ON SOME INTERPOLATORY PROPERTIES OF LEGENDRE AND ULTRASPHERICAL POLYNOMIALS — II

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1. In 1907, Birkhoff treated a general problem on Interpolation and considered a system of pairs of numbers (K_i, x_i) $(i=1, 2, \ldots, n)$ where K_i are integers $\geqslant 0$ and x_i are any points in a given interval. A particular case $n=2$ was treated directly by Pólya. Moreover most of the known quadrature and interpolation formulae, viz. Lagrange's interpolation formula, Gauss-Jacobi quadrature formula — are in some way or the other particular cases of this more general formula of Birkhoff. In most of the problems of Interpolation we prescribe the values of the function at each point x_ν and also some of its consecutive derivatives there. In Lagranges interpolation formula we prescribe the values of the function at some points in a given interval while in Hermite interpolation formula the values of the function and its first derivatives at those points in a given interval are prescribed. Tchakaloff has obtained a formula for the remainder in the case where the values of the function at some points in a given interval and its $m-1$ consecutive derivatives at those points are prescribed. But the case where the prescribed derivatives are not consecutive has not been treated in general except by Birkhoff. Turán has remarked "that his point of view was so general that one cannot expect better formulae than those of Hermite".

In two of his recent papers Turán has considered the simplest case of what he calls (0,2) interpolation, where the value of the function and its second derivatives are given at some points. He considers the problem of their existence and uniqueness and also the problem of their explicit representation. He has also promised a study of the problem of convergence.

In a previous paper Dr. A. Sharma and myself have considered the (0, 1, 3) interpolation where the value of the function, its first and third derivatives are prescribed at some points. We have considered the problem of existence, uniqueness and problem of explicit representation of those polynomials.

In this paper we propose to extend the programme of previous paper with (0, 1, 2, 4) interepolation where the values of the function, its first, second and fourth derivatives are given at the same points. The general case $(0, 1, 2, \ldots, m-1, m+1)$ interpolation can similarly be attacked but already for $m=3$, the computations become cumbersome.

2. We seek a polynomial $f_{4n-1}(x)$ of degree $\leqslant 4n-1$, where the value of the function, its first, second and fourth derivatives are prescribed at the n points x_ν, where

(2.1) $$-1 \leqq x_n < x_{n-1}, \ldots, x_1 \quad 1$$

that is we are given (say):

(2.2) $$f_{4n-1}^{(k)}(x_\nu) = y_{\nu,k}, \quad (k = 0, 1, 2, 4; \quad \nu = 1, 2, 3, \ldots, n).$$

In order to simplify the proof, we choose the points (2.1) to be the same as those chosen by Turán and his associates for the case of (0, 2) interpolation, as the zeros

(2.3) $$-1 = \xi_n < \xi_{n-1} < \quad < \xi_1 = 1$$

of the polynomials

(2.4) $$\pi_n(x) = (1-x^2)\, P'_{n-1}(x)$$

where $P_{n-1}(x)$ is the $(n-1)$ th Legendre polynomial with $P_{n-1}(1) = 1$.

3. Let $n = 2k+1$, and let

(3.1) $$1 \geqq x_1 > x_2 > \cdots x_k > x_{k+1} = 0 > x_{k+2} > \quad > x_{2k+1} \geqq -1$$

with

(3.2) $$x_j = -x_{2k+2-j} \quad (j = k+2,\ k+3, \ldots,\ 2k+1).$$

We shall prove

Theorem I. If $n = 2k+1$ and the points $x_1, x_2, \ldots, x_n$ satisfy (3.1) and (3.2) there is in general no polynomial $f(x)$ of degree $4n-1$, such that for given $y_{\nu,k}$ $(k = 0, 1, 2, 4)$,

(3.3) $$f^{(k)}(x_\nu) = y_{\nu,k} \quad (k = 0, 1, 2, 4; \quad \nu = 1, 2, \ldots, n).$$

If there exists one such polynomial, then there is an infinity of them.

Thus in the case of an odd number of distinct symmetrical points x_ν, both the problems of existence and uniqueness have a negative solution.

Taking x_ν as the points (2.3) we shall show that in the case of infinitely many solutions in Theorem I for $n \geqq 3$, the general form of the solution is

(3.4) $$f(x) = f_0(x) + c f_1(x),$$

where $f_0(x)$ and $f_1(x)$ are fixed polynomials of degres $4n-1$ and c is an arbitrary complex number.

4. In the case of n even $(=2k)$ the situation changes. We shall show the

Theorem II. If $n = 2k$, then to prescribed values $y_{\nu,k}$ $(k = 0, 1, 2, 4)$ there is a uniquely determined polynomial $f(x)$ of degree $4n-1$, such that

(4.1) $$f^{(k)}(\xi_\nu) = y_{\nu,k} \quad (k = 0, 1, 2, 4;\ \nu = 1, 2, \ldots, n)$$

where ξ_ν stands as in (2.3) for the zeros of $\pi_n(x)$. This means, of course, that in the case

(4.2) $$y_{\nu,0} = y_{\nu,1} = y_{\nu,2} = y_{\nu,4} = 0 \quad (\nu = 1, 2, \ldots, n)$$

n even, the only solution of (4.1) is $f(x) \equiv 0$. This result is curious because the polynomial $Q_{3n}(x)=\pi_n^3(x)$ has the property

$$Q_{3n}(\xi_\nu)=Q'_{3n}(\xi_\nu)=Q''_{3n}(\xi_\nu)=0,$$

but $Q_{3n}^{IV}(\xi_\nu)=0$, for $\nu=2, 3, \ldots, n-1$, i. e., there is a non-trivial polynomial of degree $4n$ satisfying almost the requirements (4.2) i. e., all except the two conditions $f^{IV}(\xi_1)=f^{IV}(\xi_n)=0$. Theorem II shows that for n even, there is no polynomial of degree $4n-1$ satisfying condition (4.2) except $f(x)\equiv 0$.

5. If we replace in the interpolation problem (2.2) the points x_ν by the zeros η_ν^λ of the nth Ultraspherical polynomial $P_n^{(\lambda)}(x)$ with $\lambda > -1/2$ (for $\lambda = -1/2$, we come back to the case of Theorem II) which satisfies the differential equation

$$(1-x^2)y''-(2\lambda+1)xy'+n(n+2\lambda)y=0 \tag{5.1}$$

with

$$P_n^{(\lambda)}(1)=\binom{n+\lambda+1}{n} \quad \text{for } \lambda \neq 0$$
$$= \quad 1 \qquad \text{for } \lambda = 0.$$

Then by the method used by Turán and Surányi, we get the following

Theorem III. If the polynomials $g(x)$ of degree $4n-1$ satisfy the condition $(\lambda > -1/2)$

$$g^{(i)}(\eta_{\nu,i}^\lambda)=0, \qquad i=0,\ 1,\ 2,\ 4,$$

for $\nu=0, 1, \ldots, n$, then the following cases occur:

(a) If $n\geqslant 4$, $n \neq 3\lambda+{}^7/_2$, $3\lambda+{}^5/_2 \neq$ odd integer, then

$$g(x)\equiv 0$$

(b) If $n=$ even and $\geqslant 3\lambda+{}^7/_2$, $n\geqslant 4$, $3\lambda+{}^5/_2=$ odd integer, then $g(x)=cK(x)$ where c is arbitrary and $K(x)$ is of degree $3n+3\lambda+{}^3/_2$.

(c) If $3\lambda+{}^5/_2=$ odd integer, $n=$ even and $\leqslant 3\lambda+{}^3/_2$, then $g(x)\equiv 0$. The proof of this theorem is omitted as it can be easily carried out on the same pattern as that of the corresponding theorem of Turán.

6. We now prove Theorem I. We decompose the polynomial

$$f(x)=\sum_{\nu=0}^{4n-1} c_\nu x^\nu$$

into an even and an odd part:

$$s(x)=\sum_{\nu=0}^{4k+1} c_{2\nu}x^{2\nu}; \quad t(x)=\sum_{\nu=0}^{4k+1} c_{2\nu+1}x^{2\nu+1}.$$

The first part of the condition (2.2) will then give

$$(6.1)\qquad \left.\begin{aligned} f(x_\nu) &= s(x_\nu)+t(x_\nu)=y_{\nu,0}\\ f(-x_\nu) &= s(x_\nu)-t(x_\nu)=y_{n-\nu+1,0}\end{aligned}\right\}\ \nu=1,2,\dots,k$$

since $s'(x)$ and $t'(x)$ are odd and even polynomials respectively, from the second part of the condition follows:

$$(6.2)\qquad \left.\begin{aligned} f'(x_\nu) &= s'(x_\nu)+t'(x_\nu)=y_{\nu,1}\\ f'(-x_\nu) &= -s'(x_\nu)+t'(x_\nu)=y_{n-\nu+1,1}\end{aligned}\right\}\ \nu=1,2,\dots,k$$

since $s''(x)$ and $t''(x)$ are now even and odd polynomials respectively, from the third part of the condition follows:

$$(6.3)\qquad \left.\begin{aligned} f''(x_\nu) &= s''(x_\nu)+t''(x_\nu)=y_{\nu,2}\\ f''(-x_\nu) &= s''(x_\nu)-t''(x_\nu)=y_{n-\nu+1,2}\end{aligned}\right\}\ \nu=1,2,\dots,k$$

the fourth condition gives

$$(6.4)\qquad \left.\begin{aligned} f^{IV}(x_\nu) &= s^{IV}(x_\nu)+t^{IV}(x_\nu)=y_{\nu,4}\\ f^{IV}(-x_\nu) &= s^{IV}(x_\nu)-t^{IV}(x_\nu)=y_{n-\nu+1,4}\end{aligned}\right\}\ \nu=1,2,\dots,k.$$

From (6.1), (6.2), (6.3) and (6.4) we get for $t(x)$

$$\left.\begin{aligned} t(x_\nu) &= \frac{1}{2}(y_{\nu,0}-y_{n-\nu+1,0})\\ t'(x_\nu) &= \frac{1}{2}(y_{\nu,1}+y_{\nu-n+1,1})\\ t''(x_\nu) &= \frac{1}{2}(y_{\nu,2}-y_{n-\nu+1,2})\\ t^{IV}(x_\nu) &= \frac{1}{2}(y_{\nu,4}-y_{n-\nu+1,4})\end{aligned}\right.\qquad \nu=1,2,\dots,k$$

and of course

$$(6.6)\qquad t(0)=0,\quad t'(0)=c_1,\quad t''(0)=0,\quad t^{IV}(0)=0.$$

The coefficients to be determined are $4k+2$ out of which one is obtained from (6.6). For the remaining $4k+1$ coefficients we have $4k$ equations which is one less than the number of coefficients to be determined. Hence there are infinitely many solutions.

For $s(x)$ the equations (6.1), (6.2), (6.3) and (6.4) give

$$(6.7)\qquad \left.\begin{aligned} s(x_\nu) &= \frac{1}{2}(y_{\nu,0}+y_{n-\nu+1,0})\\ s'(x_\nu) &= \frac{1}{2}(y_{\nu,1}-y_{n-\nu+1,1})\\ s''(x_\nu) &= \frac{1}{2}(y_{\nu,2}+y_{n-\nu+1,2})\\ s^{IV}(x_\nu) &= \frac{1}{2}(y_{\nu,4}+y_{n-\nu+1,4})\end{aligned}\right.\qquad \nu=1,2,\dots,k$$

and

(6.8) $$s(0) = c_0, \quad s'(0) = 0, \quad s''(0) = c_2, \quad s^{(IV)}(0) = 24c_4.$$

The coefficients to be determined are $4k+2$, out of which 3 are determined from (6.8) and for the remaining $4k-1$ coefficients are furnished $4k$ equations, which is one more than the number of coefficients to be determined.

7. As stated in § 3, we shall consider with an odd n, in the case of $x_\nu = \xi_\nu$, the general structure of the solution. The assertion is obviously proved if we show that in our case all polynomials $f(x)$ of degree $\leqslant 4n-1$ satisfying

(7.1) $$f^{(k)}(\xi_\nu) = 0 \qquad (k=0, 1, 2, 4; \quad \nu=1, 2, \ldots, n)$$

are

(7.2) $$c\pi_n^3(x)(P_{n-1}(x) - {}^5/_3)$$

with arbitrary numerical c. If $f(x)$ satisfies (7.1), the first three conditions lead us to write

(7.3) $$f(x) = \pi_n^3(x)\, q_{n-1}(x)$$

where $q_{n-1}(x)$ is a polynomial of degree $\leqslant n-1$. But the last part of the condition gives from $f^{IV}(\xi_\nu) = 0$, $(\nu = 1, 2, \ldots, n)$

(7.4) $$24\pi_n'^3(\xi_\nu)\, q'_{n-1}(\xi_\nu) + 36\pi_n'^2(\xi_\nu)\,\pi_n''(\xi_\nu)\, q_{n-1}(\xi_\nu) = 0.$$

The differential equation satisfied by

(7.5) $$\pi_n(x) = (1-x^2)\, P'_{n-1}(x)$$

is

(7.6) $$(1-x^2)\,\pi_n''(x) + n(n-1)\,\pi_n(x) = 0 \qquad n \geqslant 2.$$

This gives

(7.6a) $$\pi_n''(\xi_\nu) = 0, \quad (\nu=2, 3, \ldots, n-1).$$

Therefore since ξ_ν are simple zeros of $\pi_n(x)$, (7.4) gives

(7.7) $$q'_{n-1}(\xi_\nu) = 0, \quad (\nu = 2, 3, \ldots, n-1).$$

But this means that $q'_{n-1}(x)$ has all its zeros common with $P'_{n-1}(x)$ i. e., $q'_{n-1}(x) = cP'_{n-1}(x)$ with a numerical c. Hence if $c \neq 0$,

$$q_{n-1}(x) = c\,(P_{n-1}(x) + c_1)$$

that is

(7.8) $$f(x) = c\pi_n^3(x)\,(P_{n-1}(x) + c_1).$$

To determine c and c_1, we shall make use of $f^{IV}(\pm 1) = 0$. We shall now require the following results which are known and easy to verify.

$$P'_{n-1}(1) = \frac{1}{2}n(n-1) = (-1)^n P'_{n-1}(-1) \tag{7.9}$$

$$\pi'_n(1) = -n(n-1) = (-1)^{n-1}\pi'_n(-1) \tag{7.10}$$

$$\pi''_n(1) = -\frac{1}{2}n^2(n-1)^2 = (-1)^n \pi''_n(-1) \tag{7.11}$$

$$\pi'''_n(1) = -\frac{1}{8}n^2(n-1)^2(n-2)(n+1) = (-1)^{n-1}\pi'''_n(-1) \tag{7.12}$$

$$\pi^{\mathrm{IV}}_n(1) = \frac{(n-3)(n+2)}{6}\pi'''_n(1), \quad \pi^{\mathrm{IV}}_n(-1) = -\frac{(n-3)(n+2)}{6}\pi'''_n(-1). \tag{7.13}$$

The requirement $f^{\mathrm{IV}}(1) = 0$, then gives with the help of (7.9), (7.10) and (7.11) $c_1 = -5/3$. Hence

$$f(x) = c\pi^3_n(x)(P_{n-1}(x) - 5/3). \tag{7.14}$$

For n odd it is easy to verify with (7.9), (7.10) and (7.11) that the polynomial (7.14) satisfies also the condition $f^{\mathrm{IV}}(-1) = 0$. If $c = 0$, then $q_{n-1}(x) =$ constant, i. e., $f(x)$ would be $c\pi^3_n(x)$ with numerical c; but then $f^{\mathrm{IV}}(1) \neq 0$. Hence only the polynomials (7.14) fulfil our requirement.

8. The proof of Theorem II in the case of n even runs parallel to that of Theorem I. In the case of n even, it is proved also that if

$$f^{(k)}(\xi_\nu) = 0, \quad (k = 0, 1, 2, 4) \tag{8.1}$$

then $f(x)$ has necessarily the form (7.14) with numerical c, without using the requirement $f^{\mathrm{IV}}(-1) = 0$.

For even n, it can be seen with the help of formulae (7.9), (7.10) and (7.11) that

$$\left[\frac{d^4}{dx^4}\left\{\pi^3_n(x)\left(P_{n-1}(x) - \frac{5}{3}\right)\right\}\right] \neq 0. \qquad x = -1$$

Hence $c = 0$, i. e.,

$$f(x) \quad 0. \tag{8.2}$$

Now this means that writing out (8.1) as a linear system, the determinant is not zero. Considering the general problem

$$f^{(k)}(\xi_\nu) = y_{\nu,k} \quad (k = 0, 1, 2, 4; \quad \nu = 1, 2, \ldots, n) \tag{8.3}$$

this shows that corresponding linear system is always uniquely soluble, and the Theorem II is proved.

9. Explicit representation of the Interpolatory Polynomials: We now consider the following problem of (0, 1, 2, 4) interpolation:

Given $n(=2k)$ distinct points $\xi_1, \xi_2, \ldots, \xi_n$ satisfying (2.3) and arbitrary $a_1, a_2, \ldots, a_n$; $b, b_2, \ldots, b_n$; $c_1, c_2, \ldots, c_n$; $d_1, d_2, \ldots, d_n$ we want to find explicit form of the polynomial $R_n(x)$ of degree $\leqslant 4n-1$ such that

(9.1) $$R_n(\xi_\nu)=a_\nu;\quad R_n'(\xi_\nu)=b_\nu;\quad R_n''(\xi_\nu)=c_\nu;\quad R_n^{\mathrm{IV}}(\xi_\nu)=d_\nu \quad (\nu=1,2,\ldots,n).$$

The existence and uniqueness has already been proved in Theorem II. For convenience we select the numbers ξ_ν as n different real x-Zeros

(9.2) $$-1=x_n<x_{n-1}\ldots x_2<x_1=1$$

of the polynomial $\pi_n(x)$.

For $R_{2k}(x)$ we have evidently the form

(9.3) $$R_{2k}(x)=\sum_{\nu=1}^{2k}a_\nu A_\nu(x)+\sum_{\nu=1}^{2k}b_\nu B_\nu(x)+\sum_{\nu=1}^{2k}c_\nu C_\nu(x)+\sum_{\nu=1}^{2k}d_\nu D_\nu(x),$$

where the polynomials $A_\nu(x)$, $B_\nu(x)$, $C_\nu(x)$ and $D_\nu(x)$, the fundamental polynomials of the first, second, third and fourth kind of (0, 1, 2, 4) interpolation belonging to the x_ν-points respectively, are polynomials of degree $\leqslant 4n-1=8k-1$, uniquely determined by the following requirements:

(9.4) $$A_\nu(x_j)=\begin{matrix}1\\0\end{matrix}\ \text{for}\ \left.\begin{matrix}j=\nu\\j\neq\nu\end{matrix}\right\},\quad A_\nu'(x_j)=A_\nu''(x_j)=A_\nu^{\mathrm{IV}}(x_j)=0$$

(9.5) $$B_\nu(x_j)=0,\quad B_\nu'(x_j)=\begin{matrix}1\\0\end{matrix}\ \text{for}\ \left.\begin{matrix}j=\nu\\j\neq\nu\end{matrix}\right\},\quad B_\nu''(x_j)=B_\nu^{\mathrm{IV}}(x_j)=0$$

(9.6) $$C_\nu(x_j)=C_\nu'(x_j)=0,\quad C_\nu''(x_j)=\begin{matrix}1\\0\end{matrix}\ \text{for}\ \left.\begin{matrix}j=\nu\\j\neq\nu\end{matrix}\right\},\quad C_\nu^{\mathrm{IV}}(x_j)=0$$

(9.7) $$D_\nu(x_j)=D_\nu'(x_j)=D_\nu''(x_j)=0,\quad D_\nu^{\mathrm{IV}}(x_j)=\begin{matrix}1\\0\end{matrix}\ \text{for}\ \left.\begin{matrix}j=\nu\\j\neq\nu\end{matrix}\right\},$$

respectively, where $j=1,2,3,\ldots,n$. In what follows we shall explicitly determine these fundamental polynomials $A_\nu(x)$, $B_\nu(x)$, $C_\nu(x)$ and $D_\nu(x)$.

10. We shall denote by $l_\nu(x)$ the fundamental polynomial of Lagrange Interpolation, i. e.,

(10.1) $$l_\nu(x)=\frac{\pi_n(x)}{(x-x_\nu)\pi_n'(x_\nu)}.$$

We shall make use of an observation of Fejer that

(10.2) $$l_\nu'(x_\nu)=0 \quad (\nu=2,3,\ldots,n-1).$$

Further, we have

(10.3) $$l_\nu(x_j)=\begin{matrix}1\\0\end{matrix}\ \text{for}\ \left.\begin{matrix}j=\nu\\j\neq\nu\end{matrix}\right\}\qquad (j=1,2,\ldots,n).$$

We shall also require the following results which are easy to verify:

(10.4) $$(1-t^2)(t-x_\nu)\,l_\nu''(t)+2(1-t^2)\,l_\nu'(t)+n(n-1)(t-x_\nu)\,l_\nu(t)=0$$

For 2 ν $n-1$,

$$l''_\nu(x_j) = -\frac{2l'_\nu(x_j)}{x_j - x_\nu} \tag{10.5}$$

$$l'_\nu(1) = -\frac{n(n-1)}{(1-x_\nu)\,\pi'_n(x_\nu)}, \quad l'_\nu(-1) = (-1)^{n+1}\frac{n(n-1)}{(1+x_\nu)\,\pi'_n(x_\nu)} \tag{10.6}$$

$$l''_\nu(1) = \left(-\frac{2}{1-x_\nu} + \frac{n(n-1)}{2}\right) l'_\nu(1), \quad l''_\nu(-1) = \left(-\frac{2}{1-x_\nu} - \frac{n(n-1)}{2}\right) l'_\nu(-1) \tag{10.7}$$

$$l'''_\nu(1) = \frac{1}{1-x_\nu}\left[\frac{\pi'''_n(1)}{\pi'_n(x_\nu)} - 3l''_\nu(1)\right], \quad l'''_\nu(-1) = \frac{1}{1-x_\nu}\left[3l''_\nu(-1) - \frac{\pi'''_n(-1)}{\pi'_n(x_\nu)}\right] \tag{10.8}$$

and

$$l'_1(1) = \frac{n(n-1)}{4}, \qquad l'_1(-1) = \frac{(-1)^n}{2} \tag{10.9}$$

$$l''_1(1) = \frac{(n-2)(n+1)}{6}\, l'_1(1), \qquad l''_1(-1) = -\frac{(n-2)(n+1)}{2}\, l'_1(-1) \tag{10.10}$$

$$l'''_1(1) = \frac{(n-3)(n+2)}{8}\, l''_1(1), \qquad l'''_\nu(-1) = -\frac{(n-3)(n+2)}{4}\, l''_1(-1) \tag{10.11}$$

$$l^{\mathrm{IV}}_1(1) = \frac{(n-4)(n+3)}{10}\, l'''_1(1), \qquad l^{\mathrm{IV}}_1(-1) = -\frac{(n-4)(n+3)}{6}\, l'''_1(-1) \tag{10.12}$$

$$l'_n(1) = \frac{(-1)^{n+1}}{2}, \qquad l'_n(-1) = -\frac{n(n-1)}{4} \tag{10.13}$$

$$l''_n(1) = \frac{(n-2)(n+1)}{2}\, l'_n(1), \qquad l''_n(-1) = -\frac{(n-2)(n+1)}{6}\, l'_n(-1) \tag{10.14}$$

$$l'''_n(1) = \frac{(r-3)(n+2)}{4}\, l''_n(1), \qquad l'''_n(-1) = -\frac{(n-3)(n+2)}{8}\, l''_n(-1) \tag{10.15}$$

$$l^{\mathrm{IV}}_n(1) = \frac{(n-4)(n+3)}{6}\, l'''_n(1), \qquad l^{\mathrm{IV}}_n(-1) = -\frac{(n-4)(n+3)}{10}\, l'''_n(-1) \tag{10.16}$$

11. Theorem IV. The fundamental polynomials $A_\nu(x)$, $B_\nu(x)$, $C_\nu(x)$ and $D_\nu(x)$ are given by the following:

(a)

$$D_1(x) = -\frac{\pi_n^3(x)}{60n^4(n-1)^4}\left(P_{n-1}(x) + \frac{5}{3}\right) \tag{11.1}$$

$$D_n(x) = \frac{\pi_n^3(x)}{60n^4(n-1)^4}\left(P_{n-1}(x) - \frac{5}{3}\right) \tag{11.2}$$

and for $2 \leqslant \nu \leqslant n-1$,

(11.3)
$$D_\nu(x) = \frac{\pi_n^3(x)}{24P''_{n-1}(x_\nu)\pi_n'^3(x_\nu)}\left\{\int_{-1}^{x}\frac{P'_{n-1}(t)}{t-x_\nu}\,dt - \left[\frac{x_\nu}{1-x_\nu^2} - \frac{3}{5(1-x_\nu^2)P_{n-1}(x_\nu)}\right]P_{n-1}(x) - \frac{2+3x_\nu}{3(1-x_\nu^2)} + \frac{1}{(1-x_\nu^2)P_{n-1}(x_\nu)}\right\}.$$

(b) Denoting $\pi_n^2(x)$ by $Q_{2n}(x)$

(11.4)
$$C_1(x) = \frac{Q_{2n}(x)}{Q''_{2n}(1)}\left[r_1(x) + \left(\frac{5}{72} - \frac{1}{18n(n-1)}\right)\pi_n(x) - \frac{Q'_{2n}(x)}{16n(n-1)}\right]$$

where

(11.5)
$$r_1(x) = \frac{3+x}{4}\,l_1^2(x) - \frac{1-x^2}{4}\,l_1(x)\,l_1'(x) + \left(\frac{5}{16} + \frac{1}{8n(n-1)}\right)\pi_n(x)\left[1 + \frac{1}{3}P_{n-1}(x)\right]$$

(11.6)
$$C_n(x) = \frac{Q_{2n}(x)}{Q''_{2n}(-1)}\left[r_n(x) + \frac{1}{18n(n-1)} - \frac{5}{72}\right)\pi_n(x) + \frac{Q'_{2n}(x)}{16n(n-1)}\right]$$

where

(11.7)
$$r_n(x) = \frac{3-x}{4}\,l_n^2(x) + \frac{1-x^2}{4}\,l_n(x)\,l_n'(x) + \left(\frac{5}{16} + \frac{1}{8n(n-1)}\right)\pi_n(x)\left[1 - \frac{1}{3}P_{n-1}(x)\right]$$

and for $2 \leqslant \nu \leqslant n-1$,

(11.8)
$$C_\nu(x) = \frac{Q_{2n}(x)}{Q''_{2n}(x_\nu)}\left[r_\nu(x) + \frac{2n(n-1)}{3(1-x_\nu^2)}\,\varrho_\nu(x) + \alpha'_\nu\,\pi_n(x) + \alpha''_\nu\,Q'_{2n}(x)\right]$$

where

(11.9)
$$\alpha'_\nu = \frac{4}{9n(n-1)(1-x_\nu^2)P^2_{n-1}(x_\nu)}$$

(11.10)
$$\alpha''_\nu = \frac{1}{9n^2(n-1)^2(1-x_\nu^2)P^3_{n-1}(x_\nu)}$$

(11.11)
$$r_\nu(x) = l_\nu^2(x) + \frac{\pi_n(x)}{2n(n-1)P_{n-1}(x_\nu)}\left[\int_{1}^{x}\frac{l'_\nu(t)}{t-x_\nu}\,dt + \int_{-1}^{x}\frac{l'_\nu(t)}{t-x_\nu}\,dt + \frac{P_{n-1}(x)}{3(1-x_\nu^2)P^2_{n-1}(x_\nu)}\right],$$

and

(11.12)
$$\varrho_\nu(x) = \frac{\pi_n(x)}{4n^2(n-1)^2P^2_{n-1}(x_\nu)}\left\{(1-x_\nu^2)\int_{1}^{x}\frac{P'_{n-1}(t)}{t-x_\nu}\,dt + \right.$$

$$+(1-x_\nu^2)\int_{-1}^{x}\frac{P'_{n-1}(t)}{t-x_\nu}\,dt+2P_{n-1}(x)\Big[-x_\nu+\frac{1}{3P_{n-1}(x_n)}\Big]-4\Big\}.$$

(c)

(11.13) $$B_1(x)=\frac{\pi_n(x)}{\pi_n'(x)}[u_1(x)+\beta_1 w_1(x)+\beta_1' w_n(x)]-\frac{\pi_n''(1)}{\pi_n'(1)}C_1(x)$$

where

(11.14) $$\beta_1=\frac{n(n-1)}{576}(71n^4-142n^3+151n^2-80n-12)$$

(11.15) $$\beta_1'=\frac{n(n-1)}{576}(73n^4-146n^3+137n^2-64n+12)$$

(11.16) $$w_1(x)=\frac{Q_{2n}(x)}{6n^3(n-1)^3}\left(1+\frac{1}{2}P_{n-1}(x)\right)$$

(11.17) $$w_n(x)=-\frac{Q_{2n}(x)}{6n^3(n-1)^3}\left(1-\frac{1}{2}P_{n-1}(x)\right)$$

(11.18) $$u_1(x)=l_1^3(x)+\frac{\pi_n(x)}{3\pi_n'(1)}[l_1(x)\,l_1'(x)-10l_1'(1)\,r_1(x)]+$$
$$+k_1 w_1(x)+k_1' w_n(x)$$

$$k_1=-\frac{n(n-1)}{32}(n^4-2n^3-17n^2+18n+8)$$

$$k_1'=\frac{n(n-1)}{96}(25n^4-50n^3+35n^2-10n-48)$$

and $c_1(x)$ is a polynomial of degree $\leqslant 4n-1$, given by (11.4) and (11.5).

(11.19) $$B_n(x)=\frac{\pi_n(x)}{\pi_n'(-1)}[u_n(x)+\beta_n w_1(x)+\beta_n' w_n(x)]-\frac{\pi_n''(-1)}{\pi_n'(-1)}C_n(x)$$

where

(11.20) $$\beta_n=\frac{n(n-1)}{576}(73n^4-146n^3+137n^2-64n+12)$$

(11.21) $$\beta_n'=\frac{n(n-1)}{576}(71n^4-142n^3+151n^2-80n-12)$$

(11.22) $$u_n(x)=l_n^3(x)+\frac{\pi_n(x)}{3\pi_n'(-1)}[l_n(x)\,l_n'(x)-10l_n'(-1)\,r_n(x)]$$
$$+K_n w_1(x)+K_n' w_n(x)$$

$$K_n=\frac{n(n-1)}{32}(n^4-2n^3-17n^2+18n+8)$$

$$K_n'=-\frac{n(n-1)}{96}(25n^4-50n^3+35n^2-10n-48)$$

and $c_n(x)$ is the polynomial of degree $\leqslant 4n-1$ given by (11.6) and (11.7). For $2 \leqslant \nu \leqslant n-1$,

$$B_\nu(x) = \frac{\pi_n(x)}{\pi_n'(x_\nu)}\left[u_\nu(x) - \frac{x_\nu}{1-x_\nu^2} w_\nu(x) + \beta_\nu w_1(x) + \beta_\nu' w_n(x)\right] \tag{11.23}$$

where

$$\beta_\nu = \frac{n^3(n-1)^3 x_\nu}{30(1-x_\nu^2)\pi_n'^3(x_\nu)}\left[\frac{12n(n-1)}{(1-x_\nu^2)P_{n-1}(x_\nu)} - \frac{3}{P_{n-1}(x_\nu)} - 5\right] \tag{11.24}$$

$$\beta_\nu' = \frac{n^3(n-1)^3 x_\nu}{30(1-x_\nu^2)\pi_n'^3(x_\nu)}\left[\frac{12n(n-1)}{(1-x_\nu^2)P_{n-1}(x_\nu)} - \frac{3}{P_{n-1}(x_\nu)} + 5\right] \tag{11.25}$$

$$w_\nu(x) = \frac{1-x_\nu^2}{6\pi_n'^3(x_\nu)} Q_{2n}(x)\left[\int_{-1}^{x}\frac{P_{n-1}'(t)}{t-x_\nu}\,dt - \frac{P_{n-1}(x)}{2(1-x_\nu^2)}\left(2x_\nu - \frac{1}{P_{n-1}(x_\nu)}\right) - \frac{1}{1-x_\nu^2}\left(1+x_\nu - \frac{1}{P_{n-1}(x_\nu)}\right)\right] \tag{11.25a}$$

$$u_\nu(x) = l_\nu^3(x) + \frac{\pi_n(x) l_\nu(x) l_\nu'(x)}{3\pi_n'(x_\nu)} + \frac{4n(n-1)x_\nu}{(1-x_\nu^2)^2} w_\nu(x) + \frac{2n^4(n-1)^4}{(1-x_\nu)^2\pi_n'^3(x_\nu)} w_1(x) + \frac{2n^4(n-1)^4}{(1+x_\nu)^2\pi_n'^3(x_\nu)} w_n(x) \tag{11.26}$$

(d) Lastly for $2 \leqslant \nu \leqslant n-1$, we have

$$A_\nu(x) = l_\nu^4(x) + \frac{\pi_n(x)}{4\pi_n'(x_\nu)}\left[l_\nu^2(x) l_\nu'(x) - 9 l_\nu''(x_\nu)\gamma_\nu(x)\right] + \gamma_\nu D_\nu(x) + \gamma_\nu' D_1(x) + \gamma_\nu'' D_n(x) \tag{11.27}$$

where

$$\gamma_\nu = -\frac{3n(n-1)}{(1-x_\nu^2)^2}\left[(n-2)(n+1) - \frac{8x_\nu^2}{1-x_\nu^2}\right] \tag{11.28}$$

$$\gamma_\nu' = \frac{3n^2(n-1)^2}{4(1-x_\nu)^2 P_{n-1}^3(x_\nu)\pi_n'(x_\nu)}\left[\frac{10}{1-x_\nu} + \frac{n(n-1)}{(1+x_\nu)^2}\left(1 + \frac{2}{P_{n-1}(x_\nu)}\right)\right] \tag{11.29}$$

$$\gamma_\nu'' = \frac{3n^2(n-1)^2}{4(1+x_\nu)^2 P_{n-1}^3(x_\nu)\pi_n'(x_\nu)}\left[\frac{10}{1+x} + \frac{n(n-1)}{(1-x_\nu)^2}\left(1 - \frac{2}{P_{n-1}(x_\nu)}\right)\right] \tag{11.30}$$

$$v_\nu(x) = \frac{\pi_n(x)}{\pi_n'(x_\nu)}\left[r_\nu(x) + \frac{n(n-1)}{3(1-x_\nu^2)}\varrho_\nu(x) + \frac{\pi_n(x)}{6n(n-1)(1-x_\nu^2)P_{n-1}^2(x_\nu)} - \frac{Q_{2n}'(x)}{18n^2(n-1)^2(1-x_\nu^2)P_{n-1}^3(x_\nu)}\right. \tag{11.31}$$

and $D_1(x)$, $D_n(x)$ and $D_\nu(x)$ are the polynomials each of degree $\leqslant 4n-1$ given by (11.1), (11.2) and (11.3).

$$(11.32)\qquad A_1(x)=l_1^4(x)+\frac{\pi_n(x)}{4\pi_n'(1)}[l_1^2(x)\,l_1'(x)-(9l_1''(1)+10l_1'^2(1))\gamma_1(x)-$$
$$-17l_1'(1)\,u_1(x)]+\gamma_1\,D_1(x)+\gamma_1'\,D_n(x)$$

where

$$(11.33)\qquad \gamma_1=-\frac{n(n-1)}{4608}[3289n^6-9867n^5+11915n^4-2525n^3+960n^2-3772n-1728]$$

$$(11.34)\qquad \gamma_1'=\frac{n(n-1)}{1536}[653n^6-1959n^5+3035n^4-2805n^3+1352n^2-276n+864]$$

and

$$(11.35)\qquad v_1(x)=\frac{\pi_n(x)}{\pi_n'(1)}\left[r_1(x)+\frac{(n-2)(n+1)}{48n(n-1)}\pi_n(x)-\frac{Q_{2n}'(x)}{32n(n-1)}\right]$$

$$(11.36)\qquad A_n(x)=l_n^4(x)+\frac{\pi_n(x)}{4\pi_n'(-1)}[l_n^2(x)\,l_n'(x)-(9l_n''(-1)+$$
$$+10l_n'^2(-1))\,v_n(x)-17l_n'(-1)\,u_n(x)]+\gamma_n\,D_1(x)+\gamma_n'\,D_n(x)$$

where

$$(11.37)\qquad \gamma_n=\frac{n(n-1)}{1536}[653n^6-1959n^5+3035n^4-2805n^3+1352n^2-276n+864]$$

$$(11.38)\qquad \gamma_n'=-\frac{n(n-1)}{4608}[3289n^6-9867n^5+11915n^4-2525n^3+690n^2-3772n-1728]$$

and

$$(11.39)\qquad v_n(x)=\frac{\pi_n(x)}{\pi_n'(-1)}\left[r_n(x)+\frac{(n-2)(n+1)}{48n(n-1)}\pi_n(x)+\frac{Q_{2n}'(x)}{32n(n-1)}\right]$$

12. In order to prove part (a) of the above theorem, we observe that in view of the conditions (9.7), we take

$$D_1(x)=\lambda_1\,\pi_n^3(x)(\lambda_1'+P_{n-1}(x))$$

which is of degree $\leq 4n-1$, so that it only remains to verify that $D_1^{\mathrm{IV}}(x_j)=0$ for $j=2,3,\ldots,n-1$. This is easily done in virtue of (7.5) and (7.6a). In order to determine λ_1 and λ_1', we use the requirement

$$D_1^{\mathrm{IV}}(1)=1,\quad D_1^{\mathrm{IV}}(-1)=0,$$

which gives on simplification with the help of (7.9)—(7.11)

$$D_1^{\mathrm{IV}}(1)=\lambda_1[-12n^4(n-1)^4-18n^4(n-1)^4(1+\lambda_1')]=1$$

and

$$D_1^{\mathrm{IV}}(-1)=\lambda_1[12n^4(n-1)^4-18n^4(n-1)^4(-1+\lambda_1']=0$$

whence we get (11.1). Similarly, we get (11.2). For obtaining (11.3), for $2\leq\nu\leq n-1$, we put

$$D_\nu(x)=\lambda_\nu\,\pi_n^3(x)\left[\int_{-1}^{x}\frac{P'_{n-1}(t)}{t-x_\nu}\,dt+\lambda'_\nu P_{n-1}(x)+\lambda''_\nu\right].$$

The conditions

$$D_\nu(x_j)=D'_\nu(x_j)=D''_\nu(x_j)=0 \quad \text{for } j=1,\,2,\ldots,\,n$$

are already seen to be true. Also

$$D^{\mathrm{IV}}_\nu(x_j)=12\lambda_\nu\left\{\pi'^3_\nu(x_j)\left[2\,\frac{P'_{n-1}(x_j)}{x_j-x_\nu}+2\lambda'_\nu P'_{n-1}(x_j)\right]+\right.$$

$$\left.+\,\pi'^2_n(x_j)\,\pi''_n(x_j)\left[\int_{-1}^{x}\frac{P'_{n-1}(t)}{t-x_\nu}\,dt+\lambda'_\nu P_{n-1}(x)+\lambda''_\nu\right]\right\}$$

$$=0,\ j\neq\nu,\quad j=2,\,3,\ldots,\,n-1.$$

$$D^{\mathrm{IV}}_\nu(x_\nu)=24\lambda_\nu\,\pi'^3_n(x_\nu)\lim_{x_j=x_\nu}\frac{P'_{n-1}(x_j)}{x_j-x_\nu}=1$$

$$\lambda_\nu=\frac{1}{24\,P''_{n-1}(x_\nu)\,\pi'^3_n(x_\nu)}$$

λ'_ν and λ''_ν are determined from the two conditions $D^{\mathrm{IV}}_\nu(\pm1)=0$, which gives two equations to determine λ'_ν and λ''_ν,

$$5\lambda'_\nu+3\lambda''_\nu+\frac{2}{1-x_\nu}+\frac{6x_\nu}{1-x_\nu^2}-\frac{6}{(1-x_\nu^2)\,P_{n-1}(x_\nu)}=0$$

and
$$-5\lambda'_\nu+3\lambda''_\nu+\frac{2}{1+x_\nu}=0.$$

Thus
$$\lambda'_\nu=-\frac{x_\nu}{1-x_\nu^2}+\frac{3}{5(1-x_\nu^2)\,P_{n-1}(x_\nu)}$$

$$\lambda''_\nu=-\frac{2+3x_\nu}{3(1-x_\nu^2)}+\frac{1}{(1-x_\nu^2)\,P_{n-1}(x_\nu)}$$

and (11.13) is established.

13. In order to obtain part (b) of Theorem IV, we require the polynomials $r_\nu(x)$ and $\varrho_\nu(x)$ each of degree $\leqslant 2n-1$, obtained by P. Turan and J. Balazs for (0,2)-interpolation for $\nu=1,\,2,\ldots,\,n$, where we have

(13.1)
$$r_\nu(x_j)=\begin{matrix}1\\0\end{matrix}\ \text{for}\ \left.\begin{matrix}j=\nu\\ j\neq\nu\end{matrix}\right\}\ (j=1,2,\ldots,n)$$

$$r''_\nu(x_j)=0$$

and

(13.2)
$$\varrho_\nu(x_j)=0$$

$$\varrho_\nu''(x_j) = \begin{matrix} 1 \\ 0 \end{matrix} \text{ for } \left.\begin{matrix} j=\nu \\ j\neq\nu \end{matrix}\right\} \quad (j=1, 2, \ldots, n).$$

We shall require the following results, which are easy to verify:

(13.3) $$r_\nu'(1) = \frac{1}{3(1-x_\nu^2)P_{n-1}^3(x_\nu)} = r_\nu'(-1) \qquad 2 \leqslant \nu \leqslant n-1$$

(13.4) $$r_1'(1) = \frac{1}{12} + \frac{5n(n-1)}{24} = r_1'(-1)$$

(13.5) $$r_n'(1) = -\frac{1}{12} - \frac{5n(n-1)}{24} = r_n'(-1)$$

(13.6) $$\varrho_\nu'(1) = \frac{1}{n(n-1)P_{n-1}^2(x_\nu)}\left(1+\frac{1}{3P_{n-1}(x_\nu)}\right),$$

$$\varrho_\nu'(-1) = \frac{1}{n(n-1)P_{n-1}^2(x_\nu)}\left(-1+\frac{1}{3P_{n-1}(x_\nu)}\right)$$

$$2 \leqslant \nu \leqslant n-1$$

(13.7) $$\varrho_1'(1) = \frac{4}{3n(n-1)}, \quad \varrho_1'(-1) = -\frac{2}{3n(n-1)}$$

(13.8) $$\varrho_n'(1) = \frac{2}{3n(n-1)}, \quad \varrho_n'(-1) = -\frac{4}{3n(n-1)}$$

(13.9) $$Q_{2n}'''(x_\nu) = 0, \qquad (\nu = 2, 3, \ldots n-1)$$

(13.10) $$Q_{2n}''(1) = 2n^2(n-1)^2 = Q_{2n}''(-1)$$

(13.11) $$Q_{2n}'''(1) = 3n^3(n-1)^3 = -Q_{2n}'''(-1)$$

(13.12) $$Q_{2n}^{\mathrm{IV}}(1) = \frac{n^3(n-1)^3}{2}(5n^2-5n-4) = Q_{2n}^{\mathrm{IV}}(-1).$$

We now determine $C_\nu(x)$ of degree $\leqslant 4n-1$ for $2 \leqslant \nu \leqslant n-1$ in the form:

$$C_\nu(x) = \frac{Q_{2n}(x)}{Q_{2n}''(x_\nu)}\left[r_\nu(x) + a_\nu\varrho_\nu(x) + a_\nu'\pi_n(x) + a_\nu''Q_{2n}'(x)\right].$$

Clearly

$$C_\nu(x_j) = C_\nu'(x_j) = 0, \qquad j = 1, 2, \ldots, n.$$

Also

$$\left.\begin{matrix} C_\nu''(x_j) = 0 & j\neq\nu \\ \quad = 1 & j=\nu \end{matrix}\right\} \quad j = 1, 2, \ldots, n$$

follows from (13.1) and (13.2). With the help of relations (13.1), (13.2) and (13.9) we have

$$C_\nu^{\mathrm{IV}}(x_j) = 0, \quad j\neq\nu \quad (j = 2, 3, \ldots, n-1)$$

and

$$C_\nu^{IV}(x_\nu) = \frac{Q_{2n}^{IV}(x_\nu)}{Q_{2n}''(x_\nu)} + 6a_\nu = 0$$

when

$$a_\nu = -\frac{Q_{2n}^{IV}(x_\nu)}{6Q_{2n}''(x_\nu)} = \frac{2n(n-1)}{3(1-x_\nu^2)}.$$

The constants a_ν' and a_ν'' are determined from the condition $C_\nu^{IV}(\pm 1) = 0$, which furnish the two linear equations

$$\frac{4n(n-1)}{3(1-x_\nu^2)}\varrho_\nu'(1) + 2r_\nu'(1) - 3n(n-1)a_\nu' + 10n^2(n-1)^2 a_\nu'' = 0$$

and

$$\frac{4n(n-1)}{3(1-x_\nu^2)}\varrho_\nu'(-1) + 2r_\nu'(-1) + 3n(n-1)a_\nu' + 10n^2(n-1)^2 a_\nu'' = 0.$$

Thus from (13.3) and (13.6) we have (11.9) and (11.10) and hence the formula (11.8). In order to obtain $C_1(x)$, we take

$$C_1(x) = \frac{Q_{2n}(x)}{Q_{2n}''(1)}[r_1(x) + a_1\pi_n(x) + a_1' Q_{2n}'(x)].$$

We can easily see that

$$C_1(x_j) = C_1'(x_j) = 0, \quad j = 1, 2, \dots, n$$

and

$$C_1''(x_j) = \begin{matrix} 0 \\ 1 \end{matrix} \text{ for } \begin{matrix} j = 2, 3, \dots, n \\ j = 1 \end{matrix}$$

Also $C_1^{IV}(x_j) = 0$ for $j = 2, 3, \dots, n-1$, while $C_1^{IV}(\pm 1) = 0$ help us to determine a_1 and a_1'. With the help of (7.10), (7.11), (13.10), (13.11), (13.12), $C_1^{IV}(\pm 1) = 0$ furnish two linear equations

$$\frac{1}{12}(5n^2 - 5n - 4) + 2(r_1'(1) - r_1'(-1)) - 6n(n-1)a_1 = 0$$

and

$$\frac{1}{12}(5n^2 - 5n - 4) + 2(r_1'(1) + r_1'(-1)) + 20n^2(n-1)^2 a_1' = 0.$$

Therefore using (13.4) we have

$$a_1 = -\frac{1}{18n(n-1)} + \frac{5}{72}$$

$$a_1' = -\frac{1}{16n(n-1)}.$$

Hence we get (11.4). The formula (11.6) can be obtained in a similar manner by choosing the form of

$$C_n(x) = \frac{Q_{2n}(x)}{Q''_{2n}(-1)}[r_n(x) + a_n\pi_n(x) + a'_n Q'_{2n}(x)].$$

14. In order to obtain part (c) of Theorem IV we require the polynomials $u_\nu(x)$ and $w_\nu(x)$, each of degree $\leq 3n-1$, obtained by Dr. A. Sharma and myself for (0, 1, 3)-interpolation for $\nu = 1, 2, \ldots, n$, where we have

(14.1) $$u_\nu(x_j) = \begin{matrix} 1 \\ 0 \end{matrix} \text{ for } \left.\begin{matrix} j = \nu \\ j \neq \nu \end{matrix}\right\}, \quad U'_\nu(x_j) = U'''_\nu(x_j) = 0$$

(14.2) $$w_\nu(x_j) = W'_\nu(x_j) = 0, \quad W'''_\nu(x_j) = \begin{matrix} 1 \\ 0 \end{matrix} \text{ for } \left.\begin{matrix} j = \nu \\ j \neq \nu \end{matrix}\right\}$$

$$(j = 1, 2 \ldots, n).$$

We shall require the following results which can very easily be obtained from (11.16), (11.17), (11.18), (11.22), (11.25a) and (11.26). For even n:

(14.1) $$u''_1(1) = -u''_1(-1) = -u''_n(1) = u''_n(-1) =$$
$$\frac{1}{288}(-61n^4 + 122n^3 - 113n^2 + 5n - 12)$$

(14.2) $$u''_\nu(1) = -\frac{2n^3(n-1)^3 x_\nu}{3(1-x_\nu^2)\pi'^3_n(x_\nu)P_{n-1}(x_\nu)} = -u''_\nu(-1), \quad 2 \quad \nu \quad n-1$$

(14.3) $$w''_1(1) = \frac{1}{2n(n-1)} = 3w''_1(-1)$$

(14.4) $$w''_n(1) = -\frac{1}{6n(n-1)} = \frac{1}{3}w''_n(-1)$$

(14.5) $$w''_\nu(1) = -\frac{n^2(n-1)^2}{3\pi'^3_n(x_\nu)}\left(1 + \frac{1}{2P_{n-1}(x_\nu)}\right)$$

(14.6) $$w''_\nu(-1) = \frac{n^2(n-1)^2}{3\pi'^3_n(x_\nu)}\left(-1 + \frac{1}{2P_{n-1}(x_\nu)}\right).$$

We now determine $B_\nu(x)$ of degree $4n-1$ for $2 \leq \nu \leq n-1$ in the form:

$$B_\nu(x) = \frac{\pi_n(x)}{\pi'_n(x_\nu)}[u_\nu(x) + \beta_\nu w_\nu(x) + \beta'_\nu w_1(x) + \beta''_\nu w_n(x)].$$

Clearly $B_\nu(x_j) = 0$ for $j = 1, 2, \ldots, n$ and

$$B'_\nu(x_j) = \begin{matrix} 0 \\ 1 \end{matrix} \text{ for } \left.\begin{matrix} j \neq \nu \\ j = \nu \end{matrix}\right\} \quad j = 1, 2, \ldots, n$$

follows from (14.1) and (14.2). Also with the help of (14.1), (14.2) and (7.6a) we have

$$B''_\nu(x_j) = 0, \quad (j = 1, 2, \ldots, n); \quad B^{\mathrm{IV}}_\nu(x_j) = 0, \ j \neq \nu, \quad (j = 2, 3, \ldots, n-1),$$

while

$$B_\nu^{IV}(x_\nu)=\frac{\pi_n^{IV}(x_\nu)}{\pi_n'(x_\nu)}+4\beta_\nu=0$$

whence

$$B_\nu=-\frac{\pi_n^{IV}(x_\nu)}{4\pi_n'(x_\nu)}=-\frac{x_\nu}{1-x_\nu^2}.$$

The conditions $B_\nu^{IV}(\pm 1)=0$ furnish two linear equations to determine β_ν' and β_ν''. $B_\nu^{IV}(1)=0$ gives

$$6\pi_n''(1)[u_\nu''(1)+\beta_\nu w_\nu''(1)+\beta_\nu' w_\nu''(1)+\beta_\nu'' w_n''(1)]+4\pi_n'(1)\beta_\nu'=0$$

which when simplified with the help of (7.10), (7.11), (14.2), (14.3), (14.4) and (14.5) gives

$$(14.7)\quad 6n(n-1)u_\nu''(1)+\frac{2n^3(n-1)^3x_\nu}{(1-x_\nu^2)^2Q_{2n}'(x_\nu)P_{n-1}''(x_\nu)}\left(\frac{1}{P_{n-1}(x_\nu)}+2\right)+11\beta_\nu'-\beta_\nu''=0.$$

Similarly $B_\nu^{IV}(-1)=0$ gives

$$(14.8)\quad 6n(n-1)u_\nu''(-1)-\frac{2n^3(n-1)^3x_\nu}{(1-x_\nu^2)^2Q_{2n}''(x_\nu)P_{n-1}''(x_\nu)}\left(\frac{1}{P_{n-1}(x_\nu)}-2\right)+\beta_\nu'-11\beta_\nu''=0$$

whence we have

$$\beta_\nu'=\frac{n^3(n-1)^3x_\nu}{30(1-x_\nu^2)\pi_n'^3(x_\nu)}\left[\frac{12n(n-1)}{(1-x_\nu^2)P_{n-1}(x_\nu)}-\frac{3}{P_{n-1}(x_\nu)}-5\right]$$

$$\beta_\nu''=\frac{n^3(n-1)^3x_\nu}{30(1-x_\nu^2)\pi_n'^3(x_\nu)}\left[\frac{12n(n-1)}{(1-x_\nu^2)P_{n-1}(x_\nu)}-\frac{3}{P_{n-1}(x_\nu)}+5\right].$$

This proves the formula (11.23). For $B_1(x)$, we have the form:

$$B_1(x)=\frac{\pi_n(x)}{\pi_n'(1)}[u_1(x)+\beta_1 w_1(x)+\beta_1' w_n(x)]-\frac{n(n-1)}{2}C_1(x).$$

Obviously

$$\left.\begin{aligned} &B_1(x_j)=0, && j=1,2,\ldots,n\\ &B_1'(x_j)=\begin{matrix}0\\1\end{matrix}\ \text{for}\ \begin{matrix}j=2,3,\ldots,n\\ j=1\end{matrix}\\ &B_1''(x_j)=0, && j=1,2,\ldots,n\end{aligned}\right\}\ \text{See (9.6)}$$

Also $B_1^{IV}(x_j)=0$, $(j=2,3,\ldots,n-1)$ is obvious from (14.1) and (14.2). The constants β_1 and β_1' are determined by the conditions $B_1^{IV}(\pm 1)=0$, which give

$$\frac{\pi_n^{IV}(1)}{\pi_n'(1)}+\frac{6\pi_n''(1)}{\pi_n'(1)}[u_1''(1)+\beta_1 w_1''(1)+\beta_1' w_n''(1)]+4\beta_1=0$$

and

$$\frac{6\pi_n''(-1)}{\pi_n'(1)}[u_1''(-1)+\beta_1 w_1''(-1)+\beta_1' w_n''(-1)]-4\beta_1'=0.$$

These equations with the help of (7.10), (7.11), (7.13), (14.1), (14.3) and (14.5) give

$$\beta_1 = \frac{n(n-1)}{576}(71n^4 - 142n^3 + 151n^2 - 80n - 12)$$

$$\beta_1' = \frac{n(n-1)}{576}(73n^4 - 146n^3 + 137n^2 - 64n + 12)$$

and this proves (11.13). Similarly (11.19) can be obtained by taking the form of $B_n(x)$ as

$$B_n(x) = \frac{\pi_n(x)}{\pi_n'(-1)}\,[u_n(x) + \beta_n w_1(x) + \beta_n' w_n(x)] - \frac{n(n-1)}{2} C_n(x).$$

15. We shall now determine the explicit representation of $A_\nu(x)$. For this purpose we shall need the polynomials $v_\nu(x)$ obtained by Dr. A. Sharma and myself for (0, 1, 3)-interpolation for $\nu = 1, 2, \ldots, n$, where we have

$$(15.1)\qquad v_1(x_j) = 0,\quad v_1'(x_j) = \begin{matrix} 0 \\ 1 \end{matrix} \text{ for } \left.\begin{matrix} j \neq \nu \\ j = \nu \end{matrix}\right\},\quad v_1'''(x_j) = 0$$

$$(j = 1, 2, \ldots, n)$$

We shall also need the polynomials $D_1(x)$, $D_n(x)$ and $D_\nu(x)$ $(2 \quad \nu \quad n-1)$, each of degree $4n-1$, given by (11.1), (11.2) and (11.3). The following results can atonce be obtained from (11.31), (11.35) and (11.37).

$$(15.2)\qquad v_1''(1) = \frac{1}{4}(3n^2 - 3n + 1),\quad v_1''(-1) = -\frac{1}{12}(4n^2 - 4n + 1)$$

$$(15.3)\qquad v_n''(1) = \frac{1}{12}(4n^2 - 4n + 1),\quad v_n''(-1) = -\frac{1}{4}(3n^2 - 3n + 1)$$

$$(15.4)\qquad v_\nu''(1) = \frac{1}{3(1-x_\nu^2)P_{n-1}^3(x_\nu)}\left(1 + \frac{2}{P_{n-1}(x_n)}\right) \qquad 2 \quad \nu \quad n-1$$

$$v_\nu''(-1) = -\frac{1}{3(1-x_\nu^2)P_{n-1}^3(x_\nu)}\left(-1 + \frac{2}{P_{n-1}(x_\nu)}\right).$$

For determining $A_\nu(x)\,(2 \leqslant \nu \leqslant n-1)$ we start with the form (11.27) with constants γ_ν, γ_ν', γ_ν'', still to be determined. The first of condition (9.4) is seen to be true in consequence of (9.7) and (10.3). Also $A_\nu'(x_j) = 0$ $(j = 1, 2, 3, \ldots, n)$ follows from (9.7), (10.2), (10.3) and (15.1), and $A_\nu''(x) = 0$ $(j = 1, 2, 3, \ldots, n)$ again follows from (9.7), (10.2), (10.3) and (15.1). Further from (9.7), (10.3), (15.1), (7.6a)

$$(15.4a)\qquad A_\nu^{\mathrm{IV}}(x_j) = 24 l_\nu'^4(x_j) + \frac{12 l_\nu'^2(x_j)\, l_\nu''(x_j)\, \pi_n'(x_j)}{\pi_n'(x_\nu)} =$$

$$= 12l_\nu'^2(x_j)\left[2l_\nu'^2(x_j) - \frac{2l_\nu'(x_j)}{x_j - x_\nu}\cdot\frac{\pi_n'(x_j)}{\pi_n'(x_\nu)}\right] = 0$$

$j \neq \nu$, (See (10.5) and (10.1))

$2 \leqslant \nu \quad n-1$,

$$A_\nu^{\mathrm{IV}}(x_\nu) = 5l_\nu^{\mathrm{IV}}(x_\nu) + 42l_\nu''^2(x_\nu) - \frac{8\pi_n'''(x_\nu)}{\pi_n'(x_\nu)}\, l_\nu''(x_\nu) + \gamma_\nu = 0$$

$$(2 \leqslant \nu \leqslant n-1)$$

which can be satisfied with the help of the formulae

(15.5) $$5l_\nu^{\mathrm{IV}}(x_\nu) = \frac{\pi_n^{\mathrm{V}}(x_\nu)}{\pi_n'(x_\nu)}$$

(15.6) $$3l_\nu''(x_\nu) = \frac{\pi_n'''(x_\nu)}{\pi_n'(x_\nu)} = -\frac{n(n-1)}{1-x_\nu^2}$$

(15.7) $$\frac{\pi_\nu^{\mathrm{IV}}(x_\nu)}{\pi_n'''(x_\nu)} = \frac{4x_\nu}{1-x_\nu^2}$$

(15.8) $$\frac{\pi_n^{\mathrm{V}}(x_\nu)}{\pi_n'(x_\nu)} = -\frac{n(n-1)}{(1-x_\nu^2)^2}\left[\frac{24x_\nu^2}{1-x_\nu^2} + 6 - n(n-1)\right]$$

and the value γ_ν is found to be (11.28). The constants γ_ν' and γ_ν'' are determined from the two conditions $A_\nu^{\mathrm{IV}}(\pm 1) = 0$. $A_\nu^{\mathrm{IV}}(1) = 0$ gives

(15.9) $$24l_\nu'^4(1) + \frac{6\pi_n''(1)}{4\pi_n'(x_\nu)}\,[2l_\nu'^3(1) - 9l_\nu''(x_\nu)v_\nu''(1)] + \\ + \frac{\pi_n'(1)}{\pi_\nu'(x_\nu)}\,[12l_\nu'^2(1)\,l_\nu''(1)] + \gamma_\nu' = 0.$$

But

$$24l_\nu'^4(1) + \frac{12\pi_n'(1)}{\pi_n'(x_\nu)}\, l_\nu'^2(1)\, l_\nu''(1) = \frac{12l_\nu'^3(1)\pi_n''(1)}{\pi_n'(x_\nu)}$$

$$\frac{3}{2}\cdot\frac{\pi_n''(1)}{\pi_n'(x_\nu)}\,[10l_\nu'^3(1) - 9l_\nu''(x_\nu)v_\nu''(1)] + \gamma_\nu' = 0.$$

Similarly $A_\nu^{\mathrm{IV}}(-1) = 0$ gives

$$\frac{3}{2}\cdot\frac{\pi_n''(-1)}{\pi_n'(x_\nu)}[10l_\nu'^3(-1) - 9l_\nu''(x_\nu)\gamma_\nu''(-1)] + \gamma_\nu'' = 0.$$

These two equations therefore determine γ_ν' and γ_ν'' as (11.29) and (11.30). Heavy calculations are involved in obtaining the constants γ_1, γ_1', γ_n, γ_n' in the determination of $A_1(x)$ and $A_n(x)$ when their forms are chosen to be

(11.32) and (11.36) respectively. We shall give the proof of (11.32) and (11.36) will follow in the same manner.

We start with the form (11.32) for $A_1(x)$ and see that first, second and third of the conditions (9.4) are satisfied in consequence of the relations (9.7), (10.2) (10.3), (7.6a) etc.

$$A_1^{IV}(x_j) = 24l_1'^3(x_j) + \frac{12l_1'^2(x_j)\, l_1''(x_j)\, \pi_n'(x_j)}{\pi_n'(1)} = 0$$

$$(j = 2, 3, \ldots, n-1) \qquad\qquad \text{(See (15.4a))}$$

It only remains to obtain the constants γ_1 and γ_1' for which the conditions $A_1^{IV}(\pm 1) = 0$ furnish two linear equations:

$$5l_1^{IV}(1) + [56l_1'(1)\, l_1'''(1) + 42l_1''^2(1)] + 156l_1'^2(1)\, l_1''(1)$$

$$+ 24l_1'^4(1) - \frac{4\pi_n^{IV}(1)}{\pi_n'(1)}\, l_1'(1) - \frac{8\pi_n'''(1)}{\pi_n'(1)}[l_1''(1) + l_1'^2(1)]$$

$$+ \frac{3\pi_n''(1)}{2\pi_n'(1)}\, [l_1'''(1) + 6l_1'(1)\, l_1''(1) + 2l_1'^3(1) - (9l_1''(1) + 10l_1'^2(1))\, \gamma_1''(1)$$

$$-17l_1'(1)\, u_1''(1)] + \gamma_1 = 0$$

and

$$24l_1'^4(-1) + \frac{3\pi_n''(-1)}{2\pi_n'(1)}\, [2l_1'^3(-1) - (9l_1''(1) + 10l_1'^2(1))\, v_1''(1) -$$

$$-17l_1'(1)\, u_1''(-1)] + \frac{12\pi_n'(-1)}{\pi_n'(1)}\, l_1'^2(-1)\, l_1''(-1) + \gamma_1' = 0.$$

With the help of (7.10)—(7.13), (10.9)—(10.12), (14.1) and (15.1) the above two equations can be simplified to give γ_1 and γ_1' (11.33) and (11.34). The simplification is rather cumbersome.

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НЯКОИ ИНТЕРПОЛАЦИОННИ СВОЙСТВА НА ЛЕЖАНДРОВИТЕ И УЛТРАСФЕРИЧНИТЕ ПОЛИНОМИ

Р. Б. Саксена

РЕЗЮМЕ

Предмет на работата на автора е решението на следната интерполационна проблема: Дадени са n различни възли $x_1, x_2, \ldots, x_n$. Търси се такъв полином $f_{4n-1}(x)$ от степен $\leq 4n-1$, за който да са в сила равенствата

$$f_{4n-1}(x_\nu) = y_{\nu 0},\ f'_{4n-1}(x_\nu) = y_{\nu 1},\ f''_{4n-1}(x_\nu) = y_{\nu 2},\ f^{\mathrm{IV}}_{4n-1}(x_\nu) = y_{\nu 4};\quad \nu = 1, 2, \ldots, n.$$

Авторът получава резултати, подобни на резултатите на Туран—Шурани (1), а именно:

1. Ако $n = 2k+1$ и $x_j = -x_{2k+2-j}$ не съществува изобщо полином $f_{4n-1}(x)$ от степен $< 4n-1$, за който

$$f^{(k)}(x_\nu) = y_{\nu k} (k=0,\ 1, 2,\ 4;\quad \nu = 1,\ 2, \ldots,\ n). \tag{1}$$

Ако има един такъв полином, то съществуват безброй полиноми със същото свойство.

2. При $n = 2k$, ако възлите x_ν съвпадат с нулите на $(1-x^2)P'_{n-1}(x)$, където $P_m(x)$ означава m-тия Лежандров полином, то съществува при произволно дадени

$$y_{\nu 0},\ y_{\nu 1},\ y_{\nu 2}, y_{\nu 4},\quad \nu = 0, 1, 2, \ldots,\ n,$$

един единствен полином от степен $\leq 4n-1$, за който са в сила уравненията (1).

Авторът изследва и въпроса за случаите, при които нулите на ултрасферичния полином $P_n^{(\lambda)}(x)$ при $\nu > -\frac{1}{2}$ могат да бъдат възли на разглежданата от него интерполационна проблема.

НЕКОТОРЫЕ ИНТЕРПОЛЯЦИОННЫЕ СВОЙСТВА ЛЕЖАНДРОВЫХ И УЛЬТРАСФЕРИЧЕСКИХ ПОЛИНОМОВ

Р. Б. Саксена

РЕЗЮМЕ

Предметом этой работы является решение следующей интерполяционной проблемы:

Даны n различных узлов $x_1, x_2, \ldots, x_n$. Отыскивается такой полином $f_{4n-1}(x)$ степени не выше $4n-1$, для которого в силе равенства

$$f_{4n-1}(x_\nu)=y_{\nu 0},\ f'_{4n-1}(x_\nu)=y_{\nu 1},\ f''_{4n-1}(x_\nu)=y_{\nu 2},\ f^{\mathrm{IV}}_{4n-1}(x_\nu)=y_{\nu 4};\quad \nu=1, 2, \ldots, n.$$

Автор получает результаты, подобные результатам Туран-Шурани ([1]), а именно:

1. Если $n=2k+1$ и $x_j=-x_{2k+2-j}$, то вообще не существует полинома $f_{4n-1}(x)$ степени не выше $4n-1$, для которого

$$f^{(k)}(x_\nu)=y_{\nu k}\quad (k=0,\ 1,\ 2,\ 4;\ \ \nu=1, 2, \ldots, n). \tag{1}$$

Если же существует один такой полином, то существует бесконечное число полиномов с таким же самым свойством.

2. При $n=2k$, если узлы x_ν совпадают с нулями $(1-x^2)P'_{n-1}(x)$, где $P_m(x)$ означает m-ый полином Лежандра, то при произвольно заданных $y_{\nu 0}, y_{\nu 1}, y_{\nu 2}, y_{\nu 4}$, $\nu=1, 2, \ldots, n$ существует один единственный полином степени не выше $4n-1$, для которого в силе будут уравнения (1).

Автор исследует также вопрос со случаями, когда нули ультрасферического полинома $P_n^{(\lambda)}(x)$ при $\lambda>\frac{1}{2}$ могут быть узлами рассматриваемой интерполяционной проблемы.