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REFINED KOLMOGOROV-CRITERIONS FOR BEST RATIONAL CHEBYSHEV-APPROXIMATION

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1. Introduction. This paper deals with refinements of the well-known local and global Kolmogorov-criterions for best rational Chebyshev-approximations. These results were derived in [2] using an optimization theoretical approach, which included also rational approximation problems with certain side-conditions. In this paper we present a slightly modified proof in the case of rational Chebyshev-approximation without side-conditions. Further we give a slight extension to the space $C_o(T)$. We begin here by recalling some definitions and earlier results. In Section 2 we prove the refined Kolmogorov-criterions, in Section 3 we apply these refinements to prove a strong uniqueness theorem.

Let S be a locally compact Hausdorff-space and let $C_o(S)$ denote the real vector space of all continuous functions $x : S \longrightarrow \mathbb{R}$ vanishing at infinity, i.e. for each $\varepsilon > 0$ the set

$$\{s \in S \mid \ |x(s)| \geq \varepsilon\}$$

is compact. We assume that $C_o(S)$ is endowed with the norm

$$\|x\|_\infty := \sup \{|x(s)| \in \mathbb{R} \,|\, s \in S\}.$$

Let $g_1, g_2, \ldots, g_\ell, h_1, h_2, \ldots, h_m \in C(S)$ be such that

$$\bigcap_{s \in S} \{\beta \in \mathbb{R}^m \mid \sum_{i=1}^{m} \beta_i h_i(s) > 0\}$$

is non-empty and define $N := m + \ell$;

$$B(s) := (g_1(s), g_2(s), \ldots, g_\ell(s), 0, 0, \ldots, 0)$$

$$C(s) := (0, 0, \ldots, 0, h_1(s), h_2(s), \ldots, h_m(s)).$$

Define the sets

$$U := \bigcap_{s \in S} \{v \in \mathbb{R}^N \mid \langle C(s), v\rangle > 0\}$$

and

$$V := \{\frac{\langle B,v\rangle}{\langle C,v\rangle} \in C(S) \mid v \in U\} \quad ,$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^N$.

An element $r_o = \dfrac{\langle B,v_o\rangle}{\langle C,v_o\rangle}$ in V is said to be a best approximation to an element x in $C_o(T)$, if

$$\|x - r_o\|_\infty = \inf_{r \in V} \|x - r\|_\infty .$$

For each x in $C_o(S)$ we denote by M_x the following set

$$M_x := \{s \in S \mid |x(s)| = \|x\|_\infty\}.$$

A best approximation to x in V can be characterized by each of the following criterions:

Theorem 1 *(Local Kolmogorov criterion).*
An element $r_o \in V$ is a best approximation to x from V if and only if

$$\forall_{v \in \mathbb{R}^N} \quad \min_{s \in M_{x-r_o}} \quad (x(s) - r_o(s))(\langle B(s) - r_o(s)C(s), v\rangle) \leq 0.$$

Theorem 2 *(Global Kolmogorov criterion)*
An element $r_o \in V$ is a best approximation to x from V if and only if

$$\forall_{r \in V} \quad \min_{s \in M_{x-r_o}} \quad (x(s) - r_o(s))(r(s) - r_o(s)) \leq 0.$$

For the refinement of these criterions, we introduce critical signatures. A signature ε on S is a continuous mapping of a non-empty closed subset of S into $\{-1,1\}$. To define critical signatures, we introduce for each $r_o \in V$ the linear space

$$L(r_o) := \{\langle r_o C - b, v\rangle \in C(S) \mid v \in \mathbb{R}^N\}.$$

Choose a basis $\varphi_1, \varphi_2, \ldots, \varphi_d$ of $\mathbb{L}(r_o)$, and define for each $s \in S$ the vector

$$G(s) := (\varphi_1(s), \varphi_2(s), \ldots, \varphi_d(s)).$$

Then, a signature ε on S is called critical (with respect to r_o in V) iff

$$0 \in \text{con}\,\{\varepsilon(s)G(s) \in \mathbb{R}^d \,|\, t \in \text{DOM}(\varepsilon)\}.$$

A critical signature is called primitive, if it does not contain properly any other critical signature. For $x \neq 0$, there is a natural signature ε_x defined by $\varepsilon_x(s)x(s) = \|x\|_\infty$.

We need the following result of [2].

Lemma 3. Let A be a non-empty bounded subset of $\mathbb{R}^N$ and $(A_\lambda)_{\lambda\in\Lambda}$ be a family of subsets of A such that $A = \cup A_\lambda$ and for each $\lambda \in \Lambda$

$$0 \in \text{con}\,(A_\lambda) \;\;\&\;\; 0 \notin \text{con}\,(\tilde{A}_\lambda) \text{ if } \tilde{A}_\lambda \subsetneq A_\lambda.$$

Then there exists a constant $K_o > 0$ such that

(a) $$\forall_{v\in\mathbb{R}^N} \inf_{w\in A} \langle v,w\rangle \leq -K_o\|v\|_2 \Psi_v ,$$

(b) $$\forall_{v\in\mathbb{R}^N} \inf_{w\in A} \langle v,w\rangle \leq -K_o\|v\|_2 \sup_{\lambda\in\Lambda} \Psi_v^\lambda ,$$

where Ψ_v^λ denotes the angle between v and $H_\lambda^\perp := (\text{span}\, A_\lambda)^\perp$, and $\|\cdot\|_2$ denotes the Euclidean norm.

II. Refined Kolmogorov criterions. In the following use the abbreviation $w := B - r_oC$, where r_o is a fixed element of V. For each signature ε we define the linear space

$$V(\varepsilon) := \bigcap_{s\in\text{DOM}(\varepsilon)} \{v \in \mathbb{R}^N \,|\, \langle w(s),v\rangle = 0\} ,$$

and for each $v \in \mathbb{R}^N$ let $\varphi_v(\varepsilon)$ denote the angle between v and $V(\varepsilon)$. We denote by Λ_o the set of all primitive critical signatures contained in ε_{x-r_o}, where $x \in C(S)\backslash V$. Further we introduce the set

$$\Gamma_o := \Gamma_{x-r_o} := \bigcup_{\varepsilon\in\Lambda_o} \text{DOM}(\varepsilon)$$

and denote the restriction of ε_{x-r_o} to Γ_o by $\tilde{\varepsilon}_o$.

Then we have the following refinement of the local Kolmogorov criterion:

<u>Theorem 4.</u> Let r_o be a best approximation to $x \in C_o(S)\setminus V$ from V. Then there exists a constand $K > 0$ such that

(a) $$\forall_{v\in \mathbb{R}^N} \quad \min_{s\in M_{x-r_o}} \quad (x(s) - r_o(s))\langle B(s) - r_o(s)C(s), v\rangle \le - K \|v\|_2 \varphi_v(\tilde{\varepsilon}_o) \quad ;$$

(b) $$\forall_{v\in \mathbb{R}^N} \quad \min_{s\in M_{x-r_o}} \quad (x(s) - r_o(s))\langle B(s) - r_o(s)C(s), v\rangle \le - K \|v\|_2 \sup_{\varepsilon\in\Lambda_o} \varphi_v(\varepsilon).$$

<u>Proof.</u> The non-empty set

$$A := \{\tilde{\varepsilon}_o(s)w(s) \in \mathbb{R}^N \mid s \in \Gamma_o\}$$

is boundet, since it is contained in the compact set

$$\{\varepsilon_{x-r_o}(s)w(s) \in \mathbb{R}^N \mid s \in M_{x-r_o}\}.$$

By definition of Λ_o we have

$$A = \bigcup_{\varepsilon\in\Lambda_o} A_\varepsilon \ ,$$

where

$$A_\varepsilon := \{\varepsilon_{x-r_o}(s)w(s) \in \mathbb{R}^N \mid \in \mathrm{DOM}(\varepsilon)\}.$$

Since ε is a primitive criterical signature, we have $0 \in \mathrm{con}\,(A_\varepsilon)$ and $0 \notin \mathrm{con}\,(F)$ for each $F \subsetneq A_\varepsilon$ (for a proof compare [2]). By Lemma 3 there exists a constant $K_o > 0$ such that

(a) $$\forall_{v\in \mathbb{R}^N} \quad \min_{s\in M_{x-v_o}} \quad (x(s) - r_o(s)\langle w(s), v\rangle$$
$$\le \inf_{s\in\Gamma_{x-v_o}} \quad (x(s) - r_o(s)\langle w(s), v\rangle$$
$$\le -K_o \|x - r_o\|_\infty \|v\|_2 \varphi_v(\tilde{\varepsilon}_o)$$
$$=: K \|v\|_2 \varphi_v(\tilde{\varepsilon}_o)$$

$$
\begin{aligned}
\text{(b)} \quad \forall_{v\in\mathbb{R}^N} \ \min_{s\in M_{x-v_o}} \ & (x(s) - r_o(s))\langle B(s) - r_o(s)C(s), v\rangle \\
& \le \inf_{s\in\Gamma_{x-v_o}} \|x - r_o\|_\infty \varepsilon_{x-r_o}(s)\langle w(s), v\rangle \\
& \le - K_o\|x - r_o\|_\infty \|v\|_2 \sup_{\varepsilon\in\Lambda_o} \varphi v(\varepsilon) \\
& =: - K\|v\|_2 \sup_{\varepsilon\in\Lambda_o} \varphi v(\varepsilon).
\end{aligned}
$$

□

With the aid of Theorem 4 we derive the following refinement of the global criterion:

Theorem 5. Let r_o be a best approximation for $x \in C_o(S)\backslash V$ from V. Then there exists a constant $K_1 > 0$ such that

$$
\text{(a)} \quad \forall_{r\in V} \ \min_{s\in M_{x-v_o}} (x(s) - r_o(s))(r(s) - r_o(s)) \le - K_1\varphi_v(\tilde{\varepsilon}_o) ;
$$

$$
\text{(b)} \quad \forall_{r\in V} \ \min_{s\in M_{x-v_o}} (x(s) - r_o(s))(r(s) - r_o(s)) \le -K_1 \sup_{\varepsilon\in\Lambda_o} \varphi_v(\varepsilon) ,
$$

where $v \in U$ is such that $r = \frac{\langle B,v\rangle}{\langle C,v\rangle}$.

Proof. Let $\tilde{s} \in M_{x-v_o}$ be such that

$$
(x(\tilde{s}) - r_o(\tilde{s})) \ \langle w(\tilde{s}),v\rangle = \min_{s\in M_{x-v_o}} (x(s) - r_o(s))\langle B(s) - r_o(s)C(s),v\rangle.
$$

Then, by using Theorem 4 we have

$$
\begin{aligned}
\min_{s\in M_{x-v_o}} & (x(s) - r_o(s))(r(s) - r_o(s)) \\
& = \min \frac{(x(s) - r_o(s))\langle B(s) - r_o(s)C(s),v\rangle}{\langle C(s),v\rangle} \\
& \le \frac{(x(\tilde{s}) - r_o(\tilde{s}))\langle B(\tilde{s}) - r_o(\tilde{s})C(\tilde{s}),v\rangle}{\langle C(\tilde{s}),v\rangle} \\
& \le - \frac{K\varphi_v(\tilde{\varepsilon}_o)}{\|C\|_\infty\|v\|} =: - K_1\varphi_v(\tilde{\varepsilon}_o) ,
\end{aligned}
$$

which proves (a).

Since $V(\tilde{\varepsilon}_o) \subset V(\varepsilon)$ for each $\varepsilon \in \Lambda_o$, wa have $\varphi_V(\tilde{\varepsilon}_o) \geq \varphi_V(\varepsilon)$, which implies (b). □

In the case of linear problems the refined Kolmogorov criterion can be stated in a more simplified way. Consider the particular situation

$$B(s) := (g_1(s), g_2(s), \ldots, g_\ell(s), 0) ,$$

$$C(s) := (0, 0, \ldots, 0, 1) ,$$

where $g_1, g_2, \ldots, g_l$ are linearly independent functions of $C(S)$.

For any signature ε we introduce the linear subspaces

$$V_L(\varepsilon) := \bigcap_{s \in DOM(\varepsilon)} \{b \in \mathbb{R}^\ell \mid \sum_{i=1}^{\ell} b_i g_i(s) = 0\}$$

and

$$V_R(\varepsilon) := \bigcap_{s \in DOM(\varepsilon)} \{v \in \mathbb{R}^{\ell+1} \mid \langle B(s), v\rangle = 0\}.$$

Let $I : \mathbb{R}^\ell \longrightarrow \mathbb{R}^{\ell+1}$ be the injection defined by

$$\forall_{b \in \mathbb{R}^\ell} \quad I(b) := (b, 0) ,$$

and let $P_R : \mathbb{R}^{\ell+1} \longrightarrow V_R(\varepsilon)$ and $P_L : \mathbb{R}^\ell \longrightarrow V_L(\varepsilon)$ be the projections associated with the spaces $V_R(\varepsilon)$ and $V_L(\varepsilon)$ respectively. Then we have

$$P_R \circ I = I \circ P_L.$$

<u>Theorem 6</u> *(Refined linear Kolmogorov criterion).*
Let g_o be a best approximation to $x \in C_o(S) \setminus V$ from V. Then there exists a real number $K_2 > 0$ such that

(a) $$\forall_{g \in V} \quad \min_{s \in M_{x-g_o}} (x(s) - g_o(s) g(s) \leq - K_2 \|g\|_\infty \cdot \theta_g(\varepsilon_o)$$

(b) $$\forall_{g \in V} \quad \min_{s \in M_{x-g_o}} (x(s) - g_o(s)) g(s) \leq - K_2 \|g\|_\infty \cdot \sup_{\varepsilon \in \Lambda_o} \theta_g(\varepsilon) ,$$

where $\theta_g(\varepsilon)$ denotes the angle between $V_L(\varepsilon)$ and b, $g = \sum_{i=1}^{\ell} b_i g_i$,

<u>Proof.</u> We can assume $g_o = 0$. Let $g = \sum_{i=1}^{l} b_i g_i$ be given. By using Theorem 4 with $v = I(b) + e_{\ell+1}$ we have for a suitable $K_3 > 0$

$$\min_{s\in M_{x-g_o}} (x(s) - g_o(s))g(s)$$

$$\leq - K_3 \|I(b) + e_{\ell+1}\|_2 \sin \varphi_v(\tilde{\varepsilon}_o)$$

$$= - K_3 \|I(b) + e_{\ell+1} - P_R(I(b) + e_{\ell+1}\|_2$$

$$= - K_3 \|I(b) - P_R \circ I(b)\|_2$$

$$= - K_3 \|I(b) - I \circ P_L(b)\|_2$$

$$= - K_3 \|b - P_L(b)\|_2$$

$$= - K_3 \|b\|_2 \sin \theta_g(\tilde{\varepsilon}_o) \leq - K_2\|g\|_\infty \theta_g(\tilde{\varepsilon}) ,$$

which proves (a).

Statement (b) follows from (a) by using the fact $\theta_g(\varepsilon_o) \geq \theta_g(\varepsilon)$ for each $\varepsilon \in \Lambda_o$. □

Remark. Theorem 6 improves a result of Brosowski[1], who derived a weaker estimate with $\theta_g^2(\varepsilon)$ instead of $\theta_g(\varepsilon)$ for the case of a finite Λ_o.

III. Strong uniqueness. For each $r_o \in V$ we define the linear subspace

$$H_o := \bigcap_{s\in S} \{ v \in \mathbb{R}^N | \langle B(s) - r_o(s)C(s), v\rangle = 0 \} ,$$

and for each $v \in \mathbb{R}^N$ let φ_v be the angle between v and H_o. The linear subspace H_o has dimension $N - d$, where $d := \dim \lfloor(r_o)$.

An element r_o in V is called a strongly unique best approximation of the function x in $C_o(S)$ from V, if there exists a constant $K_3 := K_3(x) > 0$ such that for all $r = \frac{\langle B,v\rangle}{\langle C,v\rangle}$ in V one has

$$(*) \qquad \|x - r\|_\infty \geq \|x - r_o\|_\infty + K_3 \varphi_v$$

In the normal case (i.e. $\dim H_o = 1$) the condition (*) is equivalent to the usual definition of strong unicity, i.e.

$$\|x - r\|_\infty \geq \|x - r_o\|_\infty + K_4\|r - r_o\|_\infty .$$

For a proof and a discussion of earlier results compare the paper [2].

Theorem 7. An element r_o in V is a strongly unique best approximation to x in $C_o(S)\setminus V$ from V if and only if there exists points $s_1,s_2,\dots,s_d$ in Γ_o such that the vectors

$$B(s_i) - r_o(s_i)C(s_i) \in \mathbb{R}^N ,$$

$i = 1,2,\dots,d$ are linearly independent.

Proof. (1) Assume that there exist points $s_1,s_2,\dots,s_d$ in Γ_o such that the vectors

$$B(s_i) - r_o(s_i)C(s_i) ,$$

$i = 1,2,\dots,d$ are linearly independent. We show that $H_o = V(\tilde{\varepsilon}_o)$. The inclusion $H_o \subset V(\tilde{\varepsilon}_o)$ is clear. On the other hand there exist signatures $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_k$ in Λ_o such that

$$\{s_1,s_2,\dots,s_d\} \subset \bigcup_{i=1}^{k} DOM(\varepsilon_i).$$

The linear subspace

$$H^{\ddagger} := \bigcap_{i=1}^{d} \{v \in \mathbb{R}^N \mid \langle B(s_i) - r_o(s_i)C(s_i),v\rangle = 0\}$$

has dimension N - d and contains $V(\tilde{\varepsilon}_o)$. Thus we have

$$H_o \subset V(\tilde{\varepsilon}_o) \subset H^{\ddagger} .$$

Since dim $H_o = N - d$, we have

$$H_o = V(\tilde{\varepsilon}_o) = H^{\ddagger}.$$

Consequently, we have $\varphi_v = \varphi_v(\tilde{\varepsilon}_o)$ for each $v \in \mathbb{R}^N$. By Theorem 5 there exists a constant $K_1 > 0$ and a point $s \in M_{x-r_o}$ such that

$$(x(s) - r_o(s))(r(s) - r_o(s)) \leq - K_1\varphi_v(\tilde{\varepsilon}_o) = - K_1\varphi_v.$$

Then we have

$$\begin{aligned}\|x - r\|_\infty &\geq \varepsilon_{x-r_o}(s)(x(s) - r(s))\\ &= \varepsilon_{x-r_o}(s)(x(s) - r_o(s)) - \varepsilon_{x-r_o}(s)(r(s) - r_o(s))\\ &\geq \|x - r_o\|_\infty + \frac{K_1\varphi_v}{\|x - r_o\|_\infty}\\ &=: \|x - r_o\|_\infty + K_3\varphi_v.\end{aligned}$$

(2) To prove the necessity of the condition we use the following

Lemma 8. Let ε_o be a critical signature (with respect to r_o in V) on S; define the sets

$$\Gamma_o := \cup\{DOM(\varepsilon) \subset S \mid \varepsilon \subset \varepsilon_o \ \& \ \varepsilon \text{ primitive critical}\}$$

and

$$S_1 := \text{span}\{B(s) - r_o(s)C(s) \in \mathbb{R}^N \mid s \in \Gamma_o\}.$$

If $d_1 := \dim S_1 < d := \dim \mathsf{L}(r_o)$, then there exists an element

$$v \in S_1^{\perp} \cap H_o^{\perp}, v \neq 0 ,$$

such that

$$(*) \qquad \underset{s \in DOM(\varepsilon_o)}{\forall} \quad \varepsilon_o(s) \quad \langle r_o(s)C(s) - B(s), v\rangle \leq 0.$$

The proof of the lemma is similar to the first part of the proof (b) $\Longrightarrow$ (a) of Theorem 4.1 in [2].

Assume now that r_o is a strongly unique best approximation to x in $C_o(S)\backslash V$. Consider

$$S_1 := \text{span}\{r_o(s)C(s) - B(s) \in \mathbb{R}^N \mid s \in \Gamma_o\} ,$$

let $d_1 := \dim S_1$ and assume by contradiction $d_1 < d$. By Lemma 8 there exists an element

$$v \in S_1^{\perp} \cap H_o^{\perp}, v \neq 0 ,$$

which satisfies condition (*) of Lemma 8 with $\varepsilon_o = \varepsilon_{x-r_o}$.

Define a sequence of positive real numbers (τ_n) such that $\tau_n < 1$, and $\tau_n \longrightarrow 0$ for $n \longrightarrow \infty$, and

$$v_n := (1-\tau_n)v_o + \tau_n v$$

belongs to U. Since $v_o \in H_o$ and $v \in H_o^{\perp}$, we have

$$\sin\varphi_n = \frac{\|v_n - Pv_n\|_2}{\|v_n\|_2} = \frac{\tau_n\|v\|_2}{\|v_n\|_2}$$

$$= \frac{\tau_n\|v\|_2}{\|(1-\tau_n)v_o + \tau_n v\|_2} \geq K_o\tau_n ,$$

with a suitable constant $K_o > 0$, $\varphi_n := \varphi_{v_n}$, and where P denotes the projection associated with H_o. Of course we have also $\varphi_n \geqq K\tau_n$ with a suitable $K > 0$.

For each $n \in \mathbb{N}$ choose a point $s_n \in S$ and a sign $\eta_n \in \{-1,1\}$ such that

$$\|x - r_n\|_\infty = \eta_n(x(s_n) - r_n(s_n)) ,$$

where $r_n := \frac{\langle B,v_n\rangle}{\langle C,v_n\rangle}$.

There is an infinite subset $N_o \subset \mathbb{N}$ such that either

$$\{(\eta_n,s_n) \in \{-1,1\} \times S \mid n \in N_o\}$$

consists od a single point, say $(\overline{\eta},\overline{s})$, or, by compactness of $\{-1,1\} \times S$,

$$\{(\eta_n,s_n) \in \{-1,1\} \times S \mid n \in N_o\}$$

has an accumulation point $(\overline{\eta},\overline{s})$in $\{-1,1\} \times S$. By hypothesis, we have with a suitable constant $K_3 > 0$ and for all $n \in N_o$ the inequality

$$\begin{aligned} 0 < K_3K\tau_n &\leq K_3\varphi_n \\ &\leqq \|x - r_n\|_\infty - \|x - r_o\|_\infty \\ &\leq \eta_n(x(s_n) - r_n(s_n)) - \eta_n(x(s_n) - r_o(s_n)) \\ &= \eta_n(r_o(s_n) - r_n(s_n)) \\ &= \frac{\tau_n\eta_n\langle r_o(s_n)C(s_n) - B(s_n),v\rangle}{\langle C(s_n),v_n\rangle} , \end{aligned}$$

which implies

$$0 < K_3K \quad \frac{\eta_n\langle r_o(s_n)C(s_n) - B(s_n),v\rangle}{\langle C(s_n),v_n\rangle} .$$

By continuity and since $\overline{\eta}(x(\overline{s}) - r_o(\overline{s})) = \| x - r_o\|_\infty$, we have

$$0 < K_3K \leq \frac{\overline{\eta}\langle r_o(\overline{s})C(\overline{s}) - B(\overline{s}),v\rangle}{\langle C(\overline{s}),v_o\rangle} \leq 0 ,$$

a contradition. □

Literature

[1] B.Brosowski. A refinement of the Kolmogorov-criterion, in: Constructive Function Theory, '81 ,p. 241 - 247, Publishing House of the Bulgarian Academy of Sciences, Sofia, 1983.

[2] B.Brosowski and C.Guerreiro. An extension of strong uniqueness to rational approximation. To appear.

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