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NIKOLSKII CONSTANTS OF POSITIVE OPERATORS FOR LIPSCHITZ CLASSES

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Summary. Let $f \in C_{2\pi}$ and the approximation process $I_\varrho(f;x)$ be given as a convolution integral, i. e., $I_\varrho(f;x)=(1/2\pi)\int_{-\pi}^{\pi} f(x-u)k_\varrho(u)\,du$, the kernel $k_\varrho(x)$ being a positive approximate identity for $\varrho\to\infty$. If the quantity $\Delta(k_\varrho;\alpha)\equiv\sup_{f\in \mathrm{Lip}_1\alpha}\|I_\varrho(f;\cdot)-f(\cdot)\|_{C_{2\pi}}$ admits an asymptotic expansion of type $\Delta(k_\varrho;\alpha)=N(\alpha)\varphi(\varrho)+o(\varphi(\varrho))$ with $N(\alpha)>0$, $\varphi(\varrho)\downarrow 0$ as $\varrho\to\infty$ then $N(\alpha)$ is called the N i k o l s k i i c o n s t a n t of the process $I_\varrho(f;x)$ relative to the class $\mathrm{Lip}_1\,\alpha$. This constant will be determined for the general class of kernels of Fejér's type, i. e., there exists $\chi\in L^1(-\infty,\infty)$ with $\chi(x)\geqq 0$, $\int_{-\infty}^{\infty}\chi(u)du=2\pi$ such that $k_\varrho(x)=\sum_{j=-\infty}^{\infty}\varrho\chi(\varrho(x+2j\pi))$. One result is that $\varphi(\varrho)=\varrho^{-\alpha}$ and $N(\alpha)=(1/2\pi)\int_{-\infty}^{\infty}|u|^{\alpha}\chi(u)du$ in case this moment exists. This will be applied to various examples of positive operators.

The study of best asymptotic constants of (positive) approximation processes with regard to Lipschitz classes was initiated by N i k o l s k i i [5], who considered the Fejér means. Thus let $C_{2\pi}$ be the space of all 2π-periodic functions f which are continuous on the whole real axis R, endowed with the usual sup-norm: $\|f(\cdot)\|=\sup_{x\in R}|f(x)|$. Concerning the Fejér means of the Fourier series of f, i. e., the operators

$$(1)\qquad \sigma_n(f;x)=\frac{1}{2\pi(n+1)}\int_{-\pi}^{\pi} f(x-u)\left[\frac{\sin((n+1)u/2)}{\sin(u/2)}\right]^2 du,$$

the theorem of Fejér states that $\sigma_n(f;x)$ converges uniformly, i. e., in the norm of $C_{2\pi}$, to $f(x)$ for each $f\in C_{2\pi}$. Furthermore, if $f\in \mathrm{Lip}_M\alpha$ where

$$(2)\qquad \mathrm{Lip}_M\alpha=\{f\in C_{2\pi}\,|\,\|f(\cdot+h)-f(\cdot)\|\leqq M|h|^{\alpha}\},$$

then there exists a constant C_α independent of f such that

$$(3)\qquad \|\sigma_n(f;\cdot)-f(\cdot)\| \leq C_\alpha M \begin{cases} n^{-\alpha}, & 0<\alpha<1, \\ n^{-1}\log n, & \alpha=1; \end{cases}$$

it is well-known that the order of approximation in (3) cannot be improved for the whole class $\mathrm{Lip}_M\alpha$. Therefore the question as to the best possible constants is posed, and the result of Nikolskii in 1940 was that

$$(4)\qquad \sup_{f\in \mathrm{Lip}_M\alpha} \|\sigma_n(f;\cdot)-f(\cdot)\| = M \begin{cases} \dfrac{2\Gamma(\alpha)\sin(\alpha\pi/2)}{\pi(1-\alpha)} n^{-\alpha}+o(n^{-\alpha}), & 0<\alpha<1 \\ (2/\pi)n^{-1}\log n+O(n^{-1}), & \alpha=1. \end{cases}$$

In the meantime several authors have determined the constants for particular processes (see the rather complete list of references given in [3], [4]). The aim of this lecture is to present a unified, but nevertheless elementary approach to the subject, thus determining the best asymptotic constants for a general class of approximation processes.

To this end, let the approximation process for $f\in C_{2\pi}$ be given as a singular integral of Fourier convolution type:

$$(5)\qquad I_\varrho(f;x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)k_\varrho(u)\,du.$$

Here ϱ is a positive parameter tending to infinity, and the kernel $k_\varrho(x)$ is supposed to be a positive approximate identity for $\varrho\to\infty$ (cf. [2, Sec. 1.1.1]). The quantity

$$(6)\qquad \Delta(k_\varrho;\alpha)=\sup_{f\in \mathrm{Lip}_1\alpha}\|I_\varrho(f;\cdot)-f(\cdot)\|$$

is called the measure of approximation of the integral $I_\varrho(f;x)$ for the class $\mathrm{Lip}_1\alpha$. Suppose that $\Delta(k_\varrho;\alpha)$ admits an asymptotic expansion of the type

$$(7)\qquad \Delta(k_\varrho;\alpha)=N(\alpha)\varphi(\varrho)+o(\varphi(\varrho)),\quad \varrho\to\infty,$$

where $N(\alpha)$ is a constant different from zero and $\varphi(\varrho)$ a positive function of ϱ with $\lim_{\varrho\to\infty}\varphi(\varrho)=0$. Then the positive number $N(\alpha)$ is called the best asymptotic constant or Nikolskii constant of the process $I_\varrho(f;x)$ for the class $\mathrm{Lip}_1\alpha$.

A first result, already to be found essentially in [5], is that

$$(8)\qquad \Delta(k_\varrho;\alpha)=m(k_\varrho;\alpha)\equiv\frac{1}{2\pi}\int_{-\pi}^{\pi}|u|^\alpha k_\varrho(u)\,du.$$

This has been the starting point of many papers. It is to be emphasized that (8) does not solve the problem since the moment $m(k_\varrho;\alpha)$ still depends upon ϱ. Indeed, in almost all cases of interest the dependence of the kernel $\{k_\varrho(x)\}$ upon the parameter ϱ is so difficult to deal with that the actual problem is to find the expansion of $m(k_\varrho;\alpha)$ in terms of powers of ϱ, say.

Therefore we do not start with (8), but prefer to commence with a result of de La Vallée Poussin [8, p. 30 ff] (see also Butzer [1]), who rewrote the integral (1) in the alternate version

(9) $$\sigma_n(f;x)=\frac{n+1}{2\pi}\int_{-\infty}^{\infty} f(x-u)\left[\frac{\sin((n+1)u/2)}{(n+1)u/2}\right]^2 du,$$

which is a particular case of the more general integral

(10) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-u)\,\varrho\chi(\varrho u)\,du$$

with positive kernel $\varrho\chi(\varrho x)$. Indeed, the class of approximation processes for which a general solution of Nikolskii's problem can be given will be characterized by this property of the Fejér means.

Definition. Let $\chi\in L^1(R)$ be positive, i. e., $\chi(x)\geqq 0$ a. e., and normalized by $\int_{-\infty}^{\infty}\chi(u)du=2\pi$. If for $\varrho>0$

(11) $$\chi_\varrho^*(x)=\sum_{k=-\infty}^{\infty}\varrho\chi(\varrho(x+2k\pi)),$$

then the (periodic) kernel $\chi_\varrho^*(x)$ is said to be of Fejér's type. The corresponding approximation process

(12) $$I_\varrho^*(f;x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\chi_\varrho^*(u)\,du,\quad \varrho>0,$$

s called a singular integral of Fejér's type.

It is a well-known fact that under the present assumptions upon χ the kernel $\chi_\varrho^*(x)$ is a positive (periodic) approximate identity for $\varrho\to\infty$, see [2, sec. 3. 1. 2]. Moreover, it is an immediate consequence of the theorem of B. Levi that $I_\varrho^*(f;x)$ of (12) takes on the form (10), i. e.,

(13) $$I_\varrho^*(f;x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-u)\varrho\chi(\varrho u)\,du$$

for each $f\in C_{2\pi}$, $\varrho>0$.

Theorem. Let $\{\chi_\varrho^*(x)\}$ be a kernel of Fejér's type generated via (11) by some function χ for which the α-th (absolute) moment $\gamma(\chi;\alpha)$ exists for some $0<\alpha\leqq 1$, i. e.,

(14) $$\gamma(\chi;\alpha)\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty}|u|^{\alpha}\chi(u)\,du<\infty.$$

Then the Nikolskii constant of the process $I_\varrho^*(f;x)$ for the class $\mathrm{Lip}_1\alpha$ is given by $\gamma(\chi;\alpha)$. Indeed,

(15) $$\Delta(\chi_\varrho^*;\alpha)=\gamma(\chi;\alpha)\varrho^{-\alpha}+o(\varrho^{-\alpha}),\ \varrho\to\infty.$$

Proof. Let $f \in \mathrm{Lip}_1\alpha$. Since χ is normalized and positive, it follows by (13) that

$$\| I_\varrho^*(f;\cdot)-f(\cdot)\| \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} \|f(\cdot-u)-f(\cdot)\| \varrho\chi(\varrho u)\, du$$

$$\leq \frac{1}{2\pi}\int_{-\infty}^{\infty} |u|^\alpha \varrho\chi(\varrho u)\, du = \gamma(\chi;\alpha)\varrho^{-\alpha},$$

giving $\Delta(\chi_\varrho^*;\alpha) \leq \gamma(\chi;\alpha)\varrho^{-\alpha}$. On the other hand, using the function $f_\alpha(x)=|\sin x|^\alpha$ which is known to belong to $\mathrm{Lip}_1\alpha$ for $0<\alpha\leq 1$, one has

$$\Delta(\chi_\varrho^*;\alpha) \geq \| I_\varrho^*(f_\alpha;\cdot)-f_\alpha(\cdot)\| \geq | I_\varrho^*(f_\alpha;0)-f_\alpha(0)|$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} |\sin u|^\alpha \chi_\varrho^*(u)\, du = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\sin u|^\alpha \varrho\chi(\varrho u)\, du,$$

the latter equality being again a consequence of (13). Therefore $\gamma(\chi;\alpha)\varrho^{-\alpha}-R(\varrho) \leq \Delta(\chi_\varrho^*;\alpha) \leq \gamma(\chi;\alpha)\varrho^{-\alpha}$ with $R(\varrho)$ defined by

$$R(\varrho)=\frac{1}{2\pi}\int_{-\infty}^{\infty} [|u|^\alpha - |\sin u|^\alpha]\, \varrho\chi(\varrho u)\, du.$$

Hence the proof of (15) would be complete if one could show that $R(\varrho)=o(\varrho^{-\alpha})$, $\varrho\to\infty$. To this end, by an elementary substitution

$$\varrho^\alpha R(\varrho)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[1-\left|\frac{\sin(u/\varrho)}{u/\varrho}\right|^\alpha\right] |u|^\alpha \chi(u)\, du.$$

The term in brackets is bounded by 1 and tends to zero as $\varrho\to\infty$ for each $u\in R$. Since $\gamma(\chi;\alpha)$ is finite, Lebesgue's dominated convergence theorem yields $\lim_{\varrho\to\infty} \varrho^\alpha R(\varrho)=0$. This establishes (15).

To give an application, let us consider the singular integral (1) of Fejér. In view of (9), the (periodic) kernel of (1) is generated via (11) by $F(x)=[\sin(x/2)/(x/2)]^2$ with $\varrho=n+1$. Obviously, the α-th moment $\gamma(F;\alpha)$ exists for $0<\alpha<1$, and it follows by partial integration that

(16) $$\gamma(F;\alpha)=\frac{2}{\pi(1-\alpha)}\int_0^{\infty} u^{\alpha-1}\sin u\, du = \frac{2}{\pi(1-\alpha)}\Gamma(\alpha)\sin\frac{\alpha\pi}{2}\cdot$$

This yields the original result (4) of Nikolskii in case $0<\alpha<1$.

Further immediate applications can be given (see [4]) so as to obtain the Nikolskii constants for the singular integrals of Abel-Poisson, Jackson-de La Vallée Poussin, Weierstrass, for the Bessel potentials, and for the typical and Abel-Cartwright means of the Fourier series of f; of course for these means only in those cases for which they represent positive approximation processes.

Let us mention that also Nikolskii in his original paper determines the best asymptotic constants for the Fejér means (1) via the integral (16). However, he proceeds via (8) together with an estimate which relates the (particular) kernels of (1) and (9). By contrast, the technique of unrolling periodic problems via (11)—(13) onto the real line is here used from the very beginning. Indeed, using (8) for kernels of Fejér's type, one has to expand the moment $m(\chi_\varrho^*; \alpha)$ of the periodic kernel $\chi_\varrho^*(x)$ — the real task, since $\chi_\varrho^*(x)$ usually has a complicated functional dependence upon the parameter ϱ. On the other hand, the present theorem reveals that the measure of approximation $\Delta(\chi_\varrho^*; \alpha)$ is asymptotically equal to $\gamma(\varrho\chi(\varrho\cdot); \alpha)$, the α-th moment (14) of $\varrho\chi(\varrho x)$. But here no further expansion is needed since the parameter may now be separated by an elementary substitution, giving the Nikolskii constant $\gamma(\chi; \alpha)$.

The idea of unrolling periodic problems via (11)—(13) onto the real axis goes back to de La Vallée Poussin [8]. Even its application to the problem of determining best asymptotic constants is not new. Here results of Sz.-Nagy [6] and Teljakovskii [7] on polynomial summation processes of Fourier series deserve particular mention. However, these investigations are mainly interested in Nikolskii constants for the class W_β^α defined as follows (for all details one may consult the discussion in [4]): Let $f \in C_{2\pi}$ and suppose that for some $\alpha > 0$, $-\infty < \beta < \infty$

$$\sum_{k=-\infty}^{\infty} |k|^\alpha f^\wedge(k) e^{i(kx+(\beta\pi/2)\operatorname{sgn} k)}, \quad f^\wedge(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u) e^{-iku}\, du,$$

is the Fourier series of an integrable f_β^α; then $W_\beta^\alpha = \{f \in C_{2\pi} \mid |f_\beta^\alpha(x)| \leq 1 \text{ a. e.}\}$. In determining best asymptotic constants for the classes W_β^α, intricate Fourier analysis is already needed from the very beginning. By contrast, the present lecture deals with Nikolskii constants for the classical Lipschitz classes. To this end, the approximation process was assumed to be given in form of a singular integral (5). This enables the elementary, direct approach to the subject given by the theorem.

On the other hand, to verify whether a given (periodic) kernel actually satisfies the assumptions of the present theorem is often a piece of hard Fourier analysis. Indeed, whereas the property of a kernel to be of Fejér's type is usually given by the Poisson summation formula, the proof of the existence of the corresponding moment (14) may be based, in general, upon the fundamental results of Sz.-Nagy [6]. However, the applications listed above admit more immediate verification (compare [4]).

The elementary approach suggested in this lecture immediately lends itself to a number of generalizations. As already mentioned, the present theorem establishes the original Nikolskii result on the Fejér means (1) only for $0 < \alpha < 1$ since the corresponding moment (16) does not exist as a finite number for $\alpha = 1$. However, the existence of the α-th moment $\gamma(\chi; \alpha)$ may be dispensed with; an analogous assertion concerning $\Delta(\chi_\varrho^*; \alpha)$ is possible in case one only knows how rapidly $\gamma(\chi; \alpha)$ is asymptotically divergent. The corresponding result then also covers the case $\alpha = 1$ of (4). Certainly, if one deals with even kernels, the proof of the theorem may be modified so as to obtain the best asymptotic constants for the Zygmund classes

$$\{f \in C_{2\pi} \mid \|f(\cdot+h)-2f(\cdot)+(t\cdot-h)\| \leqq 2 |h|^{\alpha}\}, \quad 0<\alpha\leqq 2.$$

Last not least it may be mentioned that the present approach may be applied without serious difficulties to determine Nikolskii constants of approximation processes for Lipschitz classes of functions of several variables. However, all details are left to [4]; an extended survey of the subject may also be found there.

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