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ON n-DIMENSIONAL LAPLACE TRANSFORMATION

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1. The present paper deals with the operational calculus based on the n-dimensional Laplace transformation

$$f(p_1, \ldots, p_n) = p_1 \ldots p_n \int_0^\infty \ldots \int_0^\infty \exp\left(-\sum p_i x_i\right) h(x_1, \ldots, x_n)\, dx_1 \ldots dx_n,$$

where h is the 'original' and f its 'image', f being the operational representation of h, expressed symbolically as $f \underset{n}{\subset} h$ or $h \underset{n}{\supset} f$.

In this paper, I shall derive one theorem in n-dimensional Laplace transformation and apply this theorem (especially with $n=2$) to evaluate certain definite integrals, not easy to tackle otherwise in a neat form. In the theorem, I shall take $D(\sigma_{10}, \ldots, \sigma_{n0})$ to represent the set of points for which $R(p_i) > R(p_{i0}) = \sigma_{i0} > 0$.

In establishing the theorem it has been frequently necessary to effect changes in the orders of integrations in the muliple integrals. For the justification of the process, we can see Fubini's theorem and lemmas given below.

2. Lemma (i). If one of the repeated integrals

$$\int_a^\infty dx \int_b^\infty |f(x, y)|\, dy, \quad \int_b^\infty dy \int_a^\infty |f(x, y)|\, dx$$

be finite, then the integrals of $f(x, y)$ over the domain $(a, b; \infty, \infty)$ is finite and is equal to the repeated integrals over the domain.

3. Lemma (ii). If $f(x, y)$ be non-negative and measurable and be defined in $(a, b; \infty, \infty)$, then

$$\int_a^\infty dx \int_b^\infty f(x, y)\, dy, \quad \int_b^\infty dy \int_a^\infty f(x, y)\, dx, \quad \int_a^\infty \int_b^\infty f(x, y)\, dx dy;$$

are all finite and equal or else all infinite.

4. The domain of convergence of the n-dimensional Laplace transformation has been defined by me already in my Ph. D. thesis, submitted to B. I. T. S., Pilani.

(i) $$f(p_1, \ldots, p_n) \underset{n}{\subset} h(x_1, \ldots, x_n)$$

and let $h(x_1, \ldots, x_n)$ be continuous and absolutely integrable in each of the variables in $(0, \infty)$;

for $(p_1, \ldots, p_n) \in D(\sigma_{10}, \ldots, \sigma_{n0})$ the funciion

(ii) $$e^{-\sqrt{x}}\, x^{-n} f(x, \ldots, x)$$

be integrable L in $(0, \infty)$ and the operational variable p corresponds to x. Then

$$(5.1)\quad x^{v-n} e^{-\sqrt{x}} f(x, \ldots, x) \subset 2^{-v}\Gamma(2v+2)p \int_0^\infty \cdots \int_0^\infty \frac{\exp\left[\frac{1}{8(p+u_1+\cdots+u_n)}\right]}{(p+u_1+\cdots+u_n)^{v+1}}$$

$$\times D_{-2v-2}\left[\frac{1}{\sqrt{2(p+u_1+\cdots+u_n)}}\right] h(u_1, \ldots, u_n)\, du_1 \ldots du_n,$$

provided that $R(v) > -1$, $R(p>0$ and the integral on the right exists as an absolutely convergent double integral and where $D_v(x)$ is the parabolic cylinder function.

Since the definition integral (i) is absolutely convergent (by definition) for $(p_1, \ldots, p_n) \in D(\sigma_{10}, \ldots, \sigma_{n0})$, it must be absolutely convergent for all p_i, p_j such that $R(p_i) = R(p_j) = R(p) > 0$ holds. Hence the definition integral in (i) can be written as

$$x^{v-n} f(x, \ldots, x) = \int_0^\infty \cdots \int_0^\infty x^v \exp\left(-x \sum_{i=1}^n u_1\right) h(u_1, \ldots, u_n)\, du_1 \ldots du_n,$$

where $x > 0$, $v > -1$.

Multiplying both sides by $e^{-px-\sqrt{x}}$, $R(p) > 0$, and then integrating with respect to x between $(0, \infty)$, we obtain

$$(5.2)\quad \int_0^\infty \exp(-px-\sqrt{x})\, x^{v-n} f(x, \ldots, x)\, dx$$

$$= \int_0^\infty \exp(-px-\sqrt{x})\, dx \int_0^\infty \cdots \int_0^\infty x^v \exp\left(-x \sum_{i=1}^n u_i\right) h(u_1, \ldots, u_n)\, du_1 \ldots du_n.$$

Since $\exp(-px-\sqrt{x})\exp\left(-\sum_{i=1}^n x u_i\right) x^v h(u_1, \ldots, u_n)$ is integrable L in $(0, \infty)$ with respect to x and $u_1, \ldots, u_n$, we can change the order of integration on the right of (5.2) by Fubini's theorem, so as to obtain

$$(5.3)\quad \int_0^\infty \exp(-px-\sqrt{x})\, x^{v-n} f(x, \ldots, x)\, dx$$

$$=\int_0^\infty\ldots\int_0^\infty h(u_1,\ldots,u_n)\,du_1\ldots du_n\int_0^\infty x^v\exp\left[-\left(p+\sum_{i=1}^n u_i\right)x\right]e^{-\sqrt{x}}\,dx.$$

Evaluating the inner integral on the right of (5.3) by

$$\int_0^\infty x^v\exp(-px-\sqrt{x})\,dx=\Gamma(2v+2)\,2^{-v}p^{-1-v}\exp\left(\frac{1}{8p}\right)D_{-2v-2}\left(\frac{1}{\sqrt{2p}}\right),$$

we obtain the required result.

6. Applications

(a) Let $f(p,q)=\dfrac{q}{\sqrt{p}+\sqrt{q}}$, so that $h(x,y)=\dfrac{1}{\sqrt{\pi y}}-\dfrac{1}{\sqrt{\pi(x+y)}}$.

Hence from the theorem we get

$$\int_0^\infty\int_0^\infty(p+u+v')^{-1-v}\exp\left(\frac{1}{8(p+u+v')}\right)D_{-2v-2}\left(\frac{1}{\sqrt{2(p+u+v')}}\right)\left(\frac{1}{\sqrt{v'}}-\frac{1}{\sqrt{u+v'}}\right)du\,dv'$$

$$=[(2v+1)(2v)(2v-1)]^{-1}\sqrt{2\pi}\exp\left(\frac{1}{8p}\right)p^{1/2-v}D_{1-2v}\left(\frac{1}{\sqrt{2p}}\right);\quad R(v)>\frac{1}{2}$$

In particular, when $v=3/4$ we get

$$\int_0^\infty\int_0^\infty(p+u+v')^{-7/4}\exp\left(\frac{1}{8(p+u+v')}\right)D_{-7/2}\left(\frac{1}{\sqrt{2(p+u+v')}}\right)\left(\frac{1}{\sqrt{v'}}-\frac{1}{\sqrt{u+v'}}\right)du\,dv'$$

$$=\left(\frac{1}{15}\right)p^{-1/2}\,2^{11/4}\exp\left(\frac{1}{8p}\right)K_{1/4}\left(\frac{1}{8p}\right).$$

(b) Let $f(p,q)=\dfrac{q}{p+\sqrt{pq}}$ so that $h(x,y)=\left[\sqrt{\dfrac{x}{y}}-\tan^{-1}\sqrt{\dfrac{x}{y}}\right]\dfrac{2}{\pi}$.

Hence from the theorem we get

$$\int_0^\infty\int_0^\infty(p+u+v')^{-1-v}\exp\left(\frac{1}{8(p+u+v')}\right)D_{-2v-2}\left(\frac{1}{\sqrt{2(p+u+v')}}\right)\left[\sqrt{\frac{u}{v'}}-\tan^{-1}\sqrt{\frac{u}{v'}}\right]du\,dv'$$

$$=[\Gamma(2v+2)]^{-1}\Gamma(2v-2)\,\pi p^{1-v}\exp\left(\frac{1}{8p}\right)D_{2-2v}\left(\frac{1}{\sqrt{2p}}\right);\quad R(v)>1.$$

(c) Let $f(p,q)=\dfrac{\sqrt{pq}\,(\sqrt{p}+\sqrt{2q})}{p+q+\sqrt{2pq}}$ so that $h(x,y)=\dfrac{1}{\sqrt{\pi}}\,\dfrac{\sqrt{x+\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}$.

Hence from the theorem we get

$$\int_0^\infty\int_0^\infty\frac{\exp[1/8(p+u+v')]}{(p+u+v')^{v+1}}D_{-2v-2}\left(\frac{1}{\sqrt{2(p+u+v')}}\right)\frac{\sqrt{u+\sqrt{u^2+v'^2}}}{\sqrt{u^2+v'^2}}\,du\,dv'$$

$$=[\Gamma(2v+2)]^{-1}\,\Gamma(2v-1)\,2\pi\exp\left(\frac{1}{8p}\right)p^{1/2-v}D_{1-2v}\left(\frac{1}{\sqrt{2p}}\right);\ R(v)>\left(\frac{1}{2}\right)\cdot$$

(d) Let $f(p,q)=\Gamma_2\left(\mu\pm v+\frac{1}{2}\right)\left(\frac{p}{q}\right)^{\mu-1}W_{-\mu,v}\left(\frac{q}{p}\right)$ so that

$$h(x,y)=\left(\frac{x}{y}\right)^{\mu-1}\exp\left(-\frac{x}{2y}\right)W_{\mu-1,v}\left(\frac{x}{y}\right)\cdot$$

Hence from the theorem we get

$$\int_0^\infty\int_0^\infty\left(\frac{u}{v'}\right)^{\mu-1}(p+u+v')^{-1-v}\exp\left[\frac{1}{8(p+u+v')}-\frac{u}{2v'}\right]D_{-2v-2}\left(\frac{1}{\sqrt{2(p+u+v')}}\right)$$

$$\times W_{\mu-1,v}\left(\frac{u}{v'}\right)du\,dv'=4\,[\Gamma(2v+2)]^{-1}\Gamma(2v-2)\Gamma_2\left(\mu\pm v+\frac{1}{2}\right)p^{1-v}\exp\left(\frac{1}{2}+\frac{1}{8p}\right)$$

$$W_{-\mu,v}(1)\,D_{2-2v}\left(\frac{1}{\sqrt{2p}}\right);\ R\left(\mu+v\pm v-\frac{1}{2}\right)>0;\ R(v)>1;\ R\left(\mu\pm v+\frac{1}{2}\right)>0.$$

e) Let $f(p,\ q)=\left(\frac{\sqrt{pq}}{(\sqrt{p}+\sqrt{q})^{n+1}}\right)$ so that $h(x,y)=\frac{1}{\Gamma\left(\frac{n}{2}+1\right)\sqrt{\pi}}\,\frac{(xy)^{n/2}}{(x+y)^{\frac{n+1}{2}}}\cdot$

Hence from the theorem we get

$$\int_0^\infty\int_0^\infty(uv')^{n/2}(u+v')^{-n/2-1/2}(p+u+v')^{-1-v}\exp\left(\frac{1}{8(p+u+v')}\right)$$

$$\times D_{-2v-2}\left(\frac{1}{\sqrt{2(p+u+v')}}\right)du\,dv'=\frac{\Gamma(2v-n-1)\,\Gamma\left(\frac{n}{2}+1\right)e^{1/8p}\sqrt{\pi}}{\Gamma(2v+2)\,2^{n/2-1/2}\,p^{v-n/2-1/2}}\,D_{n-2v+1}\left(\frac{1}{\sqrt{2p}}\right);$$

$$R(n)>-2;\ R(v)>-\frac{1}{2}\cdot$$

(f) Let $h(x,y)=\exp(-x/y)-1$ so that $f(p,q)=(q/p)\exp(q/p)\,E_i(-q/p)$.

Hence from the theorem we get

$$\int_0^\infty\int_0^\infty(e^{-u/v'}-1)(p+u+v')^{-1-v}\exp\left(\frac{1}{8(p+u+v')}\right)D_{2v-2}\left(\frac{1}{\sqrt{2(p+u+v')}}\right)du\,dv'$$

$$=4[\Gamma(2v+2)]^{-1}\,\Gamma(2v-2)\,e^1E_i(-1)\,p^{1-v}\exp\left(\frac{1}{8p}\right)D_{2-2v}\left(\frac{1}{\sqrt{2p}}\right);\ R(v)>1.$$

(h) Let $h(x,y)=x^{-m-k-1/2}\,y^{2m-1}\,e^{-x/y}$ so that

$$f(p,q)=\Gamma_2\left(\frac{1}{2}\pm m-k\right)p^{k+1}q^{1/2-m}W_{k,m}\left(\frac{q}{p}\right)e^{q/2p}.$$

Hence from the theorem we get

$$\int_0^\infty \int_0^\infty v'^{2m-1} u^{-\left(m+k+\frac{1}{2}\right)} (p+u+v')^{-v-1} \exp\left(\frac{1}{8(p+u+v')} - \frac{u}{v'}\right)$$

$$\times D_{-2v-2}\left(\frac{1}{\sqrt{2(p+u+v')}}\right) du dv' = 2^{m-k+\frac{1}{2}} [\Gamma(2v+2)]^{-1} \Gamma(2v-2m+2k+1)$$

$$\times \Gamma_2\left(\frac{1}{2} \pm m - k\right) p^{m-v-k-\frac{1}{2}} \exp\left(\frac{1}{2} + \frac{1}{8p}\right) D_{2m-2k-2v-1}\left(\frac{1}{\sqrt{2p}}\right);$$

$$R(v) > -1;\ R(m) > 0;\ R(v-2m+1) > 0;\ R\left(\frac{1}{2} \pm m - k\right) > 0.$$

i) Let $h(x, y) = \left(\frac{1}{\sqrt{x^2+y^2}}\right)^{-1}$ so that $f(p, q) = \frac{pq}{\sqrt{p^2+q^2}} \log \frac{p+q+\sqrt{p^2+q^2}}{p+q-\sqrt{p^2+q^2}}$

Hence from the theorem we get

$$\int_0^\infty \int_0^\infty (u^2+v'^2)^{-1/2} (p+u+v')^{v+1} \exp\left(\frac{1}{8(p+u+v')}\right) D_{2v-2}\left(\frac{1}{\sqrt{2(p+u+v')}}\right) du dv'$$

$$= 2\sqrt{2} [\Gamma(2v+2)]^{-1} \Gamma(2v) \log(1+\sqrt{2}) p^{-v} \exp\left(\frac{1}{8p}\right) D_{-2v}\left(\frac{1}{\sqrt{2p}}\right);\ R(v) > 0.$$

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ЕДНА ТЕОРЕМА ВЪРХУ *n*-МЕРНАТА ЛАПЛАСОВА ТРАНСФОРМАЦИЯ

Р. С. Дахия

(Резюме)

Нека

$$f(p_1, \ldots, p_n)=p_1 \ldots p_n \int_0^\infty \ldots \int_0^\infty e^{-\Sigma p_i x_i} h(x_1 \ldots x_n)\, dx_1 \ldots dx_n .$$

В такъв случай при съответни предположения функцията $x^{v-n} e^{-\sqrt{x}} f(x_1 \ldots x_n)$ се преобразува с помощта на Лапласовата трансформация в

$$\frac{\Gamma(2v+2)}{2^v} p \int_0^\infty \quad \int_0^\infty \frac{\exp \frac{1}{8(p+u_1+\cdots+u_n)}}{(p+u_1+\cdots+u_n)^{v+1}} D_{-2v-2}\left[\frac{1}{\sqrt{2(p+u_1+\cdots+u_n)}}\right]$$

$$\times h(u_1, \ldots, u_n)\, du_1 \ldots du_n .$$

ОДНА ТЕОРЕМА ОБ *n*-МЕРНОМ ПРЕОБРАЗОВАНИИ ЛАПЛАСА

Р. С. Дахия

(Резюме)

Пусть

$$f(p_1, \ldots, p_n)=p_1 \ldots p_n \int_0^\infty \quad \int_0^\infty e^{-\Sigma p_i x_i} h(x_1 \ldots x_n)\, dx_1 \ldots dx_n .$$

В этом случае при соответствующих предположениях функция $x^{v-n} e^{-\sqrt{x}} f(x_1 \ldots x_n)$ преобразуется с помощью подстановки Лапласа в

$$\frac{\Gamma(2v+2)}{2^v} p \int_0^\infty \ldots \int_0^\infty \frac{\exp \frac{1}{8(p+u_1+\cdots+u_n)}}{(p+u_1+\cdots+u_n)^{v+1}} D_{-2v-2}\left[\frac{1}{\sqrt{2(p+u_1+\cdots+u_n)}}\right]$$

$$\times h(u_1 \ldots u_n)\, du_1 \ldots du_n .$$