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THE APPROXIMATION OF CONTINUOUS FUNCTIONS BY INTERPOLATORY POLYNOMIALS

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1. INTRODUCTION

The author [3] has considered earlier a new interpolation process

$$(1)\quad A_n(f, x)=\frac{1+x}{2}f(1)+\frac{1-x}{2}f(-1)+\sum_{k=1}^{n}\left[f(x_{kn})-\left\{\frac{1+x}{2}f(1)+\frac{1-x}{2}f(-1)\right\}\right]\lambda_{kn}(x),$$

where

$$(2)\qquad \lambda_{kn}(x)=\left(\frac{1-x^2}{1-x_{kn}^2}\right)^2[v_{kn}(x)l_{kn}^4(x)+2(x-x_{kn})l_{kn}^3(x)(1-x_{kn}^2)\psi_n(x_{kn}, x)];$$

here

$$(3)\qquad \psi_n(t, u)=\frac{2}{n+1}\sum_{k=1}^{n-1}U_r'(t)U_r(u),$$

and

$$(4)\qquad v_{kn}(x)=1-\frac{3x_{kn}(x-x_{kn})}{1-x_{kn}^2},$$

$$(5)\qquad l_{kn}(x)=\frac{(-1)^{k+1}(1-x_{kn}^2)}{n+1}\,\frac{U_n(x)}{x-x_{kn}}$$

being the fundamental polynomials of Lagrange interpolation constructed on the roots

$$(6)\qquad x_{kn}=\cos\frac{k\pi}{n+1}$$

of the Čebyšev polynomial $U_n(x)$ of second kind given by

$$(7)\qquad U_n(x)=\frac{\sin(n+1)\theta}{\sin\theta},\quad x=\cos\theta.$$

This process directly gives the proof of Jackson's theorem on the approximation of continuous functions by algebraic polynomials, namely the following:

Let $f(x)$ be a continuous function defined in $[-1,1]$; then, for the sequence of polynomials $A_n(f, x)$ in (1), we have

$$(8) \qquad |A_n(f, x)-f(x)|=O\left[\omega\left(f, \frac{1}{n}\right)\right]$$

uniformly in $[-1, 1]$, where $\omega(f, \delta)$ is the modulus of continuity of $f(x)$.

The formula (1) is an improvement of the interpolation formula considered by G. Freud [1] in the sense that, with the help of (1), the inequality (8) is proved to hold for the whole interval $[-1, 1]$. Later P. Vértesi [4] proved the inequality (8) by using the corresponding formula (1) built on the roots of $T_n(x)$ — the Čebyšev polynomials of first kind. As an improvement to his result, G. Freud and P. Vértesi [2] replaced (8) by

$$(9) \qquad |J_n^*(f, x)-f(x)|=O\left[\omega\left(f, \frac{\sqrt{1-x^2}}{n}\right)+\omega\left(f, \frac{1}{n^2}\right)\right].$$

Thus by proving (9), they were able to give a new proof of A. F. Timan's approximation theorem by interpolation method.

In the present work, the author shows that the same formula (1), considered in his earlier work [3], gives an improvement of inequality (8). In this way we are able to give a new interpolatory approach to the proof of the following theorem of A. F. Timan:

Theorem. Let $f(x)$ be a continuous function defined in $[-1, 1]$; then, for the sequence of polynomials in (1), we have

$$(10) \qquad |A_n(f, x)-f(x)|\leqq 384\left[\omega\left(f, \frac{\sqrt{1-x^2}}{n}\right)+\omega\left(f, \frac{|x|}{n^2}\right)\right]$$

in $[-1, 1]$, where $\omega(f, \delta)$ is the modulus of continuity of $f(x)$.

The inequality (10) follows by obtaining a more precise estimation of the fundamental polynomials in (1) and the inequality

$$(11) \qquad |g(\cos t)-g(\cos t_k)| \leqq\left[2 \sin\frac{1}{2}|t-t_k|+1\right]\omega\left(g, \frac{|\sin t|}{n}\right)$$
$$+\left[2\left\{n \sin\frac{1}{2}|t-t_k|\right\}^2+1\right]\omega\left(g, \frac{|\cos t|}{n^2}\right)$$

which follows easily from the relation

$$(12) \qquad \cos t_k-\cos t=2\sin\frac{1}{2}(t-t_k)\sin\frac{1}{2}(t+t_k)$$
$$=2\sin\frac{1}{2}(t-t_k)\left[\sin t\cos\frac{1}{2}(t-t_k)-\cos t\sin\frac{1}{2}(t-t_k)\right]$$

(cf. [5]). The inequality (10) has earlier been proved by P. Vértesi and O. Kis [5] by an interpolation method constructed on the nodes $\cos\frac{2k\pi}{2n+1}$.

2. As mentioned in the introduction, for the proof of the theorem, we need a more precise estimation of the fundamental polynomials $\lambda_{kn}(x)$ in (1). For this we first simplify $\lambda_{kn}(x)$ in a form convenient for the estimation.

For brevity we shall write x_k for x_{kn}, $l_k(x)$ for $l_{kn}(x)$ etc. Further, we set $x_0=1$, $x_{n+1}=-1$, $x=\cos\theta$, $x_k=\cos\theta_k$, so that

$$\theta_0=0;\quad \theta_k=\frac{k\pi}{n+1},\quad k=1,2,\ldots,n;\quad \theta_{n+1}=\pi.$$

Putting $x=\cos\theta$ and $x_k=\cos\theta_k$ in (3), (4) and (5), we can write (2) by using (7) as

$$(13)\qquad \lambda_k(\cos\theta)=\frac{\sin^4(n+1)\theta\sin^4\theta_k}{(n+1)^4(\cos\theta-\cos\theta_k)^4}-\frac{3\cos\theta_k\sin^2\theta_k\sin^4(n+1)\theta}{(n+1)^4(\cos\theta-\cos\theta_k)^3}$$
$$+(-1)^{k+1}\sqrt{1-x^2}\psi_n(x_k,x)\frac{\sin^3(n+1)\theta\sin^4\theta_k}{(n+1)^3(\cos\theta-\cos\theta_k)^2}$$

where [3, p. 174]

$$(14)\quad \sqrt{1-x^2}\,\psi_n(x_k,x)=\frac{\cos\theta_k}{2(n+1)\sin^3\theta_k}\left[\frac{\sin\frac{1}{2}(2n+1)(\theta-\theta_k)}{\sin\frac{1}{2}(\theta-\theta_k)}-\frac{\sin\frac{1}{2}(2n+1)(\theta+\theta_k)}{\sin\frac{1}{2}(\theta+\theta_k)}\right]$$
$$-\frac{1}{4(n+1)\sin^2\theta_k}\left[\frac{\cos\frac{1}{2}(\theta-\theta_k)\sin\frac{1}{2}(2n+1)(\theta-\theta_k)}{\sin^2\frac{1}{2}(\theta-\theta_k)}+\frac{\cos\frac{1}{2}(\theta+\theta_k)\sin\frac{2n+1}{2}(\theta+\theta_k)}{\sin^2\frac{1}{2}(\theta+\theta_k)}\right]$$
$$+\frac{2n+1}{4(n+1)\sin^2\theta_k}\left[\frac{\cos\frac{1}{2}(2n+1)(\theta-\theta_k)}{\sin\frac{1}{2}(\theta-\theta_k)}+\frac{\cos\frac{1}{2}(2n+1)(\theta+\theta_k)}{\sin\frac{1}{2}(\theta+\theta_k)}\right].$$

Now

$$(15)\qquad \sin\frac{1}{2}(2n+1)(\theta\pm\theta_k)=\sin(n+1)(\theta\pm\theta_k)\cos\frac{1}{2}(\theta\pm\theta_k)$$
$$-\cos(n+1)(\theta\pm\theta_k)\sin\frac{1}{2}(\theta\pm\theta_k)$$
$$=(-1)^k\left[\sin(n+1)\theta\cos\frac{1}{2}(\theta\pm\theta_k)-\cos(n+1)\theta\sin\frac{1}{2}(\theta\pm\theta_k)\right]$$

(since $\sin(n+1)\theta_k=0$ and $\cos(n+1)\theta_k=(-1)^k$). Similarly,

$$(16)\qquad \cos\frac{1}{2}(2n+1)(\theta\pm\theta_k)=(-1)^k\left[\cos(n+1)\theta\cos\frac{1}{2}(\theta\pm\theta_k)\right.$$
$$\left.+\sin(n+1)\theta\sin\frac{1}{2}(\theta\pm\theta_k)\right],$$

so that

$$(17)\quad \sqrt{1-x^2}\,\psi_n(x_k,x)=\frac{(-1)^k\sin(n+1)\theta\cos\theta_k}{2(n+1)\sin^3\theta_k}\left[\operatorname{cotg}\frac{1}{2}(\theta-\theta_k)-\operatorname{cotg}\frac{1}{2}(\theta+\theta_k)\right]$$
$$+\frac{(-1)^k\cos(n+1)\theta}{2\sin^2\theta_k}\left[\operatorname{cotg}\frac{1}{2}(\theta-\theta_k)+\operatorname{cotg}\frac{1}{2}(\theta+\theta_k)\right]$$
$$+\frac{(-1)^{k+1}\sin(n+1)\theta}{4(n+1)\sin^2\theta_k}\left[\operatorname{cosec}^2\frac{1}{2}(\theta-\theta_k)+\operatorname{cosec}^2\frac{1}{2}(\theta+\theta_k)\right]$$

$$+(-1)^k \frac{\sin(n+1)\theta}{\sin^2\theta_k} = \frac{(-1)^{k+1}\sin(n+1)\theta\cos\theta_k}{(n+1)\sin^2\theta_k(\cos\theta-\cos\theta_k)} + \frac{(-1)^{k+1}\cos(n+1)\theta\sin\theta}{\sin^2\theta_k(\cos\theta-\cos\theta_k)}$$

$$+\frac{(-1)^{k+1}\sin(n+1)\theta}{4(n+1)\sin^2\theta_k}\left[\operatorname{cosec}^2\frac{1}{2}(\theta-\theta_k)+\operatorname{cosec}^2\frac{1}{2}(\theta+\theta_k)\right]+(-1)^k\frac{\sin(n+1)\theta}{\sin^2\theta_k}$$

From this and (13) we have

$$\lambda_k(\cos\theta)=\lambda_k^*(\cos\theta)+\lambda_k^{**}(\cos\theta), \tag{18}$$

where

$$\lambda_k^*(\cos\theta)=\frac{\sin(n+1)^3\theta\cos(n+1)\theta\sin^2\theta_k\sin\theta}{(n+1)^3(\cos\theta-\cos\theta_k)^3} \tag{19}$$

and

$$\lambda_k^{**}(\cos\theta)=\frac{\sin^4(n+1)\theta\sin^4\theta_k}{(n+1)^4(\cos\theta-\cos\theta_k)^4}-\frac{2\sin^4(n+1)\theta\sin^2\theta_k\cos\theta_k}{(n+1)^4(\cos\theta-\cos\theta_k)^3} \tag{20}$$

$$-\frac{\sin^4(n+1)\theta\sin^2\theta_k}{(n+1)^3(\cos\theta-\cos\theta_k)^2}+\frac{\sin^4(n+1)\theta\sin^2\theta_k}{4(n+1)^4(\cos\theta-\cos\theta_k)^2}\left[\operatorname{cosec}^2\frac{1}{2}(\theta-\theta_k)+\operatorname{cosec}^2\frac{1}{2}(\theta+\theta_k)\right].$$

We shall need the following lemmas.

Lemma 1. For $0\leq\theta\leq\pi$

$$\sum_{k=1}^{n}|\lambda_k^*(\cos\theta)|\leq 20, \tag{23}$$

$$\sum_{k=1}^{n}|\lambda_k^*(\cos\theta)||\cos\theta-\cos\theta_k|\leq 12\frac{\sin\theta}{n}. \tag{22}$$

Lemma 2. For $0\leq\theta\leq\pi$

$$\sum_{k=1}^{n}|\lambda_k^{**}(\cos\theta)|\leq 156, \tag{23}$$

$$\sum_{k=1}^{n}|\lambda_k^{**}(\cos\theta)|\left|\sin\frac{1}{2}(\theta-\theta_k)\right|\leq\frac{92}{n}, \tag{24}$$

$$\sum_{k=1}^{n}|\lambda_k^{**}(\cos\theta)|\sin^2\frac{1}{2}(\theta-\theta_k)\leq\frac{49}{n^2}. \tag{25}$$

The proofs of these lemmas will be given in the next section. Besides these results we shall use the following facts already proved [3, lemma 3.2, p. 170].

$$\left|1-\sum_{k=1}^{n}\lambda_k(x)\right|\leq 3, \tag{26}$$

$$\sqrt{1-x^2}\left|1-\sum_{k=1}^{n}\lambda_k(x)\right|\leq\frac{3}{n}, \tag{27}$$

$-1\leq x\leq 1.$

We shall now prove the inequality (10). In [3, p. 182] we obtained

$$
(28)\qquad |f(x)-A_n(f,x)|=\Bigg|\frac{1+x}{2}[f(x)-f(1)]\Bigg[1-\sum_{k=1}^{n}\lambda_k(x)\Bigg]
$$

$$
+\frac{1-x}{2}[f(x)-f(-1)]\Bigg[1-\sum_{k=1}^{n}\lambda_k(x)\Bigg]+\sum_{k=1}^{n}[f(x)-f(x_k)]\lambda_k(x)\Bigg|.
$$

Using (26) and (27) we have

$$
(29)\qquad \Bigg|\frac{1+x}{2}[f(x)-f(1)]\Bigg[1-\sum_{k=1}^{n}\lambda_k(x)\Bigg]\Bigg|\leqq\frac{1+x}{2}\,\omega(|x-1|)\Bigg|1-\sum_{k=1}^{n}\lambda_k(x)\Bigg|
$$

$$
\leqq\frac{1+x}{2}\Bigg[1+n\frac{|x-1|}{\sin\theta}\Bigg]\omega\left(\frac{\sin\theta}{n}\right)\Bigg|1-\sum_{k=1}^{n}\lambda_k(x)\Bigg|
$$

$$
\omega\left(\frac{\sin\theta}{n}\right)\Bigg|1-\sum_{k=1}^{n}\lambda_k(x)\Bigg|+\frac{n(1+x)(1-x)}{\sin\theta}\,\omega\left(\frac{\sin\theta}{n}\right)\Bigg|1-\sum_{k=1}^{n}\lambda_k(x)\Bigg|\leqq 3\omega\left(\frac{\sin\theta}{n}\right)
$$

$$
+n\sqrt{1-x^2}\Bigg|1-\sum_{k=1}^{n}\lambda_k(x)\Bigg|\,\omega\left(\frac{\sin\theta}{n}\right)\leqq 6\omega\left(\frac{\sin\theta}{n}\right).
$$

Similarly, we have

$$
(30)\qquad \Bigg|\frac{1-x}{2}[f(x)-f(-1)]\Bigg[1-\sum_{k=1}^{n}\lambda_k(x)\Bigg]\Bigg|\leqq 6\omega\left(\frac{\sin\theta}{n}\right).
$$

We now estimate the remaining term

$$
(31)\qquad \Bigg|\sum_{k=1}^{n}[f(x)-f(x_k)]\lambda_k(x)\Bigg|
$$

in (28). Using (18), we have

$$
\Bigg|\sum_{k=1}^{n}[f(x)-f(x_k)]\lambda_k(x)\Bigg|\leqq\sum_{k=1}^{n}\omega(|\cos\theta-\cos\theta_k|)\,|\lambda_k(\cos\theta)|
$$

$$
\leqq\sum_{k=1}^{n}\omega(|\cos\theta-\cos\theta_k|)\,|\lambda_k^{*}(\cos\theta)|+\sum_{k=1}^{n}\omega(|\cos\theta-\cos\theta_k|)|\lambda_k^{**}(\cos\theta)|=\sum\nolimits_1+\sum\nolimits_2.
$$

As to the part $\sum_1$, we have

$$
(32)\qquad \sum\nolimits_1=\sum_{k=1}^{n}\omega(|\cos\theta-\cos\theta_k|)\,|\lambda_k^{*}(\cos\theta)|
$$

$$
\leqq\sum_{k=1}^{n}\Bigg[1+\frac{n}{\sin\theta}|\cos\theta-\cos\theta_k|\Bigg]|\lambda_k^{*}(\cos\theta)|\,\omega\left(\frac{\sin\theta}{n}\right)=\omega\left(\frac{\sin\theta}{n}\right)\sum_{k=1}^{n}|\lambda_k^{*}(\cos\theta)
$$

$$+\omega\left(\frac{\sin\theta}{n}\right)\frac{\sin\theta}{n}\sum_{k=1}^{n}|\cos\theta-\cos\theta_k||\lambda_k^*(\cos\theta)|\le 32\omega\left(\frac{\sin\theta}{n}\right)$$

because of (21) and (22).

For the part

$$\sum{}'_2=\sum_{k=1}^{n}\omega(|\cos\theta-\cos\theta_k|)|\lambda_k^{**}(\cos\theta)|,$$

using (11), (23), (24) and (25), we have

$$\text{(33)}\qquad \left|\sum_{k=1}^{n}[f(x)-f(x_k)]\lambda_k^{**}(\cos\theta)\right|\le\sum_{k=1}^{n}\left[\left(2n\sin\frac{1}{2}|\theta-\theta_k|+1\right)\omega\left(\frac{\sin\theta}{n}\right)\right.$$

$$\left.+\left(2n^2\sin^2\frac{1}{2}(\theta-\theta_k)+1\right)\omega\left(\frac{|\cos\theta|}{n^2}\right)\right]|\lambda_k^{**}(\cos\theta)|$$

$$=\left[\omega\left(\frac{\sin\theta}{n}\right)+\omega\left(\frac{|\cos\theta|}{n^2}\right)\right]\sum_{k=1}^{n}|\lambda_k^{**}(\cos\theta)|$$

$$+2n\omega\left(\frac{\sin\theta}{n}\right)\sum_{k=1}^{n}\sin\frac{1}{2}|\theta-\theta_k||\lambda_k^{**}(\cos\theta)|+2n^2\omega\left(\frac{|\cos\theta|}{n^2}\right)\sum_{k=1}^{n}\sin^2\frac{1}{2}(\theta-\theta_k)|\lambda_k^{**}(\cos\theta)|$$

$$\le 156\left[\omega\left(\frac{\sin\theta}{n}\right)+\omega\left(\frac{|\cos\theta|}{n^2}\right)\right]+184\omega\left(\frac{\sin\theta}{n}\right)+98\omega\left(\frac{|\cos\theta|}{n^2}\right)\le 340\omega\left(\frac{\sin\theta}{n}\right)$$

$$+254\omega\left(\frac{|\cos\theta|}{n^2}\right).$$

This completes the proof of the theorem.

4. PROOF OF THE INEQUALITIES (21)—(25)

Let

$$\text{(34)}\qquad E_k=\frac{\sin(n+1)\theta}{(n+1)\sin\frac{1}{2}(\theta-\theta_k)}.$$

Then, we can prove that, for $\theta\in[\theta_i,\theta_{i+1}]$,

$$\text{(35)}\qquad E_k^m\le\begin{cases}(i-k)^{-m}, & 1\le k<i\le n;\\ (k-i-1)^{-m}, & 0\le i\le n-2,\quad i+2\le k\le n;\\ 2^m, & k=i,\quad 1\le i\le n;\\ 2^m, & k=1+i,\quad 0\le i\le n-1.\end{cases}$$

In fact, for $1\le k<i\le n$, we have $\theta_i-\theta_k\le\theta-\theta_k\le\pi$, so that

$$\sin\frac{1}{2}(\theta-\theta_k)\ge\sin\frac{1}{2}(\theta_i-\theta_k)=\sin\frac{i-k}{2(n+1)}\pi>\frac{i-k}{n+1}$$

while, for $0 \leq i \leq n-2$, $i+2 \leq k \leq n$, we have $\theta_k - \theta_{i+1} \leq \theta_k - \theta \leq \pi$ and

$$\sin \frac{1}{2}(\theta_k - \theta) \geq \sin \frac{1}{2}(\theta_k - \theta_{i+1}) > \frac{k-i-1}{n+1}.$$

These inequalities prove the first two parts of (35). The last two parts follow from the fact that

$$\begin{aligned} |\sin(n+1)\theta| &= |\sin(n+1)\theta - \sin(n+1)\theta_i| \\ &= \left| 2\sin \frac{1}{2}(n+1)(\theta-\theta_i) \cos \frac{1}{2}(n+1)(\theta+\theta_i) \right| \\ &\leq 2\left| \sin(n+1)\frac{1}{2}(\theta-\theta_i) \right| \leq 2(n+1)\sin \frac{1}{2}|\theta-\theta_i|. \end{aligned}$$

From (35) we have

$$\sum_{k=2}^{n} E_k^m = \sum_{k=1}^{i-1} (i-k)^{-m} + 2^m + 2^m + \sum_{k=i+2}^{n} (k-i-1)^{-m} \tag{36}$$

$$\leq 2\sum_{k=1}^{\infty} k^{-m} + 2^{m+1} < 4 + 2^{m+1} \quad \text{if } m = 2, 3, \ldots$$

We shall now obtain proofs of inequalities (21)—(25) with the help of (36). Since

$$\sin \theta_k \leq \sin \theta_k + \sin \theta = 2\sin \frac{1}{2}(\theta+\theta_k) \cos \frac{1}{2}(\theta-\theta_k) \leq 2\sin \frac{1}{2}(\theta+\theta_k)$$

and

$$\sin \theta \leq \sin \theta + \sin \theta_k \leq 2\sin \frac{1}{2}(\theta+\theta_k),$$

therefore, from (19), we have

$$|\lambda_k^*(\cos\theta)| \leq \frac{\sin^3(n+1)\theta}{(n+1)\sin^3\left|\frac{1}{2}(\theta-\theta_k)\right|} = E_k^3 \tag{37}$$

and

$$|\cos\theta - \cos\theta_k|\,|\lambda_k^*(\cos\theta)| \leq \frac{\sin\theta}{n+1} E_k^2. \tag{38}$$

Hence, from (36), (37) and (38), inequalities (21) and (22) follow.

Similarly, from (20),

$$\begin{aligned} |\lambda_k^{**}(\cos\theta)| \leq{} & \frac{\sin^4(n+1)\theta}{(n+1)^4 \sin^4 \frac{1}{2}(\theta-\theta_k)} + \frac{\sin^4(n+1)\theta}{(n+1)^4 \sin \frac{1}{2}(\theta+\theta_k) \sin^3 \frac{1}{2}|\theta-\theta_k|} \\ & + \frac{\sin^4(n+1)\theta}{(n+1)^3 \sin^2 \frac{1}{2}(\theta-\theta_k)} + \frac{\sin^4(n+1)\theta}{(n+1)^4 \sin^2 \frac{1}{2}(\theta-\theta_k)} \left[\frac{1}{\sin^2 \frac{1}{2}(\theta-\theta_k)} + \frac{1}{\sin^2 \frac{1}{2}(\theta+\theta_k)} \right]. \end{aligned}$$

But

therefore

$$\lambda_k^{**}(\cos \theta) \leq 4 \frac{\sin^4 (n+1)\theta}{(n+1)\sin^4 \frac{\theta-\theta_k}{2}} + \frac{\sin^4 (n+1)\theta}{(n+1)^3 \sin^2 \frac{1}{2}(\theta-\theta_k)}$$

Thus

$$\lambda_k^{**}(\cos \theta) \mid \leq 4E_k^4 + \frac{\sin (n+1)\theta}{n+1} E_k^2 \leq 4E_k^4 + E_k^2. \tag{39}$$

$$\lambda_k^{**}(\cos \theta) \sin \frac{1}{2} \theta - \theta_k \mid \leq \frac{4 \sin (n+1)\theta}{n+1} E_k^3 + \frac{\sin^2 (n+1)\theta}{n+1} E_k^2 \leq \frac{1}{n+1} (4E_k^3 + E_k^2) \tag{40}$$

and

$$\lambda_k^{**}(\cos \theta) \sin^2 \frac{1}{2} (\theta - \theta_k) \leq \frac{4\sin^2 (n+1)\theta}{(n+1)^2} E_k^2 + \frac{\sin^4 (n+1)\theta}{(n+1)^3} \leq \frac{1}{(n+1)^2} [4 E_k^2 + 1]. \tag{41}$$

Hence, from (39), (40) and (41), on using (36), we get inequalities (23), (24), and (25).

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АПРОКСИМИРАНЕ НА НЕПРЕКЪСНАТИ ФУНКЦИИ С ИНТЕРПОЛАЦИОННИ ПОЛИНОМИ

Р. Б. Саксена

(Резюме)

Съгласно класическата теория на Джексон за всяка непрекъсната в $[-1, 1]$ функция $f(x)$ с модул на непрекъснатост $\omega_f(\delta)$ съществува полином $P(x)$ най-много от степен n, за който

$$\max_{|x|\leq 1} |f(x) - P(x)| = O\left(\omega_f\left(\frac{1}{n}\right)\right).$$

В една предишна своя работа [4] авторът доказва горната теорема на Джексон с един експлицитно построен интерполационен полином.

В работите на С. М. Никольский и А. Ф. Тиман се появиха подобрения на теоремата на Джексон, от които се вижда, че приближаващият полином може да се избере така, че в краищата на интервала да осъществява по-добро приближение при запазване на порядъка на общото равномерно приближение.

В тази работа авторът доказва, че построеният от него в споменатата му предишна работа интерполационен полином $\lambda_n(f;\, x)$ удовлетворява за всяка непрекъсната функция $f(x)$ и интервала $[-1, 1]$ неравенството

$$|f(x)-\lambda_n(f;\, x)| \leq 384\left[\omega_f\left(\frac{\sqrt{1-x^2}}{n}\right)+\omega_f\left(\frac{|x|}{n}\right)\right]\cdot$$

Този резултат представлява ново доказателство на теоремата на А. Ф. Тиман чрез използване на интерполационен полином.

ПРИБЛИЖЕНИЕ НЕПРЕРЫВНЫХ ФУНКЦИЙ ИНТЕРПОЛЯЦИОННЫМИ МНОГОЧЛЕНАМИ

Р. Б. Саксена

(Резюме)

Согласно классической теории Джексона для любой непрерывной в $[-1, 1]$ функции $f(x)$ с модулем непрерывности $\omega_f(\delta)$ существует многочлен $P(x)$, степени не больше n, для которого

$$\max_{|x|\leq 1}|f(x)-P(x)| = O\left(\omega_f\left(\frac{1}{n}\right)\right)\cdot$$

В одной из предыдущих своих работ [4] автор доказывает вышеуказанную теорему Джексона с помощью явно построенного интерполяционного многочлена.

В работах С. М. Никольского и А. Ф. Тимана появились улучшения теоремы Джексона, из которых видно, что приближающий многочлен можно подобрать так, что в концах интервала осуществляется лучшее приближение при сохранении порядка общего равномерного приближения.

В этой работе автор доказывает, что построенный им в упомянутой его работе интерполяционный многочлен $\lambda_n(f;\, x)$ удовлетворяет для любой непрерывной функции $f(x)$ в интервале $[-1, 1]$ неравенству

$$|f(x)-\lambda_n(f;\, x)| \leq 384\left[\omega_f\left(\frac{\sqrt{1-x^2}}{n}\right)+\omega_f\left(\frac{|x|}{n^2}\right)\right]\cdot$$

Этот результат представляет новое доказательство теоремы Тимана с помощью использования интерполяционного многочлена.