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CONVOLUTION AND COMMUTANT OF GELFOND-LEONTIEV OPERATOR OF INTEGRATION

I. H. Dimovski, V. S. Kiryakova

Summary. The paper deals with the construction of a convolution for Gelfond-Leontiev generalized operator of integration

$$l_{\rho,\mu} f(z) = z\,\Gamma^{-1}(\rho^{-1}) \int_0^1 (1-\sigma)^{\frac{1}{\rho}-1} \sigma^{\mu-1} f(z\sigma^{1/\rho})\,d\sigma$$

in the space $\mathscr{H}(G)$ of functions, analytic in a starlike domain $G \subset \boldsymbol{C}$. This operator is the initial right inverse of the operator $D_{\rho,\mu}$ of generalized differentiation, which is defined as

$$D_{\rho,\mu} f(z) = \sum_{n=1}^{\infty} a_n \{\Gamma(\mu+\frac{n}{\rho}) / \Gamma(\mu+\frac{n-1}{\rho})\} z^{n-1}$$

for functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$, analytic in a disk $\Delta_R: |z| < R,\ R > 0$.

The following convolution for $l_{\rho,\mu}$, $\mu \in \boldsymbol{R}$:

$$f \overset{(\rho,\mu)}{*} g(z) = \{\frac{1}{\Gamma(\mu)} t^{1-\mu} (\frac{d}{dt})^{\mu} \int_0^t (t-x)^{\mu-1} f[(t-x)^{1/\rho}]\, x^{\mu-1} g(x^{1/\rho})\,dx\}_{t=z^\rho}$$

is found, with the property $l_{\rho,\mu} f(z) = \{\rho\Gamma(\mu) z / \Gamma(\mu+\rho^{-1})\} \overset{(\rho,\mu)}{*} f(z)$.

Each linear operator $M: \mathscr{H}(G) \to \mathscr{H}(G)$, which commutes with the operator $l_{\rho,\mu}$, has the form $Mf(z) = m \overset{(\rho,\mu)}{*} f(z)$ with a function $m \in \mathscr{H}(G)$.

In [1] Gelfond and Leontiev had introduced the following generalization of the differentiation $D = d/dz$:

$$D_\rho f(z) = \sum_{n=1}^{\infty} a_n \{\Gamma(1+\frac{n}{\rho}) / \Gamma(1+\frac{n-1}{\rho})\} z^{n-1} \tag{1}$$

for functions $f \in \mathscr{H}(\Delta_R)$, analytic in a disk $\Delta_R: |z| < R,\ R > 0$:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{2}$$

where $\rho>0$ is a fixed constant and obviously $D_1=D$. The right inverse operator l_ρ of D_ρ, defined as

$$l_\rho f(z)=\sum_{n=0}^{\infty} a_n\{\Gamma(1+(n/\rho)/\Gamma(1+(n+1)/\rho)\}\, z^{n+1},$$

is said to be the Gelfond-Leontiev operator of integration.

The entire function of Mittag-Leffler type $E_\rho(z,1)$ satisfies the relation $D_\rho E_\rho(\alpha z,1)=\alpha E_\rho(\alpha z,1)$ with $\alpha\neq 0$. If $\rho=1$, we have $E_1(\alpha z,1)=e^{\alpha z}$: Later we make use of the general definition of these functions ([2, p. 117]). $E_\rho(z,\mu)=\sum_{n=0}^{\infty} z^n/\Gamma(\mu+(n/\rho))$, $\mu\in\mathbf{R}$.

In [3] the following convolution for the operator l_ρ, extended to the larger space $\mathcal{H}(G)$ of functions, analytic in a domain G starlike with respect to the origin $z=0$:

$$(3)\qquad f\overset{(\rho)}{*}g(z)=(\rho^{-1}z\frac{d}{dz}+1)\int_0^1 f[z(1-\tau)^{1/\rho}]g(z\tau^{1/\rho})d\tau$$

is found. By means of this operation explicit representations of commutants of operator l_ρ and its integer powers l_ρ^m are given. These representations are simpler than those of Tkachenko [4] and generalize the results of Raichinov [5].

In this paper we consider a more general operator $l_{\rho,\mu}$ of Gelfond-Leontiev type.

Definition 1. The operator

$$(4)\qquad D_{\rho,\mu}f(z)=\sum_{n=1}^{\infty} a_n\{\Gamma(\mu+\frac{n}{\rho})\,/\,\Gamma(\mu+\frac{n-1}{\rho})\}\, z^{n-1}$$

in the space $\mathcal{H}(\Delta_R)$ of functions, analytic in a disk $\Delta_R:|z|<R$, $f(z)=\sum_{n=0}^{\infty} a_n z^n$, is said to be Gelfond-Leontiev generalized operator of differentiation. The initial right inverse operator of $D_{\rho,\mu}$

$$(5)\qquad l_{\rho,\mu}f(z)=\sum_{n=0}^{\infty} a_n\{\Gamma(\mu+\frac{n}{\rho})/\Gamma(\mu+\frac{n+1}{\rho})\}\, z^{n+1}$$

is said to be *Gelfond-Leontiev generalized operator of integration*. Here and in what follows $\rho>0$, $-\infty<\mu<\infty$.

It is obvious that $D_{\rho,\mu}E_\rho(\alpha z,\mu)=\alpha E_\rho(\alpha z,\mu)$, $\alpha\neq 0$. The initial value operator $F_{\rho,\mu}=I-l_{\rho,\mu}D_{\rho,\mu}$ $(D_{\rho,\mu}l_{\rho,\mu}=I)$ of $l_{\rho,\mu}$ with respect to $D_{\rho,\mu}$ has the form $F_{\rho,\mu}f(z)=f(0)$.

At first we shall obtain an integral representation of an extension of $l_{\rho,\mu}$ (5) to a more general space. Without loss of generality we can assume $\mu\geqq 1$ for the sake of brevity.

Theorem 1. *Gelfond-Leontiev operator of integration $l_{\rho,\mu}$ has a representation*

$$(6)\qquad l_{\rho,\mu}f(z)=z\Gamma^{-1}(\rho^{-1})\int_0^1(1-\sigma)^{1/\rho-1}\sigma^{\mu-1}f(z\sigma^{1/\rho})d\sigma\,;\ \mu\geqq 1,$$

in the space $\mathcal{H}(G)$ of functions, analytic in a domain G starlike with respect to the origin $z=0$.

Proof. It is sufficient to verify the validity of (6) for an arbitrary integer non-negative power z^n of z. We have

$$l_{\rho,\mu}\{z^n\} = z^{n+1}\Gamma^{-1}(\rho^{-1})\int_0^1 (1-\sigma)^{1/\rho-1}\sigma^{\mu+n/\rho-1}\,d\sigma = \{\Gamma(\mu+\frac{n}{\rho})/\Gamma(\mu+\frac{n+1}{\rho})\}z^{n+1}.$$

Then, if $f(z)=\Sigma_{n=0}^{\infty} a_n z^n$ is an arbitrary function of $\mathscr{H}(\Delta_R)$, we get

$$l_{\rho,\mu} f(z) = \sum_{n=0}^{\infty} a_n l_{\rho,\mu}\{z^n\} = \sum_{n=0}^{\infty} a_n \{\Gamma(\mu+\frac{n}{\rho})/\Gamma(\mu+\frac{n+1}{\rho})\} z^{n+1},$$

which proves the theorem.

The representation (6) defines the operator $l_{\rho,\mu}$ not only in $\mathscr{H}(\Delta_R)$, but in $\mathscr{H}(G)$ as well, where G is an arbitrary domain starlike with respect to $z=0$.

It is useful to receive an analogous representation of the generalized differentiation $D_{\rho,\mu}$ in the space $\mathscr{H}(G)$. It is possible to do this for every $\rho>0$, but the result is simpler in the case $\rho\geq 1$.

Lemma 1. *If* $\mu\geq 1$, $\rho\geq 1$, *the integro-differential operator*

$$(7)\quad D_{\rho,\mu}f(z) = \begin{cases} -\dfrac{f(0)\Gamma(\mu)}{z\Gamma(\mu-\rho^{-1})} + (\rho^{-1}z\dfrac{d}{dz}+\mu)z^{-1}\displaystyle\int_0^1 \dfrac{(1-\sigma)^{-1/\rho}\sigma^{\mu-1}}{\Gamma(1-\rho^{-1})} f(z\sigma^{1/\rho})d\sigma, & \text{if } \rho>1 \\ -f(0)(\mu-1)z^{-1} + (z\dfrac{d}{dz}+\mu)z^{-1}f(z), & \text{if } \rho=1 \end{cases}$$

defined in $\mathscr{H}(G)$, *coincides with Gelfond-Leontiev operator of differentiation* (4) *in the space* $\mathscr{H}(\Delta_R)$, *when* $\Delta_R\subseteq G$.

Proof. As in the previous theorem we shall establish the coincidence of (7) and (4) only for functions of the type $f(z)=z^n$, $n\geq 0$. Due to (7), we have for $\rho>1$ (i. e. $[\rho^{-1}]=0$)

$$D_{\rho,\mu}\{z^n\} = (\rho^{-1}z\frac{d}{dz}+\mu)z^{n-1}\int_0^1 \frac{(1-\sigma)^{-1/\rho}}{\Gamma(1-\rho^{-1})}\sigma^{\mu+(n/\rho-1)}d\sigma$$

$$= \{\Gamma(\mu+(n/\rho))/\Gamma(\mu+(n-1)/\rho)\}z^{n-1},\ n\geq 1$$

and for $n=0$

$$D_{\rho,\mu}\{1\} = -\frac{\Gamma(\mu)}{z\Gamma(\mu-\rho^{-1})} + (\rho^{-1}z\frac{d}{dz}+\mu)z^{-1}\frac{\Gamma(\mu)}{\Gamma(\mu-\rho^{-1}+1)} = 0.$$

By analogy, if $\rho=1$, according to (7) $D_{1,\mu}\{z^n\}=(\mu+n-1)z^{n-1}$, $n\geq 1$; $D_{1,\mu}\{1\}=0$, $n=0$. In this case we have to bear in mind that by (4)

$$D_{1,\mu} f(z) = \sum_{n=1}^{\infty} a_n\{\Gamma(\mu+n)/\Gamma(\mu+n-1)\}z^{n-1} = \sum_{n=1}^{\infty} a_n(\mu+n-1)z^{n-1}$$

in $\mathscr{H}(\Delta_R)$.

Corollary 1. *For* $\mu=1$ *we receive the representation of* D_ρ *given in* [3]:

$$D_\rho f(z) = \begin{cases} -\dfrac{f(0)}{z\Gamma(1-\rho^{-1})} + (\rho^{-1}z\dfrac{d}{dz}+1)z^{-1}\displaystyle\int_0^1 \dfrac{(1-\sigma)^{-1/\rho}}{\Gamma(1-\rho^{-1})} f(z\sigma^{1/\rho})d\sigma, & \rho>1, \\ \dfrac{d}{dz}f(z), & \rho=1. \end{cases}$$

Now let us mention the general definition for a convolution of a linear operator.

Definition 2 ([3, 6]). Let L be a linear operator $L:\mathcal{H}\to\mathcal{H}$, which maps the space $\mathcal{H}$ into itself. A convolution of L in $\mathcal{H}$ is said to be each bilinear, commutative and associative operation $*:\mathcal{H}\times\mathcal{H}\to\mathcal{H}$ with the property $L(f*g)=(Lf)*g$, $f, g\in\mathcal{H}$.

In our case Gelfond-Leontiev operator of integration $l_{\rho,\mu}$ is linear and such that $l_{\rho,\mu}:\mathcal{H}(G)\to\mathcal{H}(G)$.

In the abstract of this paper we have given a representation of a convolution for $l_{\rho,\mu}$, namely

$$(8)\quad f\overset{(\rho,\mu)}{*}g(z)=\{\frac{1}{\Gamma(\mu)}t^{1-\mu}\left(\frac{d}{dt}\right)^{\mu}\int_0^t(t-x)^{\mu-1}f[(t-x)^{1/\rho}]x^{\mu-1}g(x^{1/\rho})dx\}_{t:=z^\rho},$$

where we make use of the following definition of the operator of fractional differentiation of order $\mu>0$:

$$(9)\qquad \left(\frac{d}{dt}\right)^{\mu}f(t)=\begin{cases}\left(\frac{d}{dt}\right)^{[\mu]+1}\int_0^t\frac{(t-\tau)^{-\{\mu\}}}{\Gamma(1-\{\mu\})}f(\tau)d\tau;\ \mu\text{—noninteger}\\ f^{(\mu)}(t),\ \mu\text{ — integer.}\end{cases}$$

It is necessary to note that the expressions (8), (9) define single-valued functions in the complex plane if we restrict our considerations to the space $\mathcal{H}(\tilde{G})$. The domain $\tilde{G}$ is received from G by putting out a suitable chosen branch cut starting at $z=0$ and ending at infinity. For example we can let the branch cut be the negative real axis, i. e. $\tilde{G}=G\setminus\{-\infty, 0\}$.

Nevertheless, it is possible to give another representation of the same operation, which defines a single-valued analytic function in the whole domain G.

Theorem 2. *For* $\mu\geq 1$ *the operation*

$$(10)\quad f\overset{(\rho,\mu)}{*}g(z)=\begin{cases}B_{\rho,\mu}^{([\mu]+1)}\int_0^1\frac{(1-\sigma)^{-\{\mu\}}}{\Gamma(1-\{\mu\})}\sigma^{2\mu-1}d\tau\int_0^1\frac{(1-\tau)^{\mu-1}}{\Gamma(\mu)}f[z\sigma^{1/\rho}(1-\tau)^{1/\rho}]\tau^{\mu-1}\\ \qquad g(z\sigma^{1/\rho}\tau^{1/\rho})d\tau\ \textit{for}\ \mu\textit{ — noninteger}\\ B_{\rho,\mu}^{(\mu)}\int_0^1\frac{(1-\tau)^{\mu-1}}{\Gamma(\mu)}f[z(1-\tau)^{1/\rho}]\tau^{\mu-1}g(z\tau^{1/\rho})d\tau\ \textit{for}\ \mu\textit{ — integer,}\end{cases}$$

where we use the notation

$$B_{\rho,\mu}^{(j)}F(z)=\prod_{k=0}^{j-1}[\rho^{-1}z\frac{d}{dz}+(\mu+k)]F(z)=[\rho^{-1}z\frac{d}{dz}+\mu]\ldots[\rho^{-1}z\frac{d}{dz}+(\mu+j-1)]F(z),$$

is a convolution for $l_{\rho,\mu}$ *in the space* $\mathcal{H}(G)$.

Proof. First we shall prove that the operations (8) and (10) coincide for the functions $f(z)=z^m$, $g(z)=z^n$, $m, n\geq 0$. According to (10) the following expression

$$(11)\qquad \{z^m\}\overset{(\rho,\mu)}{*}\{z^n\}=\frac{\Gamma(\mu+\frac{m}{\rho})\Gamma(\mu+\frac{n}{\rho})}{\Gamma(\mu)\Gamma(\mu+\frac{m+n}{\rho})}z^{m+n}$$

is found by tedious, but standard calculations, which we shall omit here. But the operation (8) gives the same result for $\{z^m\} \overset{(\rho,\mu)}{*} \{z^n\}$. Hence the operations coincide also for polynomials and for all functions $f,\ g \in \mathscr{H}(G)$, due to Runge's approximation theorem.

The bilinearity and commutativity of (10) are evident. Let us establish that (10) is an associative operation in $\mathscr{H}(G)$. To this end we verify the relation $(f \overset{(\rho,\mu)}{*} g) \overset{(\rho,\mu)}{*} h = f \overset{(\rho,\mu)}{*} (g \overset{(\rho,\mu)}{*} h)$ for arbitrary integer powers $f(z)=z^m$, $g(z)=z^n$, $h(z)=z^p$, $m, n, p \geqq 0$. It is easy to get from (11) that

$$(\{z^m\} \overset{(\rho,\mu)}{*} \{z^n\}) \overset{(\rho,\mu)}{*} \{z^p\} = \frac{\Gamma(\mu+\frac{m}{\rho})\Gamma(\mu+\frac{n}{\rho})\Gamma(\mu+\frac{p}{\rho})}{\Gamma^2(\mu)\Gamma(\mu+\frac{m+n+p}{\rho})} z^{m+n+p}.$$

Because of the symmetry of this expression with respect to m, n and p, the identity

$$(\{z^m\} \overset{(\rho,\mu)}{*} \{z^n\}) \overset{(\rho,\mu)}{*} \{z^p\} = \{z^m\} \overset{(\rho,\mu)}{*} (\{z^n\} \overset{(\rho,\mu)}{*} \{z^p\})$$

is proved. On account of bilinearity and continuity of the operation (10) its associativity holds for polynomials and for all $f,\ g \in \mathscr{H}(G)$, due to Runge's theorem.

It remains to establish that

$$l_{\rho,\mu}(f \overset{(\rho,\mu)}{*} g) = (l_{\rho,\mu} f) \overset{(\rho,\mu)}{*} g. \tag{12}$$

To this end it is sufficient to verify the identity

$$l_{\rho,\mu} f(z) = \{\rho\Gamma(\mu)z/\Gamma(\mu+\rho^{-1})\} \overset{(\rho,\mu)}{*} f(z). \tag{13}$$

Since

$$l_{\rho,\mu}\{z^n\} = \{\Gamma(\mu+\frac{n}{\rho})/\Gamma(\mu+\frac{n+1}{\rho})\} z^{n+1},$$

we get (13) for $f(z)=z^n$, $n \geqq 0$ (11), according to (11). By virtue of Runge's theorem (13) holds in $\mathscr{H}(G)$. Then, if we denote by

$$r(z) = \{\rho\Gamma(\mu)z/\Gamma(\mu+\rho^{-1})\} \in \mathscr{H}(G), \tag{14}$$

it follows

$$l_{\rho,\mu}(f \overset{(\rho,\mu)}{*} g) = r \overset{(\rho,\mu)}{*} (f \overset{(\rho,\mu)}{*} g) = (r \overset{(\rho,\mu)}{*} f) \overset{(\rho,\mu)}{*} g = (l_{\rho,\mu} f) \overset{(\rho,\mu)}{*} g,$$

i. e. (12) holds.

In this way Theorem 2 is completely proved.

The following two lemmas are evident on account of the previous proof.

Lemma 2. *The operator $l_{\rho,\mu}$ can be represented by the convolution* (10) *as $l_{\rho,\mu} f(z) = r \overset{(\rho,\mu)}{*} f(z)$, where $r \in \mathscr{H}(G)$ is defined by* (14).

Lemma 3. *The convolution* (10) *has a unity in $\mathscr{H}(G)$ and this is the constant-function* $\{1\}$, *i. e.*

$$\{1\} \overset{(\rho,\mu)}{*} f(z) = f(z). \tag{15}$$

Corollary 2. *In the case* $\mu=1$ *the convolution* (10) *reduces to the convolution* (3) *for* l_ρ, *obtained in* [3].

Having a convolution for the operator $l_{\rho,\mu}$ in $\mathscr{H}(G)$, we can find a representation of the commutant of $l_{\rho,\mu}$ in this space, using a general theorem of [3]. We have

Theorem 3. *A linear operator* $M:\mathscr{H}(G)\to\mathscr{H}(G)$ *commutes with Gelfond-Leontiev operator of integration* $l_{\rho,\mu}$ *iff it possesses a representation of the type*

$$Mf(z)=m\overset{(\rho,\mu)}{*}f(z), \tag{16}$$

where $\overset{(\rho,\mu)}{*}$ *is the operation* (10) *and* $m\in\mathscr{H}(G)$, $m(z)=M\{1\}$.

Proof. For the validity of the general theorem it is necessary to show that the operator $l_{\rho,\mu}$ has a cyclic element in $\mathscr{H}(G)$. Indeed, the constant-function $\{1\}$ is such an element for $l_{\rho,\mu}$, due to the fact that the linear span of the powers $\{l^n_{\rho,\mu}\{1\}\}_{n=1}^{\infty}$ coincides with the set of polynomials in $\mathscr{H}(G)$. But this set is dense in $\mathscr{H}(G)$, according to Runge's approximation theorem.

If $M:\mathscr{H}(G)\to\mathscr{H}(G)$ is a linear operator which commutes with $l_{\rho,\mu}$ in $\mathscr{H}(G)$, i. e. $Ml_{\rho,\mu}=l_{\rho,\mu}M$, then M is a multiplier of the convolution (10) for $l_{\rho,\mu}$. From (15) we get

$$Mf(z)=(M\{1\})\overset{(\rho,\mu)}{*}f(z)=m\overset{(\rho,\mu)}{*}f(z).$$

Conversely, it is obvious that M of form (16) commutes with $l_{\rho,\mu}$ in $\mathscr{H}(G)$.

Finally we consider some special cases.

Let us note that the case $\mu=\rho^{-1}=\alpha>0$ seems to be one of the most interesting. The differential operator $D_\alpha f(z)=\Sigma_{n=1}^{\infty} a_n\{\Gamma(\alpha n)/\Gamma[\alpha(n-1)]\}z^{n-1}$, defined for functions (2) was introduced by Iliev [7]. In this case the corresponding convolution $\overset{(\alpha)}{*}$ (10) of two integer powers of z can be expressed in terms of the generalized binomial coefficients $\binom{n}{k}_\alpha$ ([7]):

$$\{z^m\}\overset{(\alpha)}{*}\{z^n\}=z^{m+n}\binom{m+n}{m}_\alpha.$$

The case is often considered also by Džrbašjan [2].

The case $\mu=1$, $\rho>0$ ($D_{\rho,1}=D_\rho$, $l_{\rho,1}=l_\rho$) was investigated in detail in [3].

For $\mu=\rho=1$ all these results refer to the operators

$$D_{1,1}=D=\frac{d}{dz},\ l_{1,1}f(z)=lf(z)=\int_0^z f(\zeta)d\zeta=z\int_0^1 f(z\tau)d\tau.$$

The convolution (10) obtained here turns into the well-known operation

$$f\overset{(1,1)}{*}g(z)=\frac{d}{dz}\int_0^z f(z-\zeta)g(\zeta)d\zeta=\frac{d}{dz}z\int_0^1 f[z(1-\tau)]g(z\tau)d\tau.$$

In conclusion we naturally come to the question of the existence of a suitable integral transform which has the operation (10) as a convolution. This transform must also reduce the action of the generalized differentiation

$D_{\rho,\mu}$ to multiplication by a fixed function of complex variable z. Such a transform has the form

$$B_{\rho,\mu}\{f(z);\ \zeta\}=\rho\zeta^{\rho\mu-1}\int_0^{+\infty}\exp(-\zeta^\rho z^\rho)z^{\mu\rho-1}f(z)dz$$

and it has already been considered in [2], [8] and [9]. For entire functions of finite order this transform coincides with the so-called "generalized Borel transformation, associated to the function $f(z)$". Its convolutional and differential properties will be discussed elsewhere.

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Centre for Mathematics and Mechanics
1090 Sofia, P. O. Box 373 Bulgaria

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