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ON GENERALIZED CONVOLUTION OPERATORS, ITERATION AND SEMI-GROUPS

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Summary. In this paper we study two sequences of generalized convolution operators, which in such a way are defined that their sum possesses a prescribed Voronovskaya-property concerning the convergence of this sum to the identity, given by a second order differential operator which degenerates on the boundary of an interval. Further we construct a semi-group of operators by means of iteration with this differential operator as the infinitesimal generator of the semi-group.

1. Introduction. Let I be an open interval with boundary ∂I and f a continuous function on the closed interval $\bar{I}$. We would like to study generalized convolution operators of the following type

(1.1) $(L_n f)(x)=\int_{-\infty}^{\infty} f(x-a(x)t)K_n(t)dt,\ x\in\bar{I},\ n\in N,\ a(x)>0 \text{ on } I,\ a(x)=0 \text{ on } \partial I$

and with some restrictions to the functions K_n, a and f.

A theorem of R. Mamedov [6] for general positive linear operators gives the so-called Voronovskaya-property:

Theorem 1 (Mamedov). *Let I be a non-empty bounded open interval of the real axis and let the functions e_0, e_1 and e_2 on $\bar{I}$ be defined by $e_k(t)=t^k (k=0, 1, 2)$. Let $(L_n) (n=1, 2, \ldots)$ be a sequence of positive linear operators $L_n\colon C(\bar{I})\to C(\bar{I})$, so that we have for each $x\in\bar{I}$*

$$(L_n e_0)(x)=1+o(1/\varphi(n))$$
$$(L_n e_1)(x)=x+\psi_1(x)/\varphi(n)+o(1/\varphi(n))$$
$$(L_n e_2)(x)=x^2+\psi_2(x)/\varphi(n)+o(1/\varphi(n)),$$

where φ is a real function defined on N, independent of x, $\varphi(n)\neq 0$ for each n and $\lim_{n\to\infty}\varphi(n)=\infty$ and let there exist an even number $k>2$, such that $(L_n h_{k,x})(x)=o(1/\varphi(n))$, where $h_{k,x} (k\in N, x\in I)$ is the function defined on $\bar{I}$ by $h_{k,x}(t)=(t-x)^k$.

Then for each function $f\in C^2(\bar{I})$ and $x\in I$ we have

(1.2) $(L_n f)(x)-f(x)=\{2\psi_1(x)f'(x)+\{\psi_2(x)-2x\psi_1(x)\}f''(x)\}/2\varphi(n)+o(1/\varphi(n)).$

Proof See P. Sikkema [9].

The differential operator $A(x, D)=\{\frac{1}{2}\psi_2(x)-x\psi_1(x)\}D^2+\psi_1(x)D$, induced by the right-hand side of (1.2), degenerates on the boundary ∂I of $\bar{I}$ if $\psi_1(x)=\psi_2(x)=0$ for $x\in\partial I$.

For the generalized convolution operators we have this property. Of special interest is the construction of a semi-group of operators by means of iteration of the approximation operators. We make use of the following

Theorem 2 (H. Trotter-T. Kato). *Let X be a Banach space, let $\{L_n\}$ be a sequence of operators $X\to X$, let φ be a real function on N with $\lim_{n\to\infty}\varphi(n)=\infty$ and let the operators B_n; $n=1, 2, \ldots$ and B be defined by $B_n f=\varphi(n)\{L_n-I\}f, f\in X$ and $Bf=\lim_{n\to\infty}B_n f$, $f\in D(B)$, $Bf\in X$. Moreover, suppose there are numbers $M>0$, $N\geqq 0$ so that $\| L_n^k \|\leqq M\cdot e^{Nk/\varphi(n)}$ and there is a number $\lambda>N$ so that s-$\lim(\lambda-B_n)^{-1}=(\lambda-B)^{-1}$.*

Then we have for each $t\geqq 0$

$$\text{s-}\lim L_n^{[\varphi(n)t]}=T(t)$$

uniformly in each finite interval of t, and $\{T(t);\ t\geqq 0\}$ is a semi-group of class C_0 with infinitesimal generator B. ($[k]$ means entire of k).

Proof. See T. Kato [5].

The semi-group with infinitesimal generator the differential operator $A(x, D)$ defined by $A(x, D)=\alpha(x)D^2+\beta(x)D$ (degenerating on the boundary ∂I and with some restrictions to α and β) is constructed by means of iteration of a combination of generalized convolution operators.

Remark. In special cases the problem of iteration of positive linear operators is studied by several authors. S. Karlin and Z. Ziegler [4], C. Micchelli [8], M. Becker and R. Nessel [2] studied iteration of Bernstein-operators and used polynomial techniques. Further in this connection Becker, Butzer and Nessel studied Favarel operators.

2. **Definitions and Notations.** I is a non-void bounded open interval of the real axis, $\bar{I}$ is the closure of I and ∂I the boundary of I. $C(I)$, $C(\bar{I})$ is the set of the real continuous functions on I, resp. $\bar{I}$. $C^k(I)$, $C^k(\bar{I})$, $k=1, 2, \ldots$ is the set of the k-times continuous differentiable real functions on I, resp. $\bar{I}$. e_k; $k=0, 1, 2, \ldots$ is the function defined on $\bar{I}$ by $e_k(t)=t^k$. $\chi_{[a,b]}$ is the characteristic function of the interval $[a, b]$, i. e. $\chi_{[a,b]}(t)=1$ if $t\in[a, b]$ and $\chi_{[a,b]}=0$ if $t\notin[a, b]$. $h_{k,x}$; $k=1, 2, \ldots, x\in\bar{I}$ is the function defined on $\bar{I}$ by $h_{k,x}(t)=(t-x)^k$. The identity-operator will be written by I. $D, D^2, \ldots$ are the usual differential operators. The supremum norm and the operator norm will be denoted by $\|\ \|$.

Because of the existence of the convolution integral, sometimes we need a continuation of the functions of the sets $C(I)$, $C(\bar{I})$ $C^k(I)$, $C^k(\bar{I})$ outside the domain I, resp. $\bar{I}$. If necessary, we continue the functions of these sets in a trivial way without changing the notation of these sets. Let f be such a function, then we define $f(x)=0$, $x\notin I$, resp. $\bar{I}$.

3. **The Operators R_n, S_n and T_n.** Let $r\in C(\bar{I})$ satisfy the conditions

(3.1) i) $r(x)>0$, $x\in I$, ii) $r(x)=0$, $x\in\partial I$.

We define the sequence of generalized convolution operators $R_n;\ C(\bar{I})\to C(\bar{I})$, $n=1, 2, \dots$, by

$$(3.2)\qquad (R_n f)(x)=\int_{-\infty}^{\infty} f(x-r(x)t)n\chi_{[-1,0]}(nt)dt,\ x\in\bar{I}.$$

The operators R_n are linear, positive, bounded and $\|R_n\|=1$, $n=1, 2, \dots$ For $f\in C(\bar{I})$ we have

$$(R_n f)(x)=\frac{n}{r(x)}\int_0^{r(x)/n} f(x+t)dt, \text{ if } x\in I,$$

$$(R_n f)(x)=f(x) \qquad \text{if } x\in\partial I.$$

It is easy to calculate that

$$(R_n e_0)(x)=1$$

$$(R_n e_2)(x)=x^2+xr(x)/n+r^2(x)/3n^2$$

$$(R_n e_3)(x)=x^3+3x^2r(x)/2n+xr^2(x)/n^2+r^3(x)/4n^3$$

$$(R_n e_4)(x)=x^4+2x^3r(x)/n+2x^2r^2(x)/n^2+xr^3(x)/n^3+r^4(x)/5n^4.$$

With these results it is easy to find that $(R_n h_{4,x})(x)=r^4(x)/5n^4$. With $\varphi(n)=n$, $\psi_1(x)=r(x)/2$ and $\psi_2(x)=xr(x)$, the conditions of the theorem of Mamedov are satisfied. So for $f\in C^2(\bar{I})$

$$(3.3)\qquad n(R_n-I)f(x)=\frac{1}{2}r(x)(Df)(x)+0(1),\ x\in I,\ n\to\infty,$$

$$n(R_n-I)f(x)=0, \qquad x\in\partial I,\ n=1, 2, \dots.$$

It is to prove that $f\in C^1(\bar{I})$ is sufficient in this case.

If we define the sequence of operators $U_n\colon C(\bar{I})\to C(\bar{I})$ by $(U_n f)(x)=n(R_n-I)f(x)$ and the operator $U\colon C^1(\bar{I})\to C(\bar{I})$ by $(Uf)(x)=\frac{1}{2}r(x)(Df)(x)$ then from (3.3) we have for $f\in C^1(I)$

$$(3.4)\qquad \lim_{n\to\infty}(U_n f)(x)=(Uf)(x).$$

Let $s\in C(\bar{I})$ satisfy the conditions (3.1), so $s(x)>0$ if $x\in I$ and $s(x)=0$ if $x\in\partial I$. We define the sequence of generalized convolution operators $S_n\colon C(\bar{I})\to C(\bar{I})$, $n=1, 2, \dots$ by

$$(3.5)\qquad (S_n f)(x)=\int_{-\infty}^{\infty} f(x-s(x)t)n\chi_{[-1/2,\,1/2]}(nt)dt,\ x\in\bar{I}.$$

The operators S_n are linear, positive, bounded with $\|S_n\|=1$, $n=1, 2, \dots$. Moreover, we have

$$(S_n f)(x)=(n/s(x))\int_{-s(x)/2n}^{s(x)/2n} f(x+t)dt, \text{ if } x\in I,$$

$$(S_n f)(x)=0, \qquad \text{if } x\in\partial I$$

and just as in the case of the operators R_n it is to calculate that

$$(S_n e_0)(x)=1, \qquad (S_n e_2)(x)=x^2+s^2(x)/12n^2,$$

$$(S_n e_1)(x)=x, \qquad (S_n h_{4,x})(x)=s^4(x)/80n^4.$$

In this case, with $\varphi(n)=n^2$, the conditions of the theorem of Mamedov are satisfied.

For $f\in C^2(\bar{I})$ we then have

$$(3.6)\qquad n^2(S_n-I)f(x)=\begin{cases}(s^2(x)/24)(D^2f)(x)+0(1), & n\to\infty\\ 0, & x\in\partial I,\ n=1,2,\dots.\end{cases}$$

We define the operators $V_n\colon C(\bar{I})\to C(\bar{I})$ by $(V_nf)(x)=n^2(S_n-I)f(x)$ and $V\colon C^2(\bar{I})\to C(\bar{I})$ by $(Vf)(x)=(s^2(x)/24)(D^2f)(x)$. It follows from (3.6) that for $f\in C^2(\bar{I})$ we have

$$(3.7)\qquad \lim_{n\to\infty}(V_nf)(x)=(Vf)(x),\quad x\in\bar{I}.$$

Now we are ready to define the sequence of operators T_n. Let r and s satisfy the conditions (3.1). We define the operators $T_n\colon C(\bar{I})\to C(\bar{I})$; $n=1,2,\dots$ by

$$(3.8)\qquad (T_nf)(x)=\frac{1}{2}\{(R_nf)(x)+S_{\sqrt{n}}f)(x)\},$$

the operators $W_n\colon C(\bar{I})\to C(\bar{I})$; $n=1,2,\dots$ by $(W_nf)(x)=n(T_n-I)f(x)$ and the operator $W\colon C^2(\bar{I})\to C(\bar{I})$ by $(Wf)(x)=(s^2(x)/48)(D^2f)(x)+(r(x)/4)(Df)(x)$. From (3.4) and (3.7) it follows that for $f\in C^2(\bar{I})$

$$(3.9)\qquad \lim_{n\to\infty}(W_nf)(x)=(W_f)(x)$$

holds. It is easy to verify that with the sup-norm the convergence is uniform.
Now we arrive at the first main theorem of this paper.

Theorem 3. *Let $\alpha,\beta\in C(\bar{I})$, $\alpha(x)>0$ and $\beta(x)>0$ on I and $\alpha(x)=\beta(x)=0$ on ∂I. Then there exists a sequence $\{T_n\}_1^\infty$ of positive linear operators such that the differential operator $A(x,D)=\alpha(x)D^2+\beta(x)D$ occurs in the Voronovskaya-type relation* $\lim_{n\to\infty}n(T_n-I)f(x)=A(x,D)f(x)$.

Proof: Take $r=4\beta$ in (3.2), $s=\sqrt{48\alpha}$ in (3.5), then (3.8) gives the sequence $\{T_n\}$ and (3.9) gives the Voronovskaya-type relation.

4. **The Resolvants $R(\lambda;\ W_n)$ and $R(\lambda;\ W)$.** In addition to (3.1) we suppose in the following that $r,s\in C^2(\bar{I})$, s^{-2} is not integrable over the neighbourhoods of the boundary points of I and rs^{-1} is bounded on I. Because of the boundedness of the operators R_n and S_n it follows that the operators W_n are bounded and $\|W_n\|\le 2n$, $n=1,2,\dots$. So W_n and $W_n-\lambda$, $\lambda\in C$ are closed operators $C(\bar{I})\to C(\bar{I})$. Let $\lambda\in\mathrm{R}$, $\lambda>0$ and $f\subset C(\bar{I})$ and let $g(x)=(W_n-\lambda)f(x)$.

Then $f(x)=-(n+\lambda)^{-1}g(x)+(n+\lambda)^{-1}n(T_nf)(x)$. With repeated substitution we find that

$$f(x)=-(n+\lambda)^{-1}\sum_{k=0}^{\infty}((n+\lambda)^{-1}n)^k(T_nf)(x)\quad\text{or}\quad f(x)=(W_n-\lambda)^{-1}g(x).$$

We define the operators $R(\lambda;\ G_n)$, $n\in N$, $\lambda>0$ by

$$R(\lambda;\ W_n)g(x)=(W_n-\lambda)^{-1}g(x)=-(n+\lambda)^{-1}\sum_{k=0}^{\infty}((n+\lambda)^{-1}n)^k(T_nf)(x).$$

Let $g \in C(\bar{I})$, then is also $F_n g \in C(\bar{I})$, and we see $F_n^k g \in D(F_n^{k+1})$, $k \in N$. Further we have

$$\| R(\lambda;\ W_n)g \| \leqq (n+\lambda)^{-1}.(1-n/(n+\lambda))^{-1}.\| g \|/\| T_n \| = \| g \|/\lambda.$$

So

$$\| R(\lambda;\ W_n) \| \leqq 1/\lambda.$$

So $R(\lambda;\ W_n)$ is defined on $C(\bar{I})$. In the same manner we define in $C(\bar{I})$ the operator $R(\lambda;\ W)$ by $R(\lambda;\ W)g=(W-\lambda)^{-1}g$.

Let g be in the range $R(W-\lambda)$ of the operator $W-\lambda$. Suppose $R(\lambda;\ W)g=f$ and define $f_n=R(\lambda;\ W_n)g$. Then $W_n f_n - \lambda f_n = Wf - \lambda f$ or $(W_n-\lambda)(f_n-f)=-(W_n-W)f$ and $f_n-f=-(W_n-\lambda)^{-1}(W_n-W)f$. So for f sufficiently large we have

$$\| f_n - f \| \leqq \| R(\lambda;\ W_n) \| \cdot \| W_n f - Wf \| < \varepsilon/\lambda$$

because of the uniform convergence of $W_n f$ to Wf. So we have

$$\lim_{n\to\infty} R(\lambda;\ W_n)g = R(\lambda;\ W)g \tag{4.1}$$

on the range $R(W-\lambda)$.

R. Martini [7] proved that under the given conditions upon r and s there exists a $\lambda_0>0$, such that for $\lambda \geqq \lambda_0$ $R(\lambda;\ W)$ exists as a bounded linear operator of $C(\bar{I})$ into $C(\bar{I})$ with $\| R(\lambda;\ W) \| \leqq 1/\lambda$. So $R(W-\lambda)=C(\bar{I})$. With this result and (4.1) we find that

$$\text{s-lim}\, R(\lambda;\ G_n) = R(\lambda;\ G) \tag{4.2}$$

on $C(\bar{I})$.

5. Iteration, Semigroups and Saturation. Now we arrive at the second main theorem

Theorem 4. *Let the sequence of operators $\{T_n\}$ be defined by* (3.8) *and r and s satisfy the conditions of section* 4. *Then*

$$\text{s-lim}\, T_n^{[nt]} = T(t), \quad t>0 \tag{5.1}$$

uniformly in each finite interval of t, where $\{T(t);\ t>0\}$ is a semigroup of class C_0 with infinitesimal generator W.

Proof. Putting $X=C(\bar{I}), \varphi(n)=n, B_n=W_n, B=W, M=1, N=0, L_n=T_n$. Because of (3.9) and (4.1) the conditions of the theorem of Trotter-Kato are satisfied. So W is the infinitesimal generator of a semi-group of class C_0 and (5.1) holds.

Remarks. The fact that W is the infinitesimal generator of a semigroup of class C_0 was already proved by R. Martini [7] in 1975. Because of the close relation between the Voronovskaya-property of the operators T_n and the infinitesimal generator W of $T(t)$ one may expect that there is a connection between the saturation classes of the operator sequence $\{T_n\}$ and the approximation process $T(t)$. In a following paper we will consider these problems.

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