Abstracts

Voronovskaja type theorems for positive linear operators related to squared basis functions

Ulrich Abel

Technische Hochschule Mittelhessen, Germany

During the last years there has been an increasing interest in monotonicity and convexity properties of sums of squared fundamental functions arising in approximation theory (see, e.g., [5, 11, 4, 1]). They are connected to the Rényi entropy and the Tsallis entropy arising in probability theory (see [6]).

For a certain interval $I \subseteq \mathbb{R}$, consider a positive linear approximation process

$$(L_n f)(x) = \sum_{\nu=0}^{\infty} \ell_{n,\nu}(x) f(x_{n,\nu}) \qquad (x \in I)$$

such that $\sum_{\nu=0}^{\infty} \ell_{n,\nu}^2(x) > 0$, for all $x \in I$ with nodes $x_{n,\nu} \in I$. In many concrete applications we have $\ell_{n,\nu}(x) = 0$, for $\nu > n$, i.e., the sum is finite. Otherwise, we apply L_n to functions for which the sum is convergent, for all $x \in I$. We associate to L_n the positive linear operators $L_n^{\hat{}2}$ defined by

$$\left(L_n^{\hat{}2} f\right)(x) = \frac{1}{\sum_{\nu=0}^{\infty} \ell_{n,\nu}^2(x)} \sum_{\nu=0}^{\infty} \ell_{n,\nu}^2(x) f(x_{n,\nu}).$$

We deal with the asymptotic properties of the sequences $\left(L_n^{\hat{}2} f\right)(x)$ as n tends to infinity. The main results are complete asymptotic expansions of the form

$$\left(L_n^{\hat{}2} f\right)(x) \sim f(x) + \sum_{k=1}^{\infty} \frac{b_k(f,x)}{n^k} \qquad (n \to \infty),$$

for sufficiently smooth functions f. The latter formula means that, for each positive integer q,

$$\left(L_n^{\hat{}2} f\right)(x) = f(x) + \sum_{k=1}^{q} \frac{b_k(f,x)}{n^k} + o\left(n^{-q}\right) \qquad (n \to \infty).$$

The initial coefficients are explicitly calculated. The special case $q = 1$ is a Voronovskaja type formula.

In order to approximate integrable functions we consider two Durrmeyer-type variants of the operators $L_n^{\hat{}2}$.

In this talk we consider various concrete examples of $L_n^{\hat{}2}$ like Bernstein polynomials ([8, 4, 2]), Favard-Szász-Mirakjan operators, Baskakov operators, and Meyer-König and Zeller operators.

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Voronovskaya type theorems for certain operators

Ana-Maria Acu[a], Voichiţa Adriana Radu[b] and Florin Sofonea[a]

[a] *Lucian Blaga University of Sibiu, Department of Mathematics and Informatics, Str. Dr. I. Ratiu, No.5-7, RO-550012 Sibiu, Romania*

[b] *Babes-Bolyai University, FSEGA, Department of Statistics-Forecasts-Mathematics, Cluj-Napoca, Romania*

Very recently, [*F. Nasaireh, I. Raşa, Journal of Mathematical Inequalities, 12(1) (2018), 95–105*] obtained some Voronovskaya type formulas for operators which are nonpositive. These results were used in order to obtain asymptotic formulas for inverse of some known operators as Beta operators $\mathbb{B}_n$, Bernstein operators B_n, the composition $F_n = \mathbb{B}_n^{-1} \circ B_n$. Later, these results were extended for a more general case by [*Heilmann, Nasaireh and Raşa, Mathematics and Computing, Springer Proceedings in Mathematics* & *Statistics 253, Springer Nature, Singapore, 2018*].

Following the direction initiated by the mentioned authors, in the present paper are given some Voronovskaja type theorems for certain operators.

Keywords: Approximation by polynomials, positive linear operators, inverse operator.

Operators preserving exponential functions in approximation theory

Ali Aral and Firat Ozsarac

Department of Mathematics, Faculty of Science and Arts
Kırıkkale University, Kırıkkale, Turkey

In this speech, we introduce a generalization of the classical Bernstein operators that reproduce the exponential functions $\exp(\mu t)$ and $\exp(2\mu t)$, $\mu > 0$. Our results include qualitative and quantitative theorems. We show their shape preserving properties by considering generalized convexity. Our results show that in a certain sense and for certain family of illustrative functions the new sequence approximates better than the classical Bernstein polynomials. Also, we give their integral extensions in Kantorovich sense by replacing the usual differential and integral operators with their more general analogues. It is shown that the operators generate an approximation process in both $C[0,1]$ and $L_{p,\mu}[0,1]$, the latter being an exponentially weighted space. Also, quantitative results are stated in terms of appropriate moduli of smoothness and K-functional.

Keywords: Bernstein operator, Kantorovich operator, modulus of continuity.

Definite quadrature formulae of order three

Ana Avdzhieva[1], Vesselin Gushev and Geno Nikolov

Faculty of Mathematics and Informatics, Sofia University
Sofia, Bulgaria

Most of quadratures used in practice (e.g. quadrature formulae of Gauss, Radau, Lobatto, Newton-Cotes) are definite of certain order. The importance of the definite quadratures of order r stems in the fact that they provide one-sided approximation to the exact value of the integral whenever the integrand has a permanent sign in the integration interval.

The compound rectangles and trapezium quadrature formulae are classical examples of positive (respectively, negative) definite quadrature formulae of order 2. Besides providing lower and upper bounds for definite integrals of convex or concave (i.e., 1–monotone) integrands, they are convenient from computational point of view as they use equidistant nodes.

Unlike the definite quadrature formulae of even order, definite quadrature formulae of odd order are never symmetric. Somewhat unexpectedly, this phenomenon turns out to be an advantage rather than disadvantage. For instance, when reflecting the nodes of a positive definite quadrature formula of odd order (keeping the weights unchanged), we obtain a negative definite quadrature formula and vice versa. This allows derivation of simple a-posteriori error estimates, i.e. estimates that do not require knowledge of the magnitude of any derivative but just few evaluations of the integrand.

We construct sequences of definite quadrature formulae of order 3. They use the nodes of either the rectangles or the trapezium quadrature formulae and, excluding the coefficients of the boundary three or four nodes, have the same coefficients. A kind of opitmization is performed for the choice of the boundary coefficients so that the error constants of the constructed quadratures are as small as possible in absolute value.

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Fast algorithms for adaptive approximation on conforming partitions

Peter Binev

Department of Mathematics, University of South Carolina
Columbia, SC, USA

We consider piecewise approximations on general triangulations motivated by the finite element methods (FEM) for numerically solving partial differential equations. Finding an approximation with the smallest error for a given number of degrees of freedom is usually computationally prohibitive and we aim at finding a near-best approximation instead.

Results for such approximations are usually formulated as tree approximations relating the process of refining the partition to a decision tree and emphasizing the fact that the algorithms are coarse-to-fine. These results, however, feature general nonconforming partitions allowing hanging nodes, i.e. cases in which the common boundary of two neighboring triangles is not a side of both of them.

The usual adaptive strategy for finding conforming partitions in FEM is ömark $\rightarrow$ subdivide $\rightarrow$ completeö. In this strategy any element can be marked for subdivision but since the resulting partition often contains hanging nodes, additional elements have to be subdivided in the completion step to get a conforming partition. This process is very well understood for triangulations received via newest vertex bisection procedure. In particular, it is proven that the number of elements in the final partition is limited by constant times the number of marked cells. This motivated us to design a marking procedure that is limited only to cells of the partition whose subdivision will result in a conforming partition and therefore no completion step is necessary. We also proved that this procedure is near-best in terms of both error of approximation and complexity with efficient constants.

On Fatou's theorem on bounded analytic functions

Arthur A. Danielyan

University of South Florida, USA

By Fatou's theorem, a bounded analytic function f in the open unit disc D has radial (non tangential) limits at the points of the unit circle T except a subset E of measure zero. If the function f has even unrestricted limits at the points of $T \setminus E$, then obviously E is an F_σ set. We prove that the converse statement is also true. Thus we have the following:

Theorem 1. *Let E be a subset on T. Then there exists a bounded analytic function in D which has no radial limits on E but has unrestricted limits at the points of $T \setminus E$ if and only if E is an F_σ set of measure zero.*

An obvious corollary of the sufficiency part of Theorem 1 is the Lohwater-Piranian theorem of 1957: If E is an F_σ set of measure zero on T then there exists a bounded analytic function in D which has no radial limits exactly on E.

Simultaneous approximation by Bernstein polynomials and their integer forms

Borislav R. Draganov[1]

Faculty of Mathematics and Informatics, Sofia University
Sofia, Bulgaria
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences, Sofia, Bulgaria

I will present a characterization of the rate of the weighted simultaneous approximation in $L_p[0,1]$, $1 < p \leq \infty$, by the Bernstein operator. As is known, it is defined for $f \in C[0,1]$ by

$$B_n(f,x) := \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big)\binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0,1].$$

The weights I consider are those of Jacobi. The characterization of the rate of the simultaneous approximation—a direct inequality and a matching strong two-term converse inequality—is stated by means of appropriate moduli of smoothness or K-functionals. Also, I will state analogous results concerning the Kantorovich operators.

I will also consider the simultaneous approximation by the Bernstein polynomials with integer coefficients in the uniform norm. L. Kantorovich introduced the first such modification. It is given by

$$\widetilde{B}_n(f,x) := \sum_{k=0}^{n} \left[f\Big(\frac{k}{n}\Big)\binom{n}{k} \right] x^k (1-x)^{n-k},$$

where $[\alpha]$ denotes the largest integer that is less than or equal to the real α. Another integer modification of the Bernstein polynomials, which actually has better approximation properties, is defined by means of the nearest integer. I will give direct and weak converse error estimates for both operators. They are established under quite restrictive assumptions, but they turn out to be necessary too. It is noteworthy that for the derivatives of order two and higher, the necessary conditions for both operators are more restrictive than for the first derivative.

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Weighted approximation of functions in L_p-norm by some Kantorovich operators

Ivan Gadjev[1], Parvan Parvanov and Rumen Uluchev

Faculty of Mathematics and Informatics, Sofia University
Sofia, Bulgaria

The weighted approximation of functions in L_p-norm by Kantorovich modifications of the classical Baskakov and Meyer-König and Zeller operators is discussed. The weights under consideration are $w(x) = (1+x)^\alpha$, $\alpha \in \mathbb{R}$, for the Kantorowich modification of the Baskakov operator and $w(x) = (1-x)^\alpha$, $\alpha \in \mathbb{R}$, for the Meyer-König and Zeller operator. The weighted error of approximation $\|w(L_n f - f)\|_p$ where L_n is the Kantorovich modification of the classical Baskakov or Meyer-König and Zeller operator is characterized by the next K-functional

$$K_w(f,t)_p = \inf\Big\{\|w(f-g)\|_p + t\big\|w\tilde{D}g\big\|_p : f-g \in L_p(w),\ g \in W_p(w)\Big\},$$

where

$$\tilde{D} = \frac{d}{dx}\Big(\varphi(x)\frac{d}{dx}\Big),$$

$\varphi(x) = x(1+x)$ for the Baskakov operator and $\varphi(x) = x(1-x)^2$ for the Meyer-König and Zeller operator,

$$L_p(w) = \big\{f : wf \in L_p(D)\big\}$$

$$W_p(w) = \Big\{f : w\tilde{D}f \in L_p(D),\ \lim_{x\to 0^+}\varphi(x)f'(x) = 0\Big\}.$$

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Remarks on classical Kantorovich operators

Heiner Gonska

University of Duisburg–Essen, 47048 Duisburg, Germany

The talk will summarize several old and some very recent results on Kantorovich operators in their classical 1930 form. Time permitting, useful and useless modifications will be addressed as well.

The material presented is based on joint work with Ana Maria Acu (Lucian Blaga University of Sibiu).

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Some results for linking Baskakov-type operators

Margareta Heilmann

School of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20, 42119 Wuppertal, Germany

We consider operators depending on a positive real parameter ρ which constitute a link between genuine Baskakov-type operators and classical Baskakov-type operators.

We present representations for the k-th order Kantorovich modifications of linking operators acting on the interval $[0, \infty)$ as a generalization of our previous results for the Bernstein case (joint work with Ioan Raşa), convergence results for the Baskakov case (joint work with Ulrich Abel, Vitaliy Kushnirevych) and some properties of the operators applied to convex functions (joint work with Ana-Maria Acu, Ioan Raşa).

Non-linear approximation in BMO($\mathbb{S}^{d-1}$)

Kamen G. Ivanov[a] and Pencho Petrushev[b]

[a]*Institute of Mathematics and Informatics*
Bulgarian Academy of Sciences, Sofia, Bulgaria

[b]*Department of Mathematics, University of South Carolina*
Columbia, SC, USA

The fundamental solution of the Laplace equation (Newtonian kernel) $\frac{1}{|x|^{d-2}}$ in dimension $d > 2$ or $\ln\frac{1}{|x|}$ if $d = 2$ is a basic building block in Potential theory. The main goal of this lecture is to study the rates of nonlinear n-term approximation of BMO functions on the unit sphere $\mathbb{S}^{d-1}$ from shifts of the Newtonian kernel with poles outside the unit ball $\overline{B^d}$. We prove

Theorem 1 (Jackson estimate)**.** *If* $f \in \mathcal{B}_\tau^{s\tau}(\mathbb{S}^{d-1})$, $1/\tau = s/(d-1)$, $s > 0$, *then* $f \in \mathrm{BMO}(\mathbb{S}^{d-1})$ *and for* $n \geq 1$

$$E_n(f)_{\mathrm{BMO}} \leq cn^{-s/(d-1)}\|f\|_{\mathcal{B}_\tau^{s\tau}},$$

where the constant $c > 0$ *depends only on* s, d.

Here $\mathrm{BMO}(\mathbb{S}^{d-1})$ denotes the spaces of Bounded mean oscillation on $\mathbb{S}^{d-1}$ and $\mathcal{B}_p^{sq}(\mathbb{S}^{d-1})$ stands for the Besov space with parameters s, q, p on the sphere. $E_n(f)_{\mathrm{BMO}}$ denotes the best nonlinear n-term approximation of f from shifts of the Newtonian kernel in the norm of BMO.

The rates of approximation in Theorem 1 are optimal in terms of the Besov spaces. The main vehicle in establishing these results is the construction of highly localized frames for Besov and Triebel-Lizorkin spaces on the sphere whose elements are linear combinations of a fixed number of shifts of the Newtonian kernel.

The result in Theorem 1 is an extension to $p = \infty$ of previous results of the authors for the best non-linear approximation in $L^p(\mathbb{S}^{d-1})$, $1 < p < \infty$, and $H^p(\mathbb{S}^{d-1})$, $0 < p < \infty$ (Hardy spaces). It turns out that BMO is a natural replacement for L^∞ here.

Recurrence equations involving different orthogonal polynomial sequence

Aletta Jooste

University of Pretoria, South Africa

Every sequence of real polynomials $\{p_n\}_{n=0}^{\infty}$, orthogonal with respect to a positive weight function $w(x)$ on the interval (a, b), satisfies a three-term recurrence equation. We discuss the role played by the polynomials associated to p_n, especially as coefficient polynomials in the three-term recurrence equation involving polynomials p_n, p_{n-1} and p_{n-m}, $m \in \{2, 3, \ldots, n-1\}$. Furthermore, we show how Christoffel's formula is used to obtain mixed three-term recurrence equations involving the polynomials p_n, p_{n-1} and $g_{n-m,k}$, $m \in \{2, 3, \ldots, n-1\}$, where the sequence $\{g_{n,k}\}_{n=0}^{\infty}$, $k \in \mathbb{N}_0$, is orthogonal with respect to $c_k(x)w(x) > 0$ on (a, b) and c_k is a polynomial of degree k in x. The equations obtained are applied in the study of the location of the zeros of the appropriate polynomials.

On some Bernstein-type inequalities for rational functions

Sergei Kalmykov

Shanghai Jiao Tong University, 200240, Shanghai, China

We will discuss recent results concerning Bernstein- and Markov-type inequalities for polynomials and rational functions. For example, we will be interested in asymptotically sharp inequalities on one C^2-smooth Jordan arc and related questions as well as sharp estimates under additional restriction on location of the zeros or when a polynomial or rational function has a curved majorant on the interval $[-1, 1]$. In the last cases, mainly, we will consider covering and multiple point distortion theorems. Finally, we will demonstrate how the extremal polynomials can be determine or constructed. Methods are based on potential theory, conformal mappings, theories of approximation and interpolation.

This is partially based on a joint work with B. Nagy and V. Totik and partially supported by SJTU start-up grant program (WF220407115) and Russian Foundation for Basic Research (grant 18-31-00101).

Weighted approximation with the Bernstein-Chlodovsky operators

Theodore Kilgore

Auburn University, Auburn AL 36849, USA

In 1937, I. Chlodovski modified the Bernstein polynomial operators for use in approximation of functions $f \in C[0, \infty)$ and produced several interesting results on convergence. However, none of his results addressed weighted approximation.

With $W(x) = e^{-x^{\alpha}}$, let $f \in C_W[0, \infty)$, the space of those continuous functions f for which $\|f\|_W = \sup_x |W(x)f(x)|$ is finite and for which $W(x)f(x) \to 0$ as $x \to \infty$. The approximation of f by similarly weighted polynomials will be investigated. Similar questions will be considered, too, for functions in $C_W(-\infty, \infty)$, in which space $W(x)f(x) \to 0$ as $|x| \to 0$, using similarly constructed approximation operators. Some recent progress will be presented, and some unsolved open problems will be described.

Descartes' rule of signs and moduli of roots

Vladimir Petrov Kostov

Université Cote d'Azur, Laboratoire de Mathématiques
Parc Valrose, Nice, France

A hyperbolic polynomial (HP) is a real univariate polynomial with all roots real. By Descartes' rule of signs a HP with all coefficients nonvanishing has exactly c positive and exactly p negative roots counted with multiplicity, where c and p are the numbers of sign changes and sign preservations in the sequence of its coefficients. For $c = 1$ and 2, we discuss the question: When the moduli of all the roots of a HP are arranged in the increasing order on the real half-line, at which positions can be the moduli of its positive roots depending on the positions of the sign changes in the sequence of coefficients?

On polyharmonic interpolation of data on parallel hyperplanes

Ognyan Kounchev[a] and Hermann Render[b]

[a] *Institute of Mathematics and Informatics*
Bulgarian Academy of Sciences, Sofia, Bulgaria

[b] *University College Dublin, School of Mathematics and Statistics*
Beleld 4, Dublin, Ireland

We study the problem of interpolation by polyharmonic functions of data prescribed on parallel hyperplanes in a strip type domain D in $\mathbb{R}^d$. As solutions of elliptic equations the polyharmonic functions are by necessity real-analytic in their domain of definition D. Hence, one has to impose very restrictive conditions on the data defined on sets which are lying strictly in the domain of definition D. We have explored the subtle conditions which have to be satisfied by the functions $f_1, ..., f_{2N}$ defined on $\mathbb{R}^d$ such that the following interpolation problem can be solved: for the real numbers $t_1 < ... < t_{2N}$ there exists a polyharmonic function u of order N defined on $D = (t_1, t_{2N}) \times \mathbb{R}^d$ satisfying the interpolation conditions $u(t_j, y) = f_j(y)$ for all $y \in \mathbb{R}^d$ and $j = 1, 2, ..., 2N$. By using Fourier methods we prove that this problem is reduced to infinitely many one-dimensional interpolation problems with exponential polynomials depending on the parameter of the Fourier transform ξ, see [1]. The main technical result is the estimation of the asymptotic properties of these one-dimensional problems with respect to ξ, see [2].

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The approximation power of deep neural networks: theory and applications

Gitta Kutyniok

Technische Universität Berlin, Germany

Despite the outstanding success of deep neural networks in real-world applications, most of the related research is empirically driven and a mathematical foundation is almost completely missing. The main goal of a neural network is to approximate a function, which for instance encodes a classification task. Thus, one theoretical approach to derive a fundamental understanding of deep neural networks focusses on their approximation abilities.

In this talk we will provide an introduction into this research area. After a general overview of mathematics of deep neural networks, we will discuss theoretical results which prove that not only do (memory-optimal) neural networks have as much approximation power as classical systems such as wavelets or shearlets, but they are also able to beat the curse of dimensionality. On the numerical side, we will then show that superior performance can typically be achieved by combining deep neural networks with classical approaches from approximation theory.

Product Besov and Triebel-Lizorkin spaces with application to nonlinear approximation

A. Georgiadis[a], George Kyriazis[a] and Pencho Petrushev[b]

[a] *University of Cyprus, Nicosia, Cyprus*

[b] *Department of Mathematics, University of South Carolina Columbia, SC, USA*

The Littlewood-Paley theory of homogeneous product Besov and Triebel-Lizorkin spaces is developed in the spirit of the φ-transform of Frazier and Jawerth. This includes the frame characterization of the product Besov and Triebel-Lizorkin spaces and the development almost diagonal operators on these spaces. The almost diagonal operators are used to obtain product wavelet decomposition of the product Besov and Triebel-Lizorkin spaces. The main application of this theory is to nonlinear m-term approximation from product wavelets in L^p and Hardy spaces. Sharp Jackson and Bernstein estimates are obtained in terms of product Besov spaces.

Visiting Whitney

D. Leviatan

Tel Aviv University, Tel Aviv, Israel

Let $x_0 < x_1 < \cdots < x_{m-1}$, and set $I := [x_0, x_{m-1}]$ and $|I| := x_{m-1} - x_0$. Assume that for some $0 < \lambda \le 1$, we have $x_{j+1} - x_j \ge \lambda|I|$, for all $0 \le j \le m-2$. The Lagrange-Hermite polynomial of a function $f \in C^r(I)$, $L_{m-1}(x; f; x_0 \ldots, x_{m-1})$ is the unique polynomial of degree $m-1$, interpolating f at the points $x_0, \ldots, x_{m-1}$. For $m \ge \max\{r+1, 2\}$, the classical Whitney estimate of how well this polynomial approximates f in I is given by

$$|f(x) - L_{m-1}(x; f; x_0 \ldots, x_{m-1})| \le C(m, \lambda)|I|^r \omega_{m-r}(f^{(r)}, |I|, I), \quad x \in I,$$

where ω_k is that kth modulus of smoothness.

We allow some of the points to coalesce, specifically, we assume $x_0 \le x_1 \le \cdots \le x_{m-1}$ such that we only have $x_{j+r+1} - x_j \ge \lambda|I|$, for all $0 \le j \le m-r-2$. In other words, we assume that the Lagrange-Hermite polynomial interpolates f and its derivatives at x_j according to the multiplicity of the appearance of x_j in the the collection of points (but note that no x_j may appear more than $r+1$ times). We prove the above Whitney inequality for this situation.

If time permits, we will discuss an extension of Whitney's inequality to a local type estimate.

Sharp estimates for the critical points of oscillating polynomials with Laguerre weight

Lozko Milev and Nikola Naidenov[1]

Faculty of Mathematics and Informatics, Sofia University
Sofia, Bulgaria

Denote by $\mathcal{V}_n(\lambda)$ the set of all weighted polynomials of the form $f(x) = e^{-\lambda x}p(x)$ $(\lambda > 0)$, where p is an algebraic polynomial of degree n which has n simple real zeros. Given $f \in \mathcal{V}_n(\lambda)$, let $x_1 < \cdots < x_n$ and $t_1 < \cdots < t_n$ be the zeros of f and f', correspondingly. Set $h_k := x_{k+1} - x_k$, $k = 1, \ldots, n-1$.

We present sharp estimates of the forms

$$x_k + c_k h_k \le t_k \le x_{k+1} - d_k h_k, \quad k = 1, \ldots, n-1,$$

and

$$x_n + c_n h_{n-1} \le t_n \le x_n + d_n h_{n-1},$$

where $\{c_k\}_{k=1}^n$ and $\{d_k\}_{k=1}^n$ are explicit expressions, depending on λ and h_k for $k = 1, \ldots, n-1$ (or h_{n-1} for $k = n$). Known estimates of the same type for algebraic polynomials can be obtained by letting $\lambda \to 0$.

We also give simpler, rational estimates for the critical points of polynomials from $\mathcal{V}_n(\lambda)$.

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Numerical methods for the discretization of anisotropic PDEs on cartesian grids

Jean-Marie Mirebeau

Mathematics Department, University Paris-Sud, France
CNRS, University Paris-Saclay, France

Anisotropy, the existence of preferred direction in the domain, is a source of difficulty in PDE discretization.

Depending on the application, anisotropy can be due to boundary layers, shocks, or other aspects of the problem structure. Numerous numerical methods have been developed to address the related numerical issues, often based on adaptive representations of the solution via special wavelet bases or elongated finite elements.

I will describe Voronoi's first reduction, a tool coming from the field of additive lattice geometry, which turns out to be particularly efficient for the discretization of anisotropic PDEs on cartesian grids. The approach is versatile, and yields monotone and second order consistent finite difference schemes for various PDEs. It can be regarded as a local reorganization of the grid connectivity depending on locally preferred directions, and it has connections with approximation theory – in particular the approximation of convex functions – that are under investigation. Numerical results illustrate the method's robustness and accuracy.

Further results on the zeros of the derivative of oscillating polynomials with Laguerre weight

Lozko Milev and Nikola Naidenov[1]

Faculty of Mathematics and Informatics, Sofia University
Sofia, Bulgaria

Denote by $\mathcal{V}_n(\lambda)$ the set of all weighted polynomials of the form $f(x) = e^{-\lambda x}p(x)$ $(\lambda > 0)$, where p is an algebraic polynomial of degree n which has n simple real zeros. Given $f \in \mathcal{V}_n(\lambda)$, let $x_1 < \cdots < x_n$ and $t_1 < \cdots < t_n$ be the zeros of f and f', correspondingly.

In a recent paper we proved sharp estimates of the form

$$x_k + c_k h_k \le t_k \le x_{k+1} - d_k h_k, \quad k = 1, \dots, n-1, \tag{1}$$

where $h_k := x_{k+1} - x_k$, $k = 1, \dots, n-1$, and $\{c_k\}_{k=1}^{n-1}$ ($\{d_k\}_{k=1}^{n-1}$) are explicit expressions, depending on λ and h_k.

Here we present improvements of (1) of the form

$$x_k + c_k^* h_k \le t_k \le x_{k+1} - d_k^* h_k, \quad k = 1, \dots, n-1,$$

where $\{c_k^*\}_{k=1}^{n-1}$ ($\{d_k^*\}_{k=1}^{n-1}$) depends on $\lambda, x_1, x_k, x_{k+1}$ ($\lambda, x_k, x_{k+1}, x_n$).

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A discrete Markov-Bernstein inequality for sequences and polynomials

Dimitar K. Dimitrov[a] and Geno Nikolov[b1]

[a]Department of Applied Mathematics, Campus IBILCE
State University of São Paulo UNESP, Brazil

[b]Faculty of Mathematics and Informatics, Sofia University
Sofia, Bulgaria

Given $c \in (0,1)$, let $\ell_{2,c}$ be the Hilbert space of real sequences $f = (f(0), f(1), \ldots)$, for which the series $\sum_{k=0}^{\infty} c^k\,[f(k)]^2$ converges. For any such sequence f its norm is naturally defined by

$$\|f\|_{2,c}^2 = \sum_{k=0}^{\infty} c^k\,[f(k)]^2\,.$$

Let us define also the variance Δf of f to be $\Delta f = (f(1)-f(0), f(2)-f(1), \ldots)$. One of our results reads as follows:

Theorem 1. *Let $c \in (0,1)$. Then the inequality*

$$\|\Delta f\|_{2,c} \le (1 + c^{-1/2})\|f\|_{2,c}$$

holds for every $f \in \ell_{2,c}$. Moreover, the constant $1 + c^{-1/2}$ cannot be replaced by a smaller one.

We were led to this (and to a more general) result by our study of the Markov-Bernstein inequality for algebraic polynomials with respect to the Meixner discrete ℓ_2 measure. We relate the sharp Markov-Bernstein constants to the smallest zeros of certain polynomials, orthogonal with respect to a measure supported on the positive semi-axis. Upper bounds for these constants are proved as well as some monotonicity properties, which imply the sharpness of Theorem 1.

The talk is based on the joint work [1] with Dimitar K. Dimitrov from State University of Sao Paolo, Brazil.

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Gaussian bound for the heat kernel induced by prolate spheroidal wave functions with applications

Pencho Petrushev

Department of Mathematics, University of South Carolina
Columbia, SC, USA

The purpose of this talk is to establish Gaussian upper bound for the heat kernel associated with the prolate spheroidal wave functions (PSWFs) of order zero. As an application of this result we develop the related smooth functional calculus, which in turn is the necessary ground work in developing the theory of Besov and Triebel-Lizorkin spaces associated with the PSWFs. One of our main result on Besov and Triebel-Lizorkin spaces associated with the PSWFs asserts that they are the same as the Besov and Triebel-Lizorkin spaces generated by the Legendre operator.

Explicit solution, for $n = 6$, to a Markov-type extremal problem initiated by Schur

Heinz-Joachim Rack[a] and Robert Vajda[b]

[a]*Hagen, Germany*

[b]*Szeged, Hungary*

We apply our recent result from [9] (see also [10]) to the solution found by P. Erdős and G. Szegő respectively by A. Shadrin to a A. A. Markov-type [3] respectively V. A. Markov-type [4] extremal problem which was initiated by I. Schur [11] a hundred years ago, see also [6, Section 5d]. In particular, the said problem is to determine, for each $k \in \{1, \ldots, n-2\}$, a ξ with $|\xi| \leq 1$ and an algebraic polynomial P_n of degree $\leq n$ ($n \geq 4$) with $|P_n(x)| \leq 1$ for $|x| \leq 1$ and satisfying $P_n^{(k+1)}(\xi) = 0$ such that $N_{n,k,\xi}$, the supremum of $|P_n^{(k)}(\xi)|$, is attained. The said solutions as given in [1], [12] are of a general character. Concrete explicit solutions to this problem are known for $n \in \{4, 5\}$, see [7], [8]. We consider here the next higher degree $n = 6$ and determine explicitly, for $k \in \{1, 2, 3, 4\}$, the sought-for supremum $N_{6,k,\xi}$ as well as the sextic extremizer polynomials (which are, except for $k = 4$, normalized proper Zolotarev polynomials). To this end we deploy *Mathematica*™ [13] and root objects of dedicated integer polynomials Q_m of degree $m \leq 18$. We then compare (with T_6 denoting the $6th$ Chebyshev polynomial of the first kind) the concrete constants $M_{6,k} = N_{6,k,1}/T_6^{(k)}(1)$ for $k \in \{1 \ldots, 4\}$ to the corresponding upper bounds as provided in [12, Theorem 7.1], and in particular we compare the constant $M_{6,1}$ to the so-called (asymptotic) Zolotarev-Schur constant $M_{\infty,1}$, see [1], [2, Section 3.9], [5].

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On interpolation with a class of bivariate polysplines for data in the plane

Ognyan Kounchev[a] and Hermann Render[b]

[a] *Institute of Mathematics and Informatics Bulgarian Academy of Sciences, Sofia, Bulgaria*

[b] *University College Dublin, School of Mathematics and Statistics Beleld 4, Dublin, Ireland*

Interpolation with radial basis functions on scattered data is a useful and powerful method but may lead to numerical problems for very large data. A different method for structured data is based on the concept of a polyspline introduced by O. Kounchev. In this talk we discuss interpolation results with respect to a finite-dimensional subspace of bivariate polysplines of order 2 where the data lie on parallel lines in the plane. This problem leads to questions about the interpolation and best approximation with certain classes of exponential splines which is interesting in its own right. The authors acknowledges the support of a grant DN 02/13 with the Bulgarian NSF.

On Rolle's theorem for complex polynomials

Blagovest Sendov[a] and Hristo Sendov[b]

[a] *Bulgarian Academy of Sciences, Sofia, Bulgaria*

[b] *The Universtity of Western Ontario, London, Ontario, Canada*

In [2] is proved the strongest Rolle's theorem for complex polynomials, if the so called Rolle's domain is an union of two disks. The proof is based on the notion locus of a complex polynomial, see [1].

In the lecture we introduce an other definition of a locus and show that it is equivalent to the original one. The new definition is more useful for constructing algorithms for calculating loci of complex polynomials. We will show the idea for an algorithm for calculating a symmetric in respect to the real axes locus of a polynomial with real zeros. This solve the problem for the sharpest Rolle's theorem for complex polynomials.

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Polar convexity and critical points of polynomials

Blagovest Sendov[a] and Hristo Sendov[b]

[a] *Bulgarian Academy of Sciences, Sofia, Bulgaria*

[b] *The Universtity of Western Ontario, London, Ontario, Canada*

A set A, in the extended complex plane, is called convex with respect to a pole u, if for any x, y in A the arc on the unique circle through x, y and u, that connects x and y but does not contain u, is in A. If the pole u is taken at infinity, then this notion reduces to the usual convexity. Polar convexity is connected with the classical Gauss-Lucas' and Laguerre's theorems for complex polynomials. If a set is convex with respect to u and contains the zeros of a polynomial, then it contains the zeros of its polar derivative with respect to u. A set may be convex with respect to more than one pole. The main goal of this talk is to present the main relationships between sets and their poles.

Optimal sampling rates for approximating analytic functions from pointwise samples

Alexei Shadrin

Cambridge, UK

We consider the problem of approximating an analytic function on a compact interval from its values at $M+1$ distinct points. When the points are equispaced, a recent result (the so-called impossibility theorem) has shown that the best possible convergence rate of a stable method is root-exponential in M, and that any method with faster exponential convergence must also be exponentially ill-conditioned at a certain rate.

Here, we present an extension of the impossibility theorem valid for general nonequispaced points, and apply it to the case of points that are equidistributed with respect to modified Jacobi measures. This leads to a necessary sampling rate for stable approximation from such points. We prove that this rate is also sufficient, and therefore exactly quantify (up to constants) the precise sampling rate for approximating analytic functions from such node distributions with stable numerical methods.

In particular, we theoretically confirm the well-known heuristic that stable least-squares approximation using polynomials of degree $N < M$ is possible only once M is sufficiently large for there to be a subset of N of the nodes that mimic the behaviour of the Nth Chebyshev nodes.

Polynomial inequalities on general sets

Vilmos Totik

University of Szeged, Szeged, Hungary

The talk will review some recent developments (due to Andrievskii, Baran, Kalmykov, Nagy and the lecturer) in the theory of Bernstein and Markov inequalities. The role of normal derivatives of Green's functions will be demonstrated in both global and local inequalities. No a priory knowledge will be required, Green's functions and their normal derivatives will be explained in details. It will also be shown that the same normal derivatives emerge in the famous problem of Bernstein on the error of polynomial approximation of the $|x|$ function.

On Hardy L_2 inequality in certain finite dimensional spaces

Dimitar K. Dimitrov[a], Ivan Gadjev[b], Geno Nikolov[b] and Rumen Uluchev[b1]

[a]*Department of Applied Mathematics, Campus IBILCE State University of São Paulo UNESP, Brazil*

[b]*Faculty of Mathematics and Informatics, Sofia University Sofia, Bulgaria*

Denote by $\mathcal{P}_n$ be the set of real-valued algebraic polynomials of degree at most n and let $\mathcal{H}_n := \{f \,:\, f(x) = e^{-x/2}\, p(x),\ p \in \mathcal{P}_n\}$.

We examine the best (i.e. the smallest possible) constant c_n in the L_2 Hardy inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^2 dx \le c_n \int\limits_0^\infty f^2(x)\,dx, \qquad f \in \mathcal{H}_n.$$

Our main result is the following two-sided estimate for c_n:

$$4 - \frac{c}{\ln n} < c_n < 4 - \frac{c}{\ln^2 n}\,, \qquad c > 0\,.$$

It confirms the expected $\lim_{n\to\infty} c_n = 4$, showing however that the convergence speed is rather slow.

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On the set of solutions of polynomial systems

H. Hakopian and D. Voskanyan

Faculty of Informatics and Applied Mathematics
Yerevan State University, Yerevan, Armenia

We prove that a set $\mathcal{X} \subset \mathbb{C}^2$, $\#\mathcal{X} = mn$, $m \leq n$, is the set of solutions of a polynomial system

$$p(x, y) = 0, \quad q(x, y) = 0,$$

where the total degrees of polynomials p and q equal m and n, respectively, if and only if the following conditions are satisfied:

a) The set $\mathcal{X}$ is essentially $(m+n-3)$-dependent, i.e., no point from $\mathcal{X}$ has a fundamental polynomial of degree $m+n-3$,

b) The set $\mathcal{X}$ contains an $(m-1)$-correct subset.

Let us mention that the conditions a) and b) in the "only if" direction of this result follow from the Ceyley-Bacharach and Noether theorems, respectively.