Asymptotic properties of Chlodovsky polynomials

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Let f be a real function on $[0,+\infty)$. For $b>0$, define the function f_b on $\left[0,b^{-1}\right]$ by $f_b(t)=f(bt)$. Furthermore, put $\|f\|_b=\sup_{0\le t\le b}|f(t)|$. In order to approximate functions on infinite intervals, Chlodovsky introduced the positive linear operators C_{n,b_n} defined by

$$(C_{n,b_n}f)(x)=(B_nf_{b_n})\left(\frac{x}{b_n}\right),$$

where B_n denote the Bernstein polynomials and the sequence (b_n) satisfies

$$b_n>0,\qquad \lim_{n\to\infty}b_n=+\infty,\qquad \text{and}\qquad \lim_{n\to\infty}\frac{b_n}{n}=0. \tag{1}$$

He showed that, if a function f satisfies

$$\lim_{n\to\infty}\exp\left(-\frac{\sigma n}{b_n}\right)\|f\|_{b_n}=0 \tag{2}$$

for every $\sigma>0$, then

$$\lim_{n\to\infty}(C_{n,b_n}f)(x)=f(x)$$

at each point x of continuity of f. Moreover, he proved convergence in each continuity point for the large class of functions f satisfying the growth condition $f(t)=O(\exp(t^p))$ as $t\to+\infty$, if the sequence (b_n) satisfies the condition

$$b_n=O\left(n^{1/(p+1+\eta)}\right)\qquad (n\to\infty), \tag{3}$$

for an arbitrary small $\eta>0$. For more results on Chlodovsky operators see the survey article [5] by Karsli.

The talk presents a pointwise complete asymptotic expansion for the sequence of Bernstein-Chlodovsky operators in the form

$$(C_{n,b_n}f)(x)\sim f(x)+\sum_{k=1}^{\infty}c_k^{[b_n]}(f,x)\left(\frac{b_n}{n}\right)^k\qquad (n\to\infty), \tag{4}$$

for sufficiently smooth functions f satisfying $f(t)=O(\exp(\alpha t^p))$ as $t\to+\infty$. The coefficients $c_k^{[b_n]}(f,x)$, which depend on f and b_n, are bounded with respect to n.

The latter formula means that, for each fixed $x>0$ and for all positive integers q,

$$(C_{n,b_n}f)(x)=f(x)+\sum_{k=1}^{q}c_k^{[b_n]}(f,x)\left(\frac{b_n}{n}\right)^k+o\left(\left(\frac{b_n}{n}\right)^q\right)\qquad (n\to\infty).$$

Explicit expressions of the coefficients $c_k^{[b_n]}(f,x)$ in terms of Stirling numbers were given by Harun Karsli [6]. He derived the asymptotic expansion if the function f satisfies condition (2) for every $\sigma > 0$.

As a second part we discuss the same problem for the Durrmeyer variant of the Chlodovsky operators given by

$$\left(\tilde{C}_{n,b_n} f\right)(x) = (M_n f_{b_n})\left(\frac{x}{b_n}\right),$$

where M_n are the Bernstein-Durrmeyer operators. The corresponding results for the multivariate Bernstein and Bernstein-Durrmeyer operators were obtained in [1, 3].

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Proof of a conjecture of M. Patrick concerning Jacobi polynomials

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If f is an algebraic polynomial of degree not exceeding n with real coefficients and real zeros, then by Jensen's formula

$$|f(x+iy)|^2 = \sum_{k=0}^{n} L_k(f;x)\, y^{2k}\,, \quad (x,y) \in \mathbb{R}^2,$$

where $L_0(f;x) = f^2(x)$ and

$$L_k(f;x) = \sum_{j=0}^{2k} (-1)^{k-j}\, \frac{f^{(j)}(x)}{j!}\, \frac{f^{(2k-j)}(x)}{(2k-j)!}, \quad k = 1, \ldots, n.$$

A conjecture of M. Patrick from 1971 (see [3]) states that if $P = P_n^{(\alpha,\beta)}$ is the n-th Jacobi polynomial and $\alpha \geq \beta > -1$, then for $k = 1, 2, \ldots, n$,

$$\max_{x\in[0,1]} L_k(P; x) = L_k(P; 1).$$

Patrick's conjecture pertains to the famous refinement of the Markov inequality found by Duffin and Schaeffer [2]. In a recent paper [1] (joint with Helge Dietert (Cambridge), Geno Nikolov (Sofia) and Veronika Pillwein (Linz) we proved Patrick's conjecture even in a stronger form. The talk will be focused on the idea of the proof.

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Self adjoint operator Korovkin type and polynomial direct approximations with rates

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Here we present self adjoint operator Korovkin type theorems, via self adjoint operator Shisha-Mond type inequalities, also we give self adjoint operator polynomial approximations. This is a quantitative treatment to determine the degree of self adjoint operator uniform approximation with rates, of sequences of self adjoint operator positive linear operators. The same kind of work is performed over important operator polynomial sequences. Our approach is direct based on Gelfand isometry.

Asymptotically optimal definite quadrature formulae of 4-th order

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The talk is based on the paper [1]. We construct several sequences of asymptotically optimal definite quadrature formulae of fourth order and evaluate their error constants. Besides the asymptotical optimality, an advantage of our quadrature formulae is the explicit form of their weights and nodes. For the remainders of these quadrature formulae monotonicity properties are established when the integrand is a 4-convex function, and a-posteriori error estimates are proven.

This work was supported by the Sofia University Research Fund through Contract 30/2016.

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Bounds for the radii of univalence of some special functions

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In this talk some tight lower and upper bounds for the radius of univalence of some normalized Bessel, Struve and Lommel functions of the first kind are obtained via Euler-Rayleigh inequalities. It is shown also that the radius of univalence of the Struve functions is greater than the corresponding radius of univalence of Bessel functions. Moreover, by using the idea of Kreyszig and Todd, and Wilf it is proved that the radii of univalence of some normalized Struve and Lommel functions are exactly the radii of starlikeness of the same functions. The Laguerre-Pólya class of entire functions plays an important role in our study. The talk is based on the paper [1]

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Universal lower bounds for energy of codes in polynomial metric spaces

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We derive and investigate lower bounds for the potential energy of codes in polynomial metric spaces. Our bounds are optimal in the following sense – they cannot be improved by employing polynomials of the same or lower degrees in pure linear programming method. However, improvements are sometimes possible and we provide a necessary and sufficient condition for the existence of such better bounds. All our bounds can be obtained in a unified manner that does not depend on the potential function, provided the potential is given by an absolutely monotone function of the inner product between pairs of points, and this is the reason for us to call them universal.

Moduli of smoothness and polynomial approximation on spheres and balls

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There are several different well studied moduli of smoothness on the unit sphere, including the classical one defined via the translation operators (i.e., averages over rims of spherical caps), the recent one introduced by Z. Ditzian via the group of rotations, and the most recent one introduced by Y. Xu and myself via finite order differences over Euler angles. In this talk, I will survey properties of these three different moduli of smoothness and some recent results related to them, such as the direct Jackson inequality and its Stechkin type inverse, the strong inverse inequality of type A and the equivalence with different K-functionals. I will also compare these moduli of smoothness and show that they are in fact equivalent in L^p spaces with $1 < p < \infty$. Finally, I will discuss how these results on the sphere can be used to establish interesting results on algebraic polynomial approximation on the unit ball. In particular, I will discuss two new moduli of smoothness on the unit ball introduced by Y. Xu and myself, one of which is in analogy with the classical Ditzian-Totik moduli of smoothness on intervals.

Many of the results in this talk are from my joint works with Z. Ditzian and my joint works with Y. Xu.

Rubel's problem on bounded analytic functions

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The purpose of this talk is to show that for any G_δ set F of Lebesgue measure zero on the unit circle T there exists a bounded analytic function f in the unit disc such that the radial limits of f exist at each point of T and vanish precisely on F. This solves a problem proposed by L. Rubel in 1973. Note that Rubel's problem has been published in W. Hayman's well-known collection *"Research problems in function theory: new problems"* of 1974; it is Problem 5.29 in Hayman's paper.

Approximation by multivariate piecewise polynomial splines

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I will give an overview of the ideas and results in the area of multivariate splines on non-uniform partitions, concentrating on the topics of stable local bases, quasi-interpolation operators, and applications to data fitting and numerical PDEs.

Jacobi translation and Nikol'skii inequality for algebraic polynomials on a closed interval

Marina Deikalova

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We will discuss the sharp Nikol'skii inequality for algebraic polynomials on the interval $[-1,1]$ between the uniform norm and the norm of the space $L_q^{(\alpha,\beta)}$, $1 \le q < \infty$, with the Jacobi weight $\phi^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha \ge \beta > -1$. In the study, we use the generalized translation operator generated by the Jacobi weight. The set of functions at which the norm of this operator in the space $L_q^{(\alpha,\beta)}$, $1 \le q < \infty$, $\alpha > \beta \ge -1/2$, is attained will be described.

The results are obtained by the author jointly with V. V. Arestov [1].

This work was supported by the Russian Foundation for Basic Research (project no. 15-01-02705).

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Nonsmooth duals for constrained best approximations

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We put together results from a variety of sourses to show that the Lagrange duals of constrained best approximation problems are associated with nonsmooth functions. The problem of best approximation in a tube will be exemplified in particular.

Estimates of the simultaneous approximation by the Bernstein and Kantorovich operators

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We shall present a characterization of the error of the weighted simultaneous approximation by the Bernstein operator and certain combinations of its iterates, which increase its approximation rate. These combinations can be regarded as iterated Boolean sums of the Bernstein operator. They are defined by

$$\mathcal{B}_{r,n} = I - (I - B_n)^r,$$

where B_n denotes the Bernstein operator, I stands for the identity, and $r \in \mathbb{N}$. It is known that the saturation order of B_n is n^{-1}, whereas that of $\mathcal{B}_{r,n}$ is n^{-r}.

Our main goal is to generalize those results for the case of weighted simultaneous approximation in L_p, $1 < p \leq \infty$. We consider Jacobi weights. Our interest in estimating the error in L_p for $p < \infty$ is motivated by the relation between the Bernstein operator and its Kantorovich modification given in terms of derivatives and antiderivatives, which allows the direct transference of results on simultaneous approximation by the Bernstein operator to the Kantorovich one.

Let $f \in C[0,1]$ be such that $f \in AC_{loc}^{s-1}(0,1)$ and $wf^{(s)} \in L_p[0,1]$ as the exponents of the Jacobi weight w satisfy certain restrictions. We shall present the following direct estimate

$$\|w(\mathcal{B}_{r,n}f - f)^{(s)}\|_p \le c\, K_{r,s}(f^{(s)}, n^{-r})_{w,p}.$$

The K-functional $K_{r,s}(f^{(s)}, t)_{w,p}$ is given by

$$K_{r,s}(f^{(s)}, t)_{w,p} = \inf_{g \in C^{2r+s}[0,1]} \left\{ \|w(f^{(s)} - g^{(s)})\|_p + t\|w(D^r g)^{(s)}\|_p \right\},$$

where $Dg(x) = x(1-x)g''(x)$, and c is a constant, which is independent of f and n. Also, a matching two-term strong converse estimate holds. We were able to strengthen the latter for low-order derivatives in the uniform norm.

The involved K-functional $K_{r,s}(f,t)_{w,p}$ is further characterized by simpler ones and the Ditzian-Totik modulus of smoothness.

On the checkmark function polynomial approximation

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We consider the best uniform approximation of the "checkmark function" $|x - a|$ by polynomials on the interval $[-1, 1]$. The dynamic behavior of the points of alternation in terms of the parameter a is investigated. For certain values of this parameter the approximation gives rise to the classical polynomials of Chebyshev and Zolotarev. We include numerical computations to illustrate the results.

Approximation of the differentiation operator by linear bounded operators on the class of smooth functions in $L_2(0,\infty)$

Maria Filatova

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The solution of the problem of the best approximation of the differentiation operator by linear bounded operators on the class of twice differentiable functions in the space $L_2(0,\infty)$ have been obtained in [1].

We will discuss the problem of the best approximation of the (first-order) differentiation operator by linear bounded operators on the class of three times differentiable functions in the space $L_2(0,\infty)$. The upper estimate for the value of the best approximation will be presented. To prove the upper estimate, we consider a specific operator.

The results are obtained by the author jointly with V. V. Arestov.

This work was supported by the Russian Foundation for Basic Research (project no. 15-01-02705).

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Permutations fixing a k-set

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Let $1 \le k \le n/2$. For a randomly chosen permutation $\pi \in S_n$, what is the probability that π fixes a set of size k, that is, there is some subset $I \subset S_n$ of size k such that π permutes the elements of I? Equivalently, what is the probability that the cycle decomposition of π contains disjoint cycles with lengths summing to k? When $k = 1$, this is the classical derangement problem. We show that, uniformly for all n and k, the probability lies between two constant multiples of $k^{-c}(1+\log k)^{-3/2}$, where $c = 1 - \frac{1+\log\log 2}{\log 2} = 0.08607...$, improving on earlier bounds on Luczak-Pyber and Diaconis-Fulman-Guralnick. The proof uses a probabilistic model for permutations, the approximation of a random permutation by a vector of independent Poisson random variables, as well as ideas from number theory motivated by an analogy between the cycle decomposition of permutations and the prime factorization of integers. Some applications will be briefly discussed (invariable generation of S_n, permutations contained in transitive subgroups of S_n). Joint work with Sean Eberhard and Ben Green.

Approximation of functions in L_p-norm by Kantorovich modifications of some classical operators

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The approximation of functions in L_p-norm by Kantorovich modifications of classical Baskakov and Meyer-König and Zeller operators is discussed. By defining appropriate K-functionals the direct theorems and some strong converse inequalities of type B in terms of the K-functionals are proved.

An algorithm for low-rank approximation of bivariate functions using splines

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We present an algorithm for the approximation of bivariate functions by "low-rank splines", that is, sums of outer products of univariate splines. Our approach is motivated by the Adaptive Cross Approximation (ACA) algorithm for low-rank matrix approximation as well as the use of low-rank function approximation in the recent extension of the chebfun package to two dimensions. The resulting approximants lie in tensor product spline spaces, but typically require the storage of far fewer coefficients than tensor product interpolants. We analyze the complexity and show that our proposed algorithm can be efficiently implemented in terms of the cross approximation algorithm for matrices using either full or row pivoting.

We present several numerical examples which show that the performance of the algorithm is reasonably close to the best low-rank approximation using truncated singular value decomposition and leads to dramatic savings compared to full tensor product spline interpolation.

The presented algorithm has interesting applications in isogeometric analysis as a data compression scheme, as an efficient representation format for geometries, and in view of possible solution methods which operate on tensor approximations.

Grüss–Voronovskaya estimates for Bernstein–type polynomials

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We first give a brief survey of recent progress in Grüss-type inequalities for positive linear operators and then proceed by proving a so-called Grüss-Voronovskaya estimate for classical Bernstein operators and for a certain scale of Bernstein-Durrmeyer mappings on C[0,1].

Moreover, Grüss and Grüss-Voronovskaya inequalities for the corresponding operators defined for functions of a a complex variable on compact disks are obtained.

The results are extended to Bernstein-Faber polynomials attached to compact sets in the complex plane.

The talk is based on joint work with Sorin Gal (University of Oradea, Romania).

Keywords: Bernstein polynomials of real and complex variables, Bernstein-Durrmeyer-type polynomials of real and complex variables, Faber polynomials, Bernstein-Faber polynomials, Grüss-type estimate, Grüss-Voronovskaya-type estimate, analytic functions.

MSC: 41A10, 41A25, 30E10.

Numerical approximation of the inverse principal square root of an M-matrix, using univariate polynomial techniques

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Fractional powers of sparse matrices are usually dense, so their explicit numerical computation and memory storage is extremely expensive for large-scale problems. On the other hand, such nonlocal operators appear naturally in the mathematical modeling of various real-life processes. In this talk, we focus on the inverse square root $A^{-1/2}$ of an M-matrix A. We use the best univariate polynomial approximation $U_k(x)$ of $x^{-1/2}$ on the spectral interval $[\lambda_{min}, \lambda_{max}]$ of A, and show that the action of the operator $A^{-1/2}$ can be well-approximated by the action of $U_k(A)$, when A is well-conditioned. Due to this result, we develop a general framework for 3D 2-phase image segmentation, that is volume-preserving and allows for adding arbitrary linear terms into the optimization function in order to engineer the physical characteristics of the output phases. The approximation error $\|U_k(A)v - A^{-1/2}v\|_{A^{1/2}}/\|v\|_{A^{1/2}}$ is independent of the domain size, thus our approach is applicable to high resolution data. The theoretical analysis is backed up with numerical experiments.

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A nice representation for a link between Bernstein-Durrmeyer and Kantorovich operators

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In [1] Păltănea introduced a class of operators $B_{n,\rho}$ depending on a nonnegative real parameter ρ which constitute a nontrivial link between the genuine Bernstein-Durrmeyer operator and the classical Bernstein operator. We denote by $\mathcal{L}_B[0,1]$ the space of bounded Lebesgue integrable functions on $[0,1]$.

For $j \in \mathbb{N}_0$, $0 \le j \le n$, the Bernstein basis functions are given by

$$p_{n,j}(x) = \binom{n}{j} x^j (1-x)^{n-j},\ 0 \le j \le n,\ x \in [0,1].$$

Moreover, for $1 \le j \le n-1$,

$$\mu_{n,j,\rho}(t) = \frac{t^{j\rho-1}(1-t)^{(n-j)\rho-1}}{B(j\rho,(n-j)\rho)}$$

with Euler's Beta function $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $x, y > 0$. For $n \in \mathbb{N}$, $n \ge 2$, $\rho \in \mathbb{R}_+$, Păltănea [1, Definition 2.1] defined the operators $B_{n,\rho} : \mathcal{L}_B[0,1] \longrightarrow \mathcal{P}_n$ by

$$\begin{aligned} B_{n,\rho}(f;x) &= p_{n,0}(x)f(0) + p_{n,n}(x)f(1) \\ &\quad + \sum_{j=1}^{n-1} p_{n,j}(x) \int_0^1 \mu_{n,j,\rho}(t)f(t)dt. \end{aligned}$$

In [2, Theorem 2.3] Gonska and Păltănea proved the convergence of the operators $B_{n,\rho}$ to the classical Bernstein operator $B_{n,\infty}$, i.e., they proved that for every $f \in C[0,1]$

$$\lim_{\rho\to\infty} B_{n,\rho} f = B_{n,\infty} f \text{ uniformly on } [0,1].$$

In our talk we consider the first order Kantorovich modifications, i.e., $B_{n,\rho}^{(1)} = D \circ B_{n,\rho} \circ I$, where D denotes the ordinary differential operator and $I(f,x) = \int_0^x f(t)dt$; in other words, we look at a link between Bernstein-Durrmeyer and Kantorovich operators. We will show how these linking operators can be represented in terms of well-known classical operators.

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Highly localized kernels on the sphere induced by discrete Newtonian potentials

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In the lecture we present constructions of linear combinations f of several Newtonian kernels,

$$f(x) = \sum_{k=1}^{m} \frac{a_k}{|x-y_k|^{d-2}}, \quad \text{if } d > 2;$$

$$f(x) = \sum_{k=1}^{m} a_k \ln \frac{1}{|x-y_k|}, \quad \text{if } d = 2,$$

with complex coefficients a_k and poles y_k outside the unit ball B^d in $\mathbb{R}^d$ with *high localization on the unit sphere* $\mathbb{S}^{d-1}$.

We say that a function f defined on $\mathbb{S}^{d-1}$ is *localized around* $\eta \in \mathbb{S}^{d-1}$ with dilation factor $N \geq 1$ and decay rate $M > d-1$ if the estimate

$$|f(x)| \leq cN^{d-1}(1 + N\rho(\eta, x))^{-M}, \quad x \in \mathbb{S}^{d-1},$$

holds for some constant $c > 0$ depending only on d and M but not on N and η. Here ρ denotes the geodesic distance on $\mathbb{S}^{d-1}$. We also require f to be normalized by

$$\int_{\mathbb{S}^{d-1}} f(y)\, d\sigma(y) = 1.$$

For any $m \geq 2$ we achieve the decay rate $M = 2m+d-4$ by m-term combinations of Newtonian kernels depending on arbitrary $N \geq 1$ and $\eta \in \mathbb{S}^{d-1}$.

We emphasize that the localization is only *on the boundary* $\mathbb{S}^{d-1}$ of the unit ball. As far as every such f is a harmonic function in B^d it cannot be localized *inside* the ball (in the sense of $\mathbb{R}^d$).

The results are applied to the construction of frames for Besov and Triebel-Lizorkin spaces with all elements consisting of one and the same number of Newtonian kernels and to the best nonlinear n-term approximation of harmonic functions from Newtonian potentials.

General types of spherical mean operator and K-functionals of fractional orders

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We design a general type of spherical mean operators, depending on a real number as parameter, and employ them to approximate L_p class functions. We show that optimal orders of approximation are achieved via appropriately defined K-functionals of fractional orders. Asymptotic relations between the rate of approximation of the new operator and the K-functional of fractional order were established. When the parameter we work with is taken as a natural number the general type of spherical mean operator, the K-funcional and also the result relating such objects turn out the same as in Dai & Ditzian (2004) which introduces a class of "multi-layered" spherical mean operators. More details can be found in Jordão & Sun (2015).

This research is partially supported by FAPESP #2014/06209-1.

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Inequalities for series in q-shifted factorials and q-Gamma functions

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Mainly we will discuss power series with general nonnegative coefficients dependent on an additional parameter included as an argument of q-shifted factorial or q-Gamma function. Several types of such series will be considered. We will demonstrate how non-negativity or logarithmic concavity of the coefficients leads to q-log-convexity or q-log-concavity as well as multiplicative convexity or concavity for the sum of the series. Applications to modified q-Bessel functions and other q-hypergeometric functions will be given as well. Some proofs depend on curious q-hypergeometric identities that may be of independent interest.

This is based on a joint work with Dmitrii Karp.

The spectrum of the partial theta function

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The *partial theta function* is the sum of the series $\theta(q,z) := \sum_{j=0}^{\infty} q^{j(j+1)/2} z^j$ defined for $q \in \mathbb{D}_1$ (i.e. $|q| < 1$), $z \in \mathbb{C}$. Its *spectrum* is the set of values of the parameter q for which $\theta(q,.)$ has a multiple zero. We prove the following

Theorem 1.

(1) *The number* $\tilde{q} := 0.3092493386\ldots$ *is the element of the spectrum closest to* 0 *and the only one in the disk* $\mathbb{D}_{0.31}$. *The function* $\theta(\tilde{q},.)$ *has a double zero* $-7.5032559833\ldots$*; its other zeros are simple.*

(2) *For any disk* $\mathbb{D}_a$, $a \in (0,1)$, *there exists* $b > 0$ *such that* θ *has no multiple zeros in* $B_{a,b} := \mathbb{D}_a \times (\mathbb{C} \backslash \mathbb{D}_b)$, *it has only isolated spectral numbers for* $q \in \mathbb{D}_a$, *and in* $E_{a,b} := \mathbb{D}_a \times \mathbb{D}_b$ *it has at most finitely-many multiple zeros.*

(3) *The function* θ *has no multiple zeros outside the disk* $\mathbb{D}_{8^{11}}$, *for any fixed* $q \in \mathbb{D}_1$.

(4) *The zeros* ξ_j *of* θ *can be expanded in Laurent series in* q, $\xi_j = -1/q^j + O(q^{j(j-1)/2})$, *convergent for* $q \in \mathbb{D}_{\rho_j} \backslash 0$, $\rho_j \geq 0.108$. *The sequence* ρ_j *tends to* 1 *as* j *tends to* ∞.

(5) *Set* $\alpha_0 := \sqrt{3}/2\pi = 0.2756644477\ldots$ *For* $n \geq 5$, *for* $|q| \leq 1 - 1/(\alpha_0 n)$ *and for* $k \geq n$ *there exists a unique zero* ξ_k *of* θ *(which is a simple one) such that* $|q|^{-k+1/2} < |\xi_k| < |q|^{-k-1/2}$.

(6) *For* $q \in (0,1)$ *there exists a sequence of spectral numbers* $\tilde{q}_j$ $(\tilde{q}_1 = \tilde{q})$ *such that* $\theta(\tilde{q}_j, .)$ *has a double zero* y_j *its other zeros being simple. One has* $\tilde{q}_j = 1 - \pi/2j + (\ln j)/8j^2 + O(1/j^2)$, $y_j = -e^{\pi}e^{-\ln j/4j + O(1/j)}$.

(7) *For* $q \in (-1,0)$ *the spectral values are of the form* $\bar{q}_j = -1 + (\pi/8j) + o(1/j)$, $\theta(\bar{q}_j, .)$ *has a double zero* $\bar{y}_j$ *(its other zeros being simple), where* $|\bar{y}_j| \to e^{\pi/2} = 4.810477382\ldots$ *as* $j \to \infty$ *and* $\operatorname{sgn}(\bar{y}_j) = (-1)^j$.

An X-greedy algorithm with weakness parameters

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Currently, there are no results for convergence of the X-greedy algorithm for Banach spaces for general dictionaries. We give examples of Banach spaces for which the algorithm converges for special dictionaries.

Homogeneous spaces of distributions associated with operators

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Homogeneous Besov and Triebel-Lizorkin spaces with complete set of indices are introduced in the setting of a doubling metric measure space in the presence of a non-negative self-adjoint operator whose heat kernel has Gaussian localization and the Markov property. Almost diagonal operators on the associated sequence homogeneous Besov and Triebel-Lizorkin spaces are developed. The boundedness of almost diagonal operators is utilized for establishing smooth atomic and molecular decomposition of homogeneous Besov and Triebel-Lizorkin spaces as well as for development of spectral multipliers.

On a potential theoretic minimax problem on the torus. II

Bálint Farkas[1], *Béla Nagy*[2], *and Szilárd Gy. Révész*[3,4]

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In this talk some new results will be presented based on the joint work with Bálint Farkas and Szilárd Gy. Révész. The talk is natural continuation of the talk of Szilárd Gy. Révész and the results are also available at the `http://arxiv.org/abs/1512.09169` webpage.

First, we recall some general results and notation on general potential theory (e.g. potentials, Chebyshev constants), and then some well known phenomena originating in interpolation theory (e.g. equioscillation property, sandwich property). We also cite some new results on minimax problems on the torus.

Then we discuss a result of Bojanov on algebraic polynomials with prescribed zero order having minimal sup-norm on interval. As an application of our general framework we show a generalization of Bojanov's result for generalized algebraic polynomials (GAP) and generalized trigonometric polynomials (GTP) as well.

Some exact inequalities for polynomials on the standard triangle

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An actual field of research in the theory of approximations is to extend the classical polynomial inequalities to the multivariate setting. This question concerning Bernstein and Markov inequalities is satisfactorily settled for centrally symmetric convex bodies. In spite of the presence of good estimates, exact inequalities of Bernstein's type for nonsymmetric convex bodies and pointwise Markov's type inequalities are not known.

We prove that the approach based on the Krein-Milman theorem can be applied to maximize the nonlinear functional, which corresponds to the estimate of Bernstein-Szegő type for the gradients of arbitrary polynomials on convex bodies.

As applications we prove exact Bernstein-Szegő inequalities for some classes of bivariate polynomials on the standard triangle Δ. Note that in a certain sense Δ is the least symmetric convex body in $\mathbb{R}^2$.

We also present a sharp pointwise estimation of Markov's type in a set of strictly definite polynomials on Δ.

Markov-type inequality in the L_2-norm induced by Laguerre weight

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Let $w_\alpha(t) = t^\alpha e^{-t}$, $\alpha > -1$, be the Laguerre weight function, and $\|\cdot\|_{w_\alpha}$ denote the associated L_2-norm, i.e.,

$$\|f\|_{w_\alpha} := \Big(\int_0^\infty w_\alpha(t)|f(t)|^2\,dt\Big)^{1/2}.$$

Denote by $\mathcal{P}_n$ the set of algebraic polynomials of degree not exceeding n. We study the best constant $c_n(\alpha)$ in the Markov inequality in this norm,

$$\|p'\|_{w_\alpha} \le c_n(\alpha)\,\|p\|_{w_\alpha}\,, \quad p \in \mathcal{P}_n\,,$$

namely the constant

$$c_n(\alpha) = \sup_{\substack{p\in\mathcal{P}_n\\ p\neq 0}} \frac{\|p'\|_{w_\alpha}}{\|p\|_{w_\alpha}}\,,$$

and we are also interested in its asymptotic value

$$c(\alpha) = \lim_{n\to\infty} \frac{c_n(\alpha)}{n}\,.$$

We give tight lower and upper bounds for both $c_n(\alpha)$ and $c(\alpha)$. Note that according to a result of P. Dörfler from 2002, $c(\alpha) = [j_{(\alpha-1)/2,1}]^{-1}$, with $j_{\nu,1}$ being the first positive zero of the Bessel function $J_\nu(z)$, hence our bounds for $c(\alpha)$ imply bounds for $j_{(\alpha-1)/2,1}$ as well.

The derivatives of polynomials as a linear combination of finite differences

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In this paper the derivatives of any polynomial are represented as a linear combination of finite differences of the polynomial. The coefficients are found explicitly. This representation is useful when one works with discrete orthogonal polynomials where the finite differences are known.

Orthogonal projection operators onto spline spaces

Markus Passenbrunner

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We prove that the spline orthoprojector onto spaces of periodic splines is L^∞-bounded independently of the knot sequence. The idea is to reduce the periodic case to the case of spline projectors on intervals, in which the result is a famous theorem by A. Shadrin [Acta Math., 187 (2001), 59-137].

Nonlinear n-term approximation of harmonic functions from discrete Newtonian potentials

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We consider the problem for approximation of harmonic functions on the unit ball B^d in $\mathbb{R}^d$ from finite linear combinations of shifts of the fundamental solution of the Laplace equation (Newtonian potential) $\frac{1}{|x|^{d-2}}$ in dimensions $d > 2$ or $\ln \frac{1}{|x|}$ if

$d = 2$. More explicitly, the problem is for a given harmonic function U on B^d and $n \geq 1$ to find n locations $\{y_j\}$ in $\mathbb{R}^d \setminus \overline{B^d}$ and coefficients $\{c_j\}$ such that

$$c_0 + \sum_{j=1}^{n} \frac{c_j}{|x - y_j|^{d-2}} \quad \text{if } d > 2 \quad \text{or} \quad c_0 + \sum_{j=1}^{n} c_j \ln \frac{1}{|x - y_j|} \quad \text{if } d = 2$$

approximates U well (near best) in the harmonic Hardy space $\mathscr{H}^p(B^d)$. Our main results is a Jackson estimate for nonlinear n-term approximation in $\mathscr{H}^p(B^d)$, $1 < p < \infty$, from shifts of the Newtonian potential in terms of certain harmonic Besov spaces. To achieve this we first study the harmonic Besov and Triebel-Lizorkin spaces on B^d. It is shown that these spaces can be identified with respective Besov and Triebel-Lizorkin spaces of distributions on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$. Frames consisting of harmonic functions are also developed and frame characterization of the harmonic Besov and Triebel-Lizorkin spaces is established. The key idea is to construct by small perturbation of an existing frame a new frame whose elements are linear combinations of small (uniformly bounded) number of shifts of the Newtonian potential, they are highly localized on $\mathbb{S}^{d-1}$, and the frame coefficients characterize the Besov spaces of interest. A Jackson estimate for nonlinear n-term approximation in $\mathscr{H}^p(B^d)$ from this frame is established, which in turn implies the desired Jackson estimate for nonlinear n-term approximation from shifts of the Newtonian potential.

Best rational approximation of functions with logarithmic singularities

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I will report on the results of my recent work with Dmitri Yafaev (Rennes-1), to appear in Constructive Approximation. We consider functions ω on the unit circle with a finite number of logarithmic singularities. We study the approximation of ω by rational functions in the BMO norm. We find the leading term of the asymptotics of the distance in the BMO norm between ω and the set of rational functions of degree n as n goes to infinity. Our approach relies on the Adamyan-Arov-Krein theorem and on the study of the asymptotic behaviour of singular values of Hankel operators.

A potential theoretic minimax problem on the torus

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We extend some equilibrium-type results first conjectured by Ambrus, Ball and Erdélyi and then proved recently by Hardin, Kendall and Saff. Similarly to them, we too work on the torus $\mathbb{T} \simeq [0, 2\pi)$ (unit circle), but a motivation comes from an analogous setup on the unit interval, investigated earlier by Fenton.

The problem is to minimize–with respect to the arbitrary translates $y_0 = 0, y_j \in \mathbb{T}$, $j = 1, \ldots, n$–the maximum of the sum function $F := K_0 + \sum_{j=1}^{n} K_j(\cdot - y_j)$, where the K_j's are certain fixed "kernel functions". If they are concave on $\mathbb{T}$, except for having possible singularities or cusps at zero, then the translates by y_j will have singularities at y_j (while in between these nodes the sum function F still behaves regularly). So one can consider the maxima m_i on each subinterval between the nodes y_j, and look for the minimization of $\max F = \max_i m_i$. Also the dual question of maximization of $\min_i m_i$ arises.

Hardin, Kendall and Saff considered one single *even* kernel, $K_j = K$ for $j = 0, \ldots, n$, and Fenton considered the case of the interval $[-1, 1]$ with *two* fixed kernels $K_0 = J$ and $K_j = K$ for $j = 1, \ldots, n$. Here we build up a systematic treatment of the situation when *all the kernel functions can be different* without assuming them to be even.

MSC: Primary 31D05 ; Secondary 90C47

On rational Fourier series for some elementary functions

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In the present work we consider Fourier series with respect to two systems of rational functions. First, we study Chebyshev – Markov rational fractions. In general case these functions do not possess the orthogonality property. However,

we prove that in case of special choice of poles these functions form orthogonal system over interval $[-1,1]$. Then we obtain Fourier series for the function $|x|$ with respect to these system. It should be noted that the coefficients of the Fourier series are found explicitly.

In the second part of the work we consider orthogonal system of Dzhrbashyan – Kitbalyan rational functions. And again we get Fourier series for the function $|x|$ with respect to these system.

Also we study some properties of the obtained decompositions.

Refinement of Gauss-Lucas theorem for polynomials with real coefficients

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The celebrated Gauss-Lucas theorem states:

Theorem 1 (Gauss-Lucas). *Let $p(z)$ be a polynomial of degree $n \geq 2$ and $H(p)$ be the convex hull of the zeros of $p(z)$. Then all critical points of $p(z)$ (the zeros of $p'(z)$) are in $H(p)$.*

Let $p(z)$ be a polynomial with real coefficients and let its zeros be

$$\{z_1, z_2, \ldots, z_k, \overline{z_1}, \overline{z_2}, \ldots, \overline{z_k}, x_1, x_2, \ldots, x_{n-2k}\},$$

where $\Im(z_s) > 0$; $s = 1, 2, \ldots, k$ and x_ℓ; $\ell = 1, 2, \ldots, n-2k$ are real. Denote by $b = b(p)$ the smallest real number for which the polynomial $p(z+b)$ is with only non negative coefficients and by $a = a(p)$ the smallest real number for which the polynomial $p(-z+a)$ is with only non negative coefficients. Let $\Omega^+(p)$ be the convex hull of the points $\{z_1, z_2, \ldots, z_k, -a, b\}$ and $\Omega^-(p)$ be the convex hull of the points $\{\overline{z_1}, \overline{z_2}, \ldots, \overline{z_k}, -a, b\}$. Our refinement of Gauss-Lucas theorem is the following:

Theorem 2. *All critical points of the polynomial $p(z)$ are in $\Omega(p) = \Omega^-(p) \bigcup \Omega^+(p)$.*

It is obvious that $\Omega(p) \subset H(p)$. The proof of Theorem 2 is based on the:

Theorem 3 (Sectore theorem). *Let $p(z)$ be a polynomial with only real and non-negative coefficients and $p(z) \neq 0$ for every z in the sector $S(\phi) = \{z : |\arg(z)| \geq \phi\}$. Then $p'(z) \neq 0$ for every z in the sector $S(\phi)$.*

Markov-type inequalities and their applications

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The classical Markov inequality estimates the norm of the k-th derivative of an algebraic polynomial of degree n in terms of the polynomial itself, so that $|p(x)| \leq 1$ on $[-1, 1]$ implies $\|p'\| \leq n^2$, $\|p''\| \leq \frac{1}{3}n^2(n^2 - 1)$, etc.

In our talk, we discuss recent and not very recent advances in Markov-type inequalities which include Markov inequalities for splines and other Chebyshev systems, polynomial inequalities with weighted norms and inequalities for discrete sets.

Some applications of those inequalities to the problems of optimal recovery of functions from their values or Fourier samples will be considered including numerical stability issues.

Voronovskaja's theorem for functions with exponential growth

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In the present talk we establish general form of Voronovskajas theorem for functions defined on unbounded interval and having exponential growth. The case of aproximation by linear combinations is also considered. Applications are given for some Szasz-Mirakyan and Baskakov-type operators.This talk is based on joint research with prof. Vijay Gupta

Algorithms for comparison for coincidence of Bézier curves and surfaces

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It is known that Bézier curves and surfaces may have multiple representations by different control polygons. The polygons may have different number of control points and may even be disjoint. This phenomenon causes difficulties in variety of applications where it is important to recognize cases where different representations define same curve (surface) or partially coincident curves (surfaces). The problem of finding whether two arbitrary polynomial curves are the same has been addressed in Pekerman et al. [Are two curves the same?, *Comput.-Aided Geom. Des. and Appl.*, 2(1-4)(2005), pp. 85-94] where the curves are reduced into canonical irreducible forms using monomial basis, then they are compared and their shared domains, if any, are identified. We present an alternative geometric algorithm based on subdivision that compares two input control polygons and reports the coincidences between the corresponding Bézier curves if they are present. We generalize the algorithm for the two basic types of Bézier surfaces: tensor-product and triangular. The algorithms are implemented and tested using Mathematica package.

On the iterates of continuous linear operators preserving constants and Jackson type operator $G_{s,n}$

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We study the limit of the iterates of a large class of linear bounded operators preserving constants. We obtain in addition the limit of the iterates of algebraic version of the trigonometric Jackson integrals. The proofs are based on spectral theory of linear operators.

Local splines on non-uniform grids generating real-time wavelet transforms

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Local cubic quasi-interpolating splines on non-uniform grids are described. The splines are computed by fast computational algorithms that utilize the relation between splines and cubic interpolation polynomials. These splines provide an efficient tool for real-time signal processing . As an input, they use either clean or noised arbitrarily-spaced samples. Exact estimations of the approximation errors are established. The capability to adapt the grid to the structure of an object and to have minimal requirements to the operating memory are of great advantages for off-line processing of signals and multidimensional data arrays. The designed splines serve as a source for generating real-time wavelet transforms for signals in scenarios where signal's samples subsequently arrive one after another at randomized times. The wavelet transforms are executed without delay. On arrival of samples, only a couple of adjacent wavelet transform coefficients are updated.