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A Pompeiu-Type Mean-Value Theorem

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We present a generalization of a Pompeiu-type mean-value theorem.

1. Introduction and Preliminary Results

In 1946, Pompeiu [5] derived a variant of the Lagrange mean-value theorem, now known as *Pompeiu's mean-value theorem* (see also [12, p. 83])

Theorem 1 (Pompeiu, 1946). *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, differentiable on (a, b) and $0 \notin [a, b]$. Then there exists a point $c \in (a, b)$ such that*

$$\frac{a\, f(b) - b\, f(a)}{a - b} = f(c) - c\, f'(c).$$

A geometric interpretation of Theorem 1 is given in Figure 1.

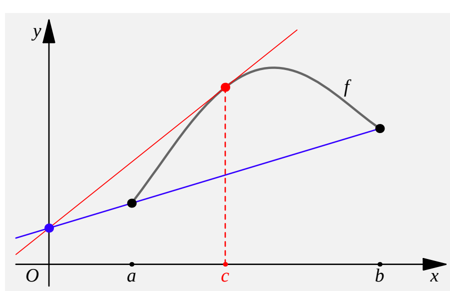

Figure 1. The graph of the Taylor polynomial $T_1(f; c)$, the graph of the Lagrange interpolating polynomial $L_1[a, b; f]$ and the Oy axis intersect at the same point.

Another Pompeiu-type mean-value theorem is given in [6].

Theorem 2 (Ivan, 1970). *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If f has no zeros in (a,b) and $f(a) \neq f(b)$, then there exists a point $c \in (a,b)$ such that*

$$\frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = c - \frac{f(c)}{f'(c)}.$$

We point out that f' may have zeros in (a,b). A geometric interpretation of Theorem 2 is given in Figure 2.

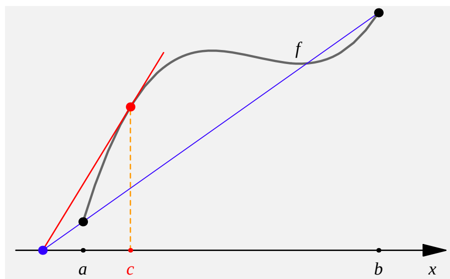

Figure 2. The graph of the Taylor polynomial $T_1(f;c)$, the graph of the Lagrange interpolating polynomial $L_1[a,b;f]$ and the Ox axis intersect at the same point.

In what follows we consider the points $a \le x_0 < \cdots < x_n \le b$, $n \ge 1$. Let $m \in \mathbb{N}$ and $\alpha_0, \ldots, \alpha_n \in \mathbb{N}^*$ be such that $\alpha_0 + \cdots + \alpha_n = m+1$. Denote by $\mathcal{D}_m$ the set of all functions possessing a derivative of order m on $[a,b]$ and let $f \in \mathcal{D}_m$. Hermite [2] proved that there exists a unique polynomial H_m of degree at most m, called nowadays the *Hermite interpolating polynomial*, such that

$$H_m^{(j)}(x_i) = f^{(j)}(x_i), \qquad i = 0, \ldots, n, \quad j = 0, \ldots, \alpha_i - 1. \tag{1}$$

We denote the polynomial (1) by

$$H[f] = H_m[f] = H_m[\underbrace{x_0, \ldots, x_0}_{\alpha_0 - \text{times}}, \ldots, \underbrace{x_n, \ldots, x_n}_{\alpha_n - \text{times}}; f] = H_m\left[\begin{smallmatrix} x_0 \\ \alpha_0 \end{smallmatrix} \cdots \begin{smallmatrix} x_n \\ \alpha_n \end{smallmatrix}; f\right].$$

With $\ell(x) := (x - x_0)^{\alpha_0} \ldots (x - x_n)^{\alpha_n}$, the interpolating polynomial (1) can be written in the form

$$H_m(x) = \sum_{i=0}^{n} \sum_{j=0}^{\alpha_i - 1} \sum_{k=0}^{\alpha_i - j - 1} f^{(j)}(x_i)\, \frac{1}{k!\,j!} \left(\frac{(x - x_i)^{\alpha_i}}{\ell(x)} \right)^{(k)}_{x = x_i} \frac{\ell(x)}{(x - x_i)^{\alpha_i - j - k}}.$$

For $\alpha_0 = \cdots = \alpha_n = 1$, we get the Lagrange polynomial,

$$H_m\left[\begin{smallmatrix} x_0 \\ 1 \end{smallmatrix} \cdots \begin{smallmatrix} x_n \\ 1 \end{smallmatrix}; f\right] = L[x_0, \ldots, x_n; f].$$

For $\alpha_0 = n$, $\alpha_1 = \cdots = \alpha_n = 0$ and $x_0 = c$, we obtain the Taylor polynomial of degree n associated to f at c,

$$H_n\,[{}^{\,c}_{n+1}; f] = T_n[f;c], \qquad T_n[f;c](x) := \sum_{i=0}^{n} \frac{f^{(i)}(c)}{i!}\,(x-c)^i.$$

Definition 1 (Popoviciu). The divided difference for coalescing points

$$[\underbrace{x_0,\dots,x_0}_{\alpha_0-\text{times}},\dots,\underbrace{x_n,\dots,x_n}_{\alpha_n-\text{times}}; f]$$

is defined to be the coefficient at x^m of the Hermite polynomial satisfying (1) (see, e.g., [1, p. 120]).

Nuernberger [3, p. 17] emphasises that the previous definition of divided differences in the case of coalescing points using quotient of determinants was firstly given by Popoviciu [4].

For $\alpha_0 = \cdots = \alpha_n = 1$, Szasz* [11] obtained the following theorem.

Theorem 3 (Szasz, 1961). *If f is continuous on $[a,b]$ and possesses a derivative on (a,b), then there exist points $\xi_0,\dots,\xi_{n-1}$, such that*

$$L'[x_0,\dots,x_n; f] = L[\xi_0,\dots,\xi_{n-1}; f']. \tag{2}$$

A natural generalization of Theorem 3 in the case of multiple knots is the following:

Theorem 4. *If $f \in \mathcal{D}_m$, then there exist points $\xi_0,\dots,\xi_{m-1} \in [a,b]$ such that*

$$H'[x_0,\dots,x_n; f] = H[\xi_0,\dots,\xi_{m-1}; f']. \tag{3}$$

Proof. Consider the auxiliary function $g\colon [a,b] \to \mathbb{R}$,

$$g(t) = H[x_0,\dots,x_n; f](t) - f(t).$$

The function g satisfies the equalities:

$$g^{(j)}(x_i) = 0, \qquad i = 0,\dots,n, \quad j = 0,\dots,\alpha_i - 1.$$

By Rolle's Theorem, there exist points $\eta_i \in (x_{i-1}, x_i)$, $i = 1,\dots,n$, such that $g'(\eta_i) = 0$. The polynomial $H'[x_0,\dots,x_m; f]$ has degree at most $m-1$ and satisfies the equalities

$$\big(H'[x_0,\dots,x_m; f]\big)^{(j-1)}(x_i) = (f')^{(j-1)}(x_i), \qquad i = 0,\dots,n, \quad 1 \le j \le \alpha_i - 1,$$
$$H'[x_0,\dots,x_m; f](\eta_i) = f'(\eta_i), \qquad \eta_i \in (x_{i-1}, x_i), \quad i = 1,\dots,n.$$

*Carol Szasz, born 1940 in Romania.

Since the Hermite interpolation polynomial is unique, we obtain:

$$H'[x_0,\dots,x_n;f] = H\left[{}^{x_0}_{\alpha_0-1}\,{}^{\eta_1}_{1}\,{}^{x_1}_{\alpha_1-1}\cdots{}^{\eta_n}_{1}\,{}^{x_n}_{\alpha_n-1};f'\right].$$

With the re-notation

$$(\xi_0,\dots,\xi_{m-1}) := (\underbrace{x_0,\dots,x_0}_{(\alpha_0-1)-\text{times}},\eta_1,\underbrace{x_1,\dots,x_1}_{(\alpha_1-1)-\text{times}},\dots,\eta_n,\underbrace{x_n,\dots,x_n}_{(\alpha_n-1)-\text{times}})$$

the theorem is proved. □

Let $k \in \{0,\dots,m\}$. Applying Theorem 4 we get:

Theorem 5. *Let $f \in \mathcal{D}_m$. Then there exist points $\xi_0,\dots,\xi_{m-k} \in [a,b]$ such that*

$$H_m^{(k)}[x_0,\dots,x_n;f] = H_m[\xi_0,\dots,\xi_{m-k};f^{(k)}].$$

For $k = m$ and $x_0 = c$, Theorem 5 gives

$$m!\,[x_0,\dots,x_n;f] = H_m^{(m)}[x_0,\dots,x_n;f] = H_m[\xi_0;f^{(m)}] = f^{(m)}(c),$$

a well-known extension of Lagrange's Mean-Value Theorem to the case of divided differences (see also [9, p. 36]):

Proposition 1. *If $f \in \mathcal{D}_m$, then there exists $c \in (a,b)$ such that*

$$[x_0,\dots,x_n;f] = \frac{f^{(m)}(c)}{m!}.$$

Among the many other extensions of the Pompeiu's Theorem we focus on that of Stamate [10].

Theorem 6 (Stamate, 1958). *If $f \in \mathcal{D}_n[a,b]$ and $0 \notin [a,b]$, then there exists a point $c \in (a,b)$ such that*

$$\sum_{i=0}^{n} f(x_i) \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{x_j}{x_j - x_i} = \sum_{i=0}^{n} (-1)^i c^i\, \frac{f^{(i)}(c)}{i!}.$$

Stamate proved his theorem by applying Proposition 1 to $\varphi(t) = t^n f(1/t)$ for $t_i = 1/x_i$, $i = 0,1,\dots,n$. Since

$$L[x_0,\dots,x_n;f](0) = \sum_{i=0}^{n} f(x_i) \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{x_j}{x_j - x_i}, \qquad T_n[f;c](0) := \sum_{i=0}^{n} \frac{f^{(i)}(c)}{i!}(-c)^i,$$

we can reformulate Stamate's Theorem 6 in the following form:

Theorem 7. *If $f \in \mathcal{D}_n[a,b]$ and $0 \notin [a,b]$, then there exists a point $c \in (a,b)$ such that*

$$L_n[x_0,\dots,x_n;f](0) = T_n[f;c](0).$$

In what follows we need the following identity for the Hermite polynomials:

Theorem 8 (see, e.g., [8]). *With the notation*

$$(t_0,\dots,t_m) := (\underbrace{x_0,\dots,x_0}_{\alpha_0-\text{times}},\dots,\underbrace{x_n,\dots,x_n}_{\alpha_n-\text{times}}),$$

the following formula

$$H_m[t_0,\dots,t_m;(x-t_0)\cdots(x-t_i)\,f] = (x-t_0)\cdots(x-t_i)\,H_m[t_{i+1},\dots,t_m;f] \quad (4)$$

holds true for $i = 0,1,\dots,m-1$.

The following reduction formula for divided differences will be used:

Proposition 2. *Let $i \in \{0,1,\dots,m-1\}$, then*

$$[t_0,\dots,t_m;(x-t_0)\cdots(x-t_i)f] = [t_{i+1},\dots,t_m;f]. \quad (5)$$

Proof. We simply identify the coefficients of x^m in (4). □

Proposition 3. *Let $u \in \mathbb{R}\setminus[a,b]$. The following formula is holds true:*

$$[u,x_0,\dots,x_n;f] = [x_0,\dots,x_n;[\,\cdot\,,u]f].$$

Proof. Since

$$f(t) = f(u) + (t-u)[t,u;f], \qquad t \in [a,b],$$

using (5), we obtain

$$[u,x_0,\dots,x_n;f] = 0 + [u,x_0,\dots,x_n;(t-u)[t,u;f]]\big|_t = [x_0,\dots,x_n;[t,u;f]]\big|_t.$$

□

We note that, by using (2) and Stamate's Theorem 6, Szasz obtained mean-value type formulas for all the coefficients of the Lagrange polynomial.

Problem 1. *Let $f \in \mathcal{D}_m[a,b]$ and $u \in \mathbb{R}\setminus[a,b]$. Find $v \in \mathbb{R}$ such that there exists $c \in (a,b)$ with $v = T_m[f;c](u)$.*

Of course, for some v, Problem 1 has no solution (see Figure 3).

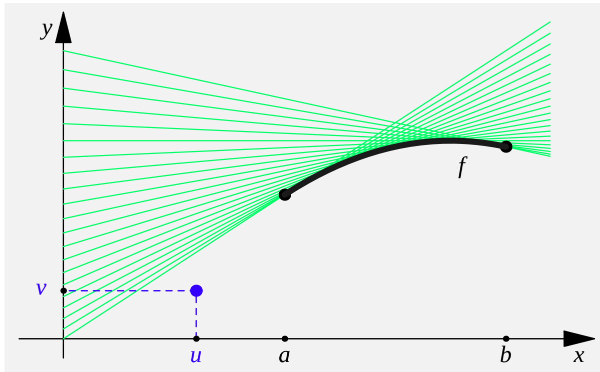

Figure 3. There exists no $c \in (a,b)$ such that $T_1[f;c](u) = v$.

2. Main Results

Let $k \in \{0,\ldots,m\}$. The following is a generalization of Stamate's Theorem 6.

Theorem 9. *If* $f \in \mathcal{D}_m[a,b]$ *and* $u \in \mathbb{R} \setminus [a,b]$, *then there exists a point* $c \in (a,b)$ *such that*

$$T_{m-k}[f^{(k)};c](u) = \big(H_m[x_0,\ldots,x_n;f]\big)^{(k)}(u).$$

Proof. For demonstration purpose, instead of $H_m[f]$, we will use the notation $H[f(t)]$ emphasizing that the variable of the function f is t.

Let $\xi_0,\ldots,\xi_{m-k}$ be given by Theorem 5. Since the degree of the polynomial $\big(H_m[x_0,\ldots,x_n;f]\big)^{(k)}(u)$ is at most $m-k$, we get:

$$\begin{aligned}
0 &= \Big[u,\xi_0,\ldots,\xi_{m-k};\big(H_m[x_0,\ldots,x_n;f]\big)^{(k)}\Big]\\
&= \left[\xi_0,\ldots,\xi_{m-k};\frac{\big(H_m[x_0,\ldots,x_n;f]\big)^{(k)}(t)-\big(H_m[x_0,\ldots,x_n;f]\big)^{(k)}(u)}{t-u}\right]_t\\
&= \left[\xi_0,\ldots,\xi_{m-k};\frac{H_m[\xi_0,\ldots,\xi_{m-k};f^{(k)}](t)-\big(H_m[x_0,\ldots,x_n;f]\big)^{(k)}(u)}{t-u}\right]_t\\
&= \left[\xi_0,\ldots,\xi_{m-k};\frac{f^{(k)}(t)-\big(H_m[x_0,\ldots,x_n;f]\big)^{(k)}(u)}{t-u}\right]_t.
\end{aligned}$$

By Proposition 1, there exists $c \in (a,b)$ such that

$$\frac{1}{(m-k)!}\left(\frac{\mathrm{d}}{\mathrm{dt}}\right)^{m-k}\left(\frac{f^{(k)}(t)-\big(H_m[x_0,\ldots,x_n;f]\big)^{(k)}(u)}{t-u}\right)\Bigg|_{t=c} = 0.$$

Application of Leibniz's derivation rule yields

$$\sum_{i=0}^{m-k}\binom{m-k}{i}\left(f^{(k)}(t)-\left(H_m[x_0,\dots,x_n;f]\right)^{(k)}(u)\right)^{(i)}\left(\frac{1}{t-u}\right)^{(m-k-i)}\Bigg|_{t=c}=0,$$

and hence

$$\left(H_m[x_0,\dots,x_n;f]\right)^{(k)}(u)=\sum_{i=0}^{m-k}\frac{f^{(k+i)}(c)}{i!}(u-c)^i=T_{m-k}[f^{(k)};c](u). \quad \square$$

The following consequence of Theorem 9 gives a sufficient condition related to Problem 1 (see Figure 4).

Corollary 1. *If $f\in\mathcal{D}_m[a,b]$, then there exists $c\in(a,b)$ such that*

$$H_m[x_0,\dots,x_n;f](u)=T_m[f;c](u).$$

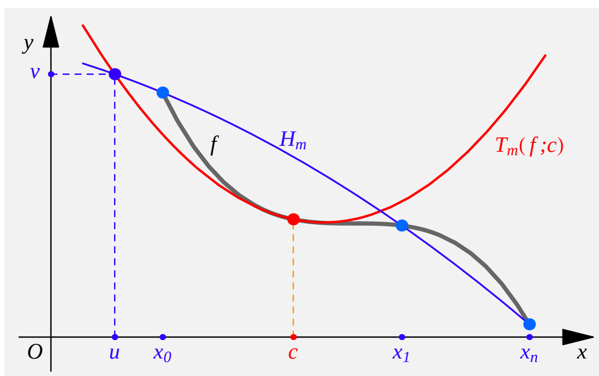

Figure 4. For $v=H_m[x_0,\dots,x_n;f](u)$ there exists $c\in(a,b)$ such that $T_m[f,c](u)=v$.

Note that, in the special case of $f=H_m[x_0,\dots,x_n;g]$, since f is a polynomial of degree at most n, we obtain:

$$T_m[f,x]=f, \qquad H_m[x_0,\dots,x_n;f](u)=f(u)=T_m[f,x](u), \qquad \forall x\in(a,b).$$

Remark 1. The following results are particular cases of Theorem 9:

- Theorem 1 of Pompeiu ($n=1$, $k=0$, $x_0=a$, $x_1=b$, $u=0$);
- Theorem 2 ($n=1$, $k=0$, $x_0=a$, $x_1=b$, $u=\frac{b\,f(a)-a\,f(b)}{f(a)-f(b)}$);
- Theorem 7 of Stamate ($k=0$, $u=0$);
- The mean value formulas for the coefficients of the Lagrange polynomial $L_n[x_0,\dots,x_n;f]$ obtained by C. Szasz ($u=0$).

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