psi \in C(\mathbb{R}^d) \cap C^{n_d}(\mathbb{R}^d\backslash\{0\})$ is positive-homogeneous of order $\kappa > 0$, $\psi(u) \neq 0$ for $u \neq 0$, $\lambda_0 \in M(\mathbb{R}^d)$ and $\widehat{d\lambda_0}(0) \neq 0$. If $g \in B$ is such that $\psi\hat{g} = \widehat{G}$ for some $G \in B$, then there holds

$$\|g - J_{\mu,\rho} g\|_B \leq c\,\rho^{-\kappa}\|G\|_B \quad \forall \rho > 0,$$

where c is a constant, whose value is independent of g and ρ.

The last theorem and its corollary do not include the important for the applications case $\psi(u) = |u_1|^{\kappa_1} + \cdots + |u_d|^{\kappa_d}$ with $\kappa_1, \ldots, \kappa_d \geq 0$. To cover it, we establish the following assertion.

Theorem 5.4. *Let B be a HBS on $\mathbb{R}^d$ and $J_{\mu,\rho}$ be defined by* (6) *with $\mu \in M(\mathbb{R}^d)$, which satisfies* (7)*. Let there hold the representation*

$$1 - \widehat{d\mu}(u) = \psi(u)\,\widehat{d\lambda_0}(u), \quad u \in \mathbb{B},$$

where $\mathbb{B} \subseteq \mathbb{R}^d$ is a neighbourhood of the origin, $\lambda_0 \in M(\mathbb{R}^d)$ as $\widehat{d\lambda_0}(0) \neq 0$, $\psi \in C(\mathbb{R}^d) \cap C^d(\mathbb{R}^d_+)$ is even in each variable, positive-homogeneous of order $\kappa > 0$, $\psi(u) \neq 0$ for $u \neq 0$,

$$\lim_{u_{j+1}\to\infty} \frac{\partial^j}{\partial u_1 \ldots \partial u_j}\left(\frac{1}{\psi(u)}\right) = 0, \quad j = 0, \ldots, d-1,$$

and

$$\int_{\mathbb{R}^d_+\backslash\mathbb{B}'} \sup_{\substack{|v_j| \geq u_j,\\ 1\leq j\leq d}} \left|\frac{\partial^d}{\partial v_1 \ldots \partial v_d}\left(\frac{1}{\psi(v)}\right)\right| du < \infty$$

for some neighbourhood of the origin $\mathbb{B}' \subsetneq \mathbb{B}$. Finally, let $\phi : (0,\infty) \to \mathbb{C}\backslash\{0\}$ be such that $\lim_{\rho\to\infty} \rho^\kappa \phi(\rho) = -\widehat{d\lambda_0}(0)$. Then there exists a constant c such that for all $f \in B$ and $\rho > 0$ we have

$$\|f - J_{\mu,\rho} f\|_B \leq c\,K(f, \rho^{-\kappa}; B, \mathcal{D}_{\mu,\phi}).$$

P r o o f. The proof is similar to the one of Theorem 5.1. We need only to consider in greater detail the verification of conditions (ii) and (iv) of Theorem 2.2. Let $w \in \mathcal{S}$ be even in each variable, equal to 1 on $\mathbb{B}'$ and equal to 0 on $\mathbb{R}^d \backslash \mathbb{B}$. Following Jan Boman [18, Appendix I, p. 259], we make use of the representation

$$\begin{aligned}\frac{1-\widehat{d\mu}(u)}{\psi(u)} &= w(u)\frac{1-\widehat{d\mu}(u)}{\psi(u)} + (1-w(u))\frac{1-\widehat{d\mu}(u)}{\psi(u)} \\ &= w(u)\,\widehat{d\lambda_0}(u) + \frac{1-w(u)}{\psi(u)}\,(1-\widehat{d\mu}(u)), \quad u \in \mathbb{R}^d.\end{aligned}$$

Since

$$w(u)\,\widehat{d\lambda_0}(u) = \big((2\pi)^{-d}\widehat{\widehat{w}} * d\lambda_0\big)\widehat{\ }(u), \quad u \in \mathbb{R}^d,$$

to establish (iv) it is enough to show that $\theta(u) = (1-w(u))/\psi(u)$ is the Fourier-Stieltjes transform of a measure of $M(\mathbb{R}^d)$. To this end, we observe that

$$\theta(u_1,\ldots,u_d) = (-1)^d \int_{|u_1|}^{\infty} \cdots \left(\int_{|u_d|}^{\infty} \frac{\partial^d \theta(v_1,\ldots,v_d)}{\partial v_1 \ldots \partial v_d}\, dv_d \right) \ldots dv_1$$

and apply the criterion [20, Theorem 4.II] (cited in [14, Theorem D]).

To establish (ii) we again shall show that $\widehat{\mathfrak{D}} \subset \mathcal{D}^{-1}_{\mu,\phi}(L(\mathbb{R}^d))$ by means of Proposition 4.2. Let $g \in \widehat{\mathfrak{D}}$ and $w \in \mathfrak{D}$ be even in each variable and equal to 1 on the support of $\hat{g}$. Then $\psi\hat{g} = \psi w \hat{g}$. Again e.g. by [20, Theorem 4.II] we prove that $\psi w = \hat{h}$ with some $h \in L(\mathbb{R}^d)$. Then $\psi\hat{g} = \widehat{h * g}$ as $h * g \in L(\mathbb{R}^d)$. ∎

Seemingly the most specific condition in Theorems 2.2, 5.1 and 5.4 is the representation (iv) (either globally or locally). Also, it has played the most important role in our considerations. Butzer and König [5, Theorem 3] (or see also [6, Proposition 12.3.8]) showed for convolution operators on $L(\mathbb{R})$ with an absolutely continuous measure μ that if $\|f - J_{\mu,\rho}f\|_L = O(\rho^{-\kappa})$ for some $f \in L(\mathbb{R})$ such that there exists $\lim_{|u|\to\infty} |u|^\kappa \hat{f}(u) \neq 0$, then there exists $\lambda \in M(\mathbb{R})$ such that (iv) of Theorem 2.2 is valid with $\psi(u) = |u|^\kappa$. Repeating verbatim their argument and applying (4), one can extend this assertion to any HBS on $\mathbb{R}$ at least for measures of compact support. The same proof can actually be carried out in the multivariate case too as we observe that Phillips' generalization [15, Theorem 1] of Bochner's well-known representation theorem [1] (see also e.g. [6, Theorems 6.2.1 and 6.2.2]) can be extended to measurable functions on $\mathbb{R}^d$. As a matter of fact a much stronger result of Rosenthal's [16] yields this generalization. Thus we arrive at the following assertion about the necessity of condition (iv).

Theorem 5.5. *Let B be a HBS on $\mathbb{R}^d$ and $J_{\mu,\rho}$ be defined by* (6) *with $\mu \in M(\mathbb{R}^d)$, which satisfies* (7) *and has compact support. Let there exist $g_0 \in L(\mathbb{R}^d)$ of compact support such that*

$$\|g_0 - J_{\mu,\rho} g_0\|_B = O(\rho^{-\kappa})$$

and there exists (as a finite number)

$$\lim_{|u|\to\infty} \psi(u)\hat{g}_0(u) \neq 0,$$

where $\psi \in C(\mathbb{R}^d)$ is positive-homogeneous of order $\kappa > 0$ and $\psi(0) = 0$ iff $u = 0$. Then there exists $\lambda \in M(\mathbb{R}^d)$ such that

$$1 - \widehat{d\mu}(u) = \psi(u)\,\widehat{d\lambda}(u), \quad u \in \mathbb{R}.$$

6. Generalized multivariate singular integral of Picard

We finish by illustrating the established general results on two versions of the Picard singular integral for multivariate functions.

Let B be HBS on $\mathbb{R}^d$. The generalized multivariate singular integral of Picard of the function $f \in B$ is defined by

$$C_{\kappa,\rho} f(x) = \rho^d \int_{\mathbb{R}^d} c_{d,\kappa}(\rho y) f(x-y)\,dy, \quad x \in \mathbb{R}^d,$$

where the kernel $c_{d,\kappa} \in L(\mathbb{R}^d)$, $\kappa > 0$, is given by its Fourier transform

$$\hat{c}_{d,\kappa}(u) = \frac{1}{1+|u|^\kappa}, \quad u \in \mathbb{R}^d.$$

The operator $C_{\kappa,\rho}$ is of the form (6) with $\mu(y) = c_{d,\kappa}(y)\,dy$. Let $a \in \mathbb{R}^d$ with $|a| = 1$. Set

$$\phi(\rho) = \hat{c}_{d,\kappa}(\rho^{-1}a) - 1 = -\frac{\rho^{-\kappa}}{1+\rho^{-\kappa}},$$

hence $|\phi(\rho)| \sim \rho^{-\kappa}$ for $\rho \geq 1$. We write that two functions are $\sim$ iff their ratio is uniformly bounded between two positive constants.

We have for each $u \in \mathbb{R}^d$

$$\psi(u) = \lim_{\rho\to\infty} \frac{\hat{c}_{d,\kappa}(\rho^{-1}u) - 1}{\phi(\rho)} = |u|^\kappa.$$

Further, we calculate

$$1-\widehat{d\mu}(u)=\frac{|u|^{\kappa}}{1+|u|^{\kappa}}=\psi(u)\,\widehat{d\mu}(u),\quad u\in\mathbb{R}^d. \tag{1}$$

Now, Theorem 5.1 with $\lambda_0=\mu$, $\mathbb{B}=\mathbb{R}^d$ and ψ and ϕ as set above implies that there exists an absolute constant c such that

$$\|f-C_{\kappa,\rho}f\|_B\le c\,K(f,\rho^{-\kappa};B,\mathcal{D}_{\mu,\phi}) \tag{2}$$

for any $f\in B$ and $\rho>0$.

Let us note that (1) means that (cf. [9, Remark 4.5])

$$g-C_{\kappa,\rho}g=\rho^{-\kappa}C_{\kappa,\rho}\mathcal{D}_{\mu,\phi}g,\quad g\in\mathcal{D}^{-1}_{\mu,\phi}(B),\quad \rho>0,$$

through which it can be shown that (see [9, Remark 3.9]) the converse of (2) also holds, that is,

$$\|f-C_{\kappa,\rho}f\|_B\ge c\,K(f,\rho^{-\kappa};B,\mathcal{D}_{\mu,\phi})$$

with some absolute positive constant c. Thus, we have established the following characterization of this multivariate form of the generalized singular integral of Picard.

Theorem 6.1. *Let B be a HBS on $\mathbb{R}^d$. Then for $f\in B$ and $\rho>0$ there holds*

$$\|f-C_{\kappa,\rho}f\|_B\sim K(f,\rho^{-\kappa};B,\mathcal{D}_{\mu,\phi}),$$

where $\mu(y)=c_{d,\kappa}(y)\,dy$.

Given any functions $k_1,\dots,k_d$, which are summable on $\mathbb{R}$ as

$$\int_{\mathbb{R}}k_j(y)\,dy=1,\quad j=1,\dots,d,$$

we can define a multivariate convolution operator J_ρ on a HBS on $\mathbb{R}^d$ by

$$J_\rho f(x)=\rho^d\int_{\mathbb{R}^d}\prod_{j=1}^{d}k_j(\rho y_j)f(x-y)\,dy,\quad x\in\mathbb{R}^d.$$

In particular, we can define a multivariate form of the generalized singular integral of Picard by setting

$$\widetilde{C}_{\kappa,\rho}f(x)=\rho^d\int_{\mathbb{R}^d}\prod_{j=1}^{d}c_{1,\kappa}(\rho y_j)f(x-y)\,dy,\quad x\in\mathbb{R}^d.$$

If we put

$$d\mu(y) = \prod_{j=1}^{d} c_{1,\kappa}(y_j)\,dy,$$

then $\widetilde{C}_{\kappa,\rho}$ adopts the form (6). The Fourier transform of μ is

$$\widehat{d\mu}(u) = \prod_{j=1}^{d} \frac{1}{1+|u_j|^{\kappa}}.$$

Let $a = (1, 0, \dots, 0) \in \mathbb{R}^d$. We have again $\phi(\rho) = \widehat{d\mu}(\rho^{-1}a) - 1 = -\rho^{-\kappa}/(1+\rho^{-\kappa})$ and $|\phi(\rho)| \sim \rho^{-\kappa}$ for $\rho \geq 1$. Further, we calculate

$$\psi(u) = \lim_{\rho\to\infty} \frac{\widehat{d\mu}(\rho^{-1}u) - 1}{\phi(\rho)} = |u_1|^{\kappa} + \dots + |u_d|^{\kappa}.$$

Similarly to the first case but by means of Theorem 5.4 we derive the following upper estimate.

Theorem 6.2. *Let B be a HBS on $\mathbb{R}^d$. Then for $f \in B$ and $\rho > 0$ there holds*

$$\|f - \widetilde{C}_{\kappa,\rho} f\|_B \leq c\, K(f, \rho^{-\kappa}; B, \mathcal{D}_{\mu,\phi}),$$

where c is an absolute constant and

$$d\mu(y) = \prod_{j=1}^{d} c_{1,\kappa}(y_j)\,dy.$$

Relation (6) holds for the differential operators $\mathcal{D}_{\mu,\phi}$ associated with the two versions of the Picard operators defined above for even κ's. In Theorem 4.4 one can set $n_{k,j} = \max\{\ell - [k/2], 0\}$, where $\kappa = 2\ell$.

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