CONSTRUCTIVE THEORY OF FUNCTIONS, Sozopol 2016
(K. Ivanov, G. Nikolov and R. Uluchev, Eds.), pp. 13-27
Prof. Marin Drinov Academic Publishing House, Sofia, 2018

Self Adjoint Operator Korovkin Type and Polynomial Direct Approximations with Rates

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Here we present self adjoint operator Korovkin type theorems, via self adjoint operator Shisha-Mond type inequalities, also we give self adjoint operator polynomial approximations. This is a quantitative treatment to determine the degree of self adjoint operator uniform approximation with rates, of sequences of self adjoint operator positive linear operators. The same kind of work is performed over important operator polynomial sequences. Our approach is direct based on Gelfand isometry.

Keywords and Phrases: Self adjoint operator, Hilbert space, Korovkin theory, Shisha-Mond inequality, positive linear operator, Bernstein type polynomials.

Mathematics Subject Classification 2010: 41A17, 41A25, 41A36, 41A80, 47A58, 47A60, 47A67.

1. Background

Let A be a self adjoint linear operator on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. The Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous functions defined on the spectrum of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see e.g. [6, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have:

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ (the operation composition is on the right) and $\Phi(\overline{f}) = (\Phi(f))^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup\limits_{t \in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \quad \text{for all} \quad f \in C(Sp(A)),$$

and we call it the continuous functional calculus for a self adjoint operator A.

If A is a self adjoint operator and f is a real valued continuous function on $Sp(A)$ then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued continuous functions on $Sp(A)$ then the following important property holds:

(P) *$f(t) \geq g(t)$ for any $t \in Sp(A)$, implies that $f(A) \geq g(A)$ in the operator order of $B(H)$.*

Equivalently, we use (see [5, pp. 7–8]).

Let U be a self adjoint operator on the complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family.

Then for any continuous function $f : [a, b] \to \mathbb{C}$, where $[m, M] \subset (a, b)$, it is well known that we have the following spectral representation in terms of the Riemann-Stieljes integral:

$$\langle f(U)x, y \rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, y \rangle),$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of bounded variation on the interval $[m, M]$, and

$$g_{x,y}(m-0) = 0 \quad \text{and} \quad g_{x,y}(M) = \langle x, y \rangle,$$

for any $x, y \in H$. Furthermore, it is known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is increasing and right continuous on $[m, M]$.

In this article we will be using a lot the formula

$$\langle f(U)x, x \rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, x \rangle), \quad \forall x \in H.$$

As a symbol we can write

$$f(U) = \int_{m-0}^{M} f(\lambda)\, dE_\lambda.$$

Above, $m = \min\{\lambda \mid \lambda \in Sp(U)\} := \min Sp(U)$, $M = \max\{\lambda \mid \lambda \in Sp(U)\} := \max Sp(U)$. The projections $\{E_\lambda\}_{\lambda \in \mathbb{R}}$, are called the spectral family of A, with the properties:

(a) $E_\lambda \leq E_{\lambda'}$ for $\lambda \leq \lambda'$;

(b) $E_{m-0} = 0_H$ (zero operator), $E_M = 1_H$ (identity operator) and $E_{\lambda+0} = E_\lambda$ for all $\lambda \in \mathbb{R}$.

Furthermore

$$E_\lambda := \varphi_\lambda(U), \quad \forall \lambda \in \mathbb{R},$$

is a projection which reduces U, with

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \leq \lambda, \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

The spectral family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ determines uniquely the self adjoint operator U and vice versa.

For more on the topic see [9, pp. 256–266], and for more detalis see there pp. 157-266. See also [4].

Some more basics are given (we follow [5, pp. 1–5]).

Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{C}$. A bounded linear operator A defined on H is selfjoint, i.e., $A = A^*$, iff $\langle Ax, x \rangle \in \mathbb{R}$, $\forall x \in H$. If A is self adjoint, then

$$\|A\| = \sup_{x \in H, \; \|x\|=1} |\langle Ax, x \rangle|.$$

Let A, B be self adjoint operators on H. Then $A \leq B$ iff $\langle Ax, x \rangle \leq \langle Bx, x \rangle$, $\forall x \in H$. In particular, A is called positive if $A \geq 0$.

Denote by

$$\mathcal{P} := \Big\{ \varphi(s) := \sum_{k=0}^{n} \alpha_k s^k \,|\, n \geq 0, \; \alpha_k \in \mathbb{C}, \; 0 \leq k \leq n \Big\}.$$

If $A \in \mathcal{B}(H)$ (the Banach algebra of all bounded linear operators defined on H, i.e. from H into itself) is self adjoint, and $\varphi(s) \in \mathcal{P}$ has real coefficients, then $\varphi(A)$ is self adjoint, and

$$\|\varphi(A)\| = \max_{\lambda \in Sp(A)} |\varphi(\lambda)|.$$

If φ is any function defined on $\mathbb{R}$ we define

$$\|\varphi\|_A := \sup_{\lambda \in Sp(A)} |\varphi(\lambda)|.$$

If A is self adjoint operator on Hilbert space H and φ is continuous and given that $\varphi(A)$ is self adjoint, then $\|\varphi(A)\| = \|\varphi\|_A$. And if φ is a continuous real valued function so it is $|\varphi|$, then $\varphi(A)$ and $|\varphi|(A) = |\varphi(A)|$ are self adjoint operators (by [5, p. 4, Theorem 7]).

Hence it holds

$$\||\varphi(A)|\| = \||\varphi|\|_A = \sup_{\lambda \in Sp(A)} \||\varphi(\lambda)\|| = \sup_{\lambda \in Sp(A)} |\varphi(\lambda)| = \|\varphi\|_A = \|\varphi(A)\|,$$

that is $\||\varphi(A)|\| = \|\varphi(A)\|$.

For a self adjoint operator $A \in \mathcal{B}(H)$ which is positive, there exists a unique positive self adjoint operator $B := \sqrt{A} \in \mathcal{B}(H)$ such that $B^2 = A$, that is $\left(\sqrt{A}\right)^2 = A$. We call B the square root of A.

Let $A \in \mathcal{B}(H)$, then A^*A is self adjoint and positive. Define the "operator absolute value" $|A| := \sqrt{A^*A}$. If $A = A^*$, then $|A| = \sqrt{A^2}$.

For a continuous real valued function φ we observe the following:

$$|\varphi|(A) \ \text{ (the functional absolute value) } = \int_{m-0}^{M} |\varphi(\lambda)|\, dE_\lambda$$
$$= \int_{m-0}^{M} \sqrt{\varphi^2(\lambda)}\, dE_\lambda = \sqrt{\varphi^2(A)} = |\varphi(A)| \ \text{ (operator absolute value),}$$

where A is a self adjoint operator.

That is we have

$$|\varphi(A)| \ \text{ (functional absolute value) } = |\varphi(A)| \ \text{ (operator absolute value).}$$

The next comes from [4, p. 3].

We say that a sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{B}(H)$ converges uniformly to A (convergence in norm), iff

$$\lim_{n\to\infty} \|A_n - A\| = 0,$$

and we denote it as $\lim\limits_{n\to\infty} A_n = A$.

We will be using Hölder's-McCarthy inequality [10]: Let A be a self adjoint positive operator on a Hilbert space H. Then

$$\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r,$$

for all $0 < r < 1$ and $x \in H$, $\|x\| = 1$.

Let $A, B \in \mathcal{B}(H)$, then

$$\|AB\| \leq \|A\|\,\|B\|,$$

by Banach algebra property.

2. Main Results

Here we derive self adjoint operator-Korovkin type theorems via operator-Shisha-Mond type inequalities. This is a quantitative approach, studying the degree of operator-uniform approximation with rates of sequences of operator-positive linear operators in the operator order of $\mathcal{B}(H)$. We continue similarly with important polynomial operators. Our approach is direct based on Gelfand's isometry.

All the functions we are dealing here are real valued. We assume that $Sp(A) \subseteq [m, M]$.

Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators from $C([m, M])$ into itself (i.e. if $f, g \in C([m, M])$ such that $f \geq g$, then $L_n f \geq L_n g$). It is interesting to study the convergence of $L_n \to I$ (unit operator, i.e. $I(f) = f$, $\forall f \in C([m, M])$). By property (i) we have that

$$\Phi(L_n f - f) = \Phi(L_n f) - \Phi(f) = (L_n f)(A) - f(A),$$

and

$$\Phi(L_n 1 \pm 1) = \Phi(L_n 1) \pm \Phi(1) = (L_n 1)(A) \pm 1_H,$$

the last comes by property (iv).

And by property (iii) we obtain

$$\|\Phi(L_n f - f)\| = \|(L_n f)(A) - f(A)\| = \|L_n f - f\|,$$

and

$$\|\Phi(L_n 1 \pm 1)\| = \|(L_n 1)(A) \pm 1_H\| = \|L_n 1 \pm 1\|.$$

We need

Theorem 1 (Shisha and Mond [12]). *Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators from $C([m, M])$ into itself. For $n = 1, 2, \ldots$, suppose $L_n 1$ is bounded. Let $f \in C([m, M])$. Then for $n = 1, 2, \ldots$, we have*

$$\|L_n f - f\|_\infty \leq \|f\|_\infty \|L_n 1 - 1\|_\infty + \|L_n 1 + 1\|_\infty \, \omega_1(f, \mu_n), \tag{1}$$

where

$$\mu_n := \left\| \left(L_n (t - x)^2 \right)(x) \right\|_\infty^{1/2},$$

$$\omega_1(f, \delta) := \sup_{\substack{x,y\in[m,M] \\ |x-y|<\delta}} |f(x) - f(y)|, \qquad \delta > 0,$$

and $\|\cdot\|_\infty$ stands for the sup-norm over $[m, M]$.

In particular, if $L_n 1 = 1$, then (1) becomes

$$\|L_n f - f\|_\infty \leq 2\omega_1(f, \mu_n).$$

Note: (i) In foming μ_n^2, x is kept fixed, however t forms the functions t, t^2 on which L_n acts.

(ii) One can easily find, for $n = 1, 2, \ldots$,

$$\mu_n^2 \leq \left\| \left(L_n t^2 \right)(x) - x^2 \right\|_\infty + 2c \|(L_n t)(x) - x\|_\infty + c^2 \|(L_n 1)(x) - 1\|_\infty,$$

where

$$c := \max(|m|, |M|).$$

So, if the Korovkin's assumptions are fulfilled, i.e. if $L_n(id^2) \xrightarrow{u} id^2$, $L_n(id) \xrightarrow{u} id$ and $L_n 1 \xrightarrow{u} 1$, as $n \to \infty$, id is the identity map and u is the uniform convergence, then $\mu_n \to 0$, and then $\omega_1(f, \mu_n) \to 0$, as $n \to +\infty$, and we obtain from (1) that $\|L_n f - f\|_\infty \to 0$, i.e. $L_n f \xrightarrow{u} f$, as $n \to \infty$, $\forall f \in C([m, M])$.

We give

Theorem 2. *All as in Theorem 1. Then*

$$\|(L_n f)(A) - f(A)\| \leq \|f(A)\| \|(L_n 1)(A) - 1_H\| + \|(L_n 1)(A) + 1_H\| \omega_1(f, \mu_n), \quad (2)$$

where

$$\mu_n := \left\| \left(L_n (t - A)^2\right)(A) \right\|^{1/2}.$$

In particular, if $(L_n 1)(A) = 1_H$, *then*

$$\|(L_n f)(A) - f(A)\| \leq 2\omega_1(f, \mu_n).$$

Furthermore it holds

$$\mu_n^2 \leq \left\| \left(L_n t^2\right)(A) - A^2 \right\| + 2c \|(L_n t)(A) - A\| + c^2 \|(L_n 1)(A) - 1_H\|. \quad (3)$$

So, if $\left(L_n t^2\right)(A) \to A^2$, $(L_n t)(A) \to A$, $(L_n 1)(A) \to 1_H$, uniformly, as $n \to \infty$, then by (3) and (2) we get $(L_n f)(A) \to f(A)$, uniformly, as $n \to \infty$.

That is establishing the self adjoint operator Korovkin theorem with rates.

Next we follow [2, pp. 273–274].

Theorem 3. *Let* $L_n : C([m, M]) \to C([m, M])$, $n \in \mathbb{N}$, *be a sequence of positive linear operators,* $f \in C([m, M])$, $g \in C([m, M])$ *and it is an* $(1 - 1)$ *function.*

Assume $\{L_n 1\}_{n \in \mathbb{N}}$ *is uniformly bounded. Then*

$$\|L_n f - f\| \leq \|f\| \, \|L_n 1 - 1\| + \left(1 + \|L_n 1\|\right) \omega_g(f, \rho_n), \quad (4)$$

where

$$\omega_g(f, h) := \sup_{x, y \in [m, M]} \left\{ |f(x) - f(y)| \, : \, |g(x) - g(y)| \leq h \right\}, \qquad h > 0,$$

$$\rho_n := \left\| \left(L_n (g - g(y))^2\right)(y) \right\|^{1/2}.$$

Here $\|\cdot\|$ *stands for the supremum norm. If* $L_n 1 = 1$, *then* (4) *simplifies to*

$$\|L_n f - f\| \leq 2\omega_g(f, \rho_n).$$

We also have that

$$\rho_n^2 \leq \|L_n g^2 - g^2\| + 2\|g\| \, \|L_n g - g\| + \|g\|^2 \, \|L_n 1 - 1\|.$$

If $L_n 1 \xrightarrow{u} 1$, $L_n g \xrightarrow{u} g$, $L_n g^2 \xrightarrow{u} g^2$, *then* $\omega_g(f, \rho_n) \to 0$, *and then* $L_n f \xrightarrow{u} f$, *as* $n \to +\infty$, $\forall f \in C([m, M])$, *where* u *stands for uniform convergence, so we get a generalization of Korovkin theorem quantitatively, and clearly by* $L_n 1 \xrightarrow{u} 1$, *we get* $\|L_n 1\| \leq K$, $\forall n \in \mathbb{N}$, *where* $K > 0$.

We present

Theorem 4. *All as in Theorem 3. Then*

$$\|(L_n f)(A) - f(A)\| \leq \|f(A)\| \, \|(L_n 1)(A) - 1_H\| + \big(1 + \|(L_n 1)(A)\|\big)\omega_g(f, \rho_n),$$

with

$$\rho_n := \big\|\big(L_n(g - g(A))^2\big)(A)\big\|^{1/2}.$$

If $(L_n 1)(A) = 1_H$*, then*

$$\|(L_n f)(A) - f(A)\| \leq 2\omega_g(f, \rho_n).$$

It holds

$$\begin{aligned}\rho_n^2 \leq \big\|\big(L_n g^2\big)(A) - g^2(A)\big\| + 2\|g(A)\| \, \|(L_n g)(A) - A\| \\ + \|g(A)\|^2 \, \|(L_n 1)(A) - 1_H\|.\end{aligned}$$

If $(L_n 1)(A) \to 1_H$*,* $(L_n g)(A) \to A$*,* $(L_n g^2)(A) \to g^2(A)$*, uniformly, as* $n \to +\infty$*, then* $(L_n f)(A) \to f(A)$*, uniformly, as* $n \to +\infty$*.*

We make

Remark 1. Next we consider the general Bernstein positive linear polynomial operators from $C([m, M])$ into itself, for $f \in C([m, M])$ we define

$$(B_N f)(s) = \sum_{k=0}^{N} f\Big(m + k\frac{M - m}{N}\Big)\binom{N}{k}\Big(\frac{s - m}{M - m}\Big)^k \Big(\frac{M - s}{M - m}\Big)^{N-k},$$

$\forall s \in [m, M]$, see [13, p. 80].

Then by [13, p. 81], we get that

$$\|B_N f - f\|_\infty \leq \frac{5}{4}\,\omega_1\Big(f, \frac{M - m}{\sqrt{N}}\Big),$$

$\forall N \in \mathbb{N}$, i.e. $B_N f \xrightarrow{u} f$, as $N \to +\infty$, $\forall f \in C([m, M])$, the convergence is given with rates.

We clearly have that

$$\|(B_N f)(A) - f(A)\|_\infty \leq \frac{5}{4}\,\omega_1\Big(f, \frac{M - m}{\sqrt{N}}\Big),$$

$\forall N \in \mathbb{N}$, i.e. $(B_N f)(A) \to f(A)$, uniformly, as $N \to +\infty$.

We need

Notation 1. Let $x \in [m, M]$. Denote

$$c(x) := \max(x - m, M - x) = \frac{1}{2}\big(M - m + |M + m - 2x|\big) > 0.$$

Let $h > 0$ be fixed, $n \in \mathbb{N}$. Define (see [1, p. 210])

$$\Phi_{*n}(x) := \frac{|x|^{n+1}}{(n+1)!h} + \frac{|x|^n}{2n!} + \frac{h|x|^{n-1}}{8(n-1)!}.$$

We need

Theorem 5 ([1, p. 219]). *Let $\{L_N\}_{N\in\mathbb{N}}$ be a sequence of positive linear operators from $C([m, M])$ into itself, $x \in [m, M]$, $f \in C^n([m, M])$.*

Here $c(x)$ and $\Phi_{*n}(x)$ are as in Notation 1. Assume that $\omega_1(f^{(n)}, h) \le w$, where w, h are fixed positive numbers, $0 < h < M - m$. Then

$$|(L_N f)(x) - f(x)| \le |f(x)|\,|(L_N 1)(x) - 1| + \sum_{k=1}^{n} \frac{|f^{(k)}(x)|}{k!}\big|\big(L_N(t-x)^k\big)(x)\big| + \frac{w\Phi_{*n}(c(x))}{c^n(x)}\big(L_N|t-x|^n\big)(x). \tag{5}$$

Inequality (5) is sharp, for details see [1, p. 220].

Clearly all functions involved in (5) are continuous, see also [3], i.e. both sides of (5) are continuous functions.

Using properties (P) and (ii) and (5) we derive

Theorem 6. *All as in Theorem 5. Then*

$$\begin{aligned}|(L_N f)(A) - f(A)| \le{}& |f(A)|\,|(L_N 1)(A) - 1_H| \\ &+ \sum_{k=1}^{n} \frac{|f^{(k)}(A)|}{k!}\big|\big(L_N(t-A)^k\big)(A)\big| \\ &+ \frac{w\Phi_{*n}(c(A))}{c^n(A)}(L_N|t-A|^n)(A).\end{aligned} \tag{6}$$

Remark 2. Inequality (6) implies

$$\begin{aligned}\|(L_N f)(A) - f(A)\| \le{}& \|f(A)\|\,\|(L_N 1)(A) - 1_H\| \\ &+ \sum_{k=1}^{n} \frac{\|f^{(k)}(A)\|}{k!}\big\|\big(L_N(t-A)^k\big)(A)\big\| \\ &+ w\Big\|\frac{\Phi_{*n}(c(A))}{c^n(A)}\Big\|\,\big\|(L_N|t-A|^n)(A)\big\|.\end{aligned} \tag{7}$$

Remark 3 (to Theorem 6 and (7)**).** Assume further

$$\|L_N 1\|_\infty \le \mu, \qquad \forall N \in \mathbb{N},\ \mu > 0.$$

By Riesz representation theorem, for each $s \in [m, M]$, there exists a positive finite measure μ_s on $[m, M]$ such that

$$(L_N f)(s) = \int_{[m,M]} f(t)\, d\mu_{sN}(t), \qquad \forall f \in C([m, M]).$$

Therefore, for $k = 1, \ldots, n-1$,

$$\big|\big(L_N(\cdot - s)^k\big)(s)\big| = \Big|\int_{[m,M]} (\lambda - s)^k\, d\mu_{sN}(\lambda)\Big| \le \int_{[m,M]} |\lambda - s|^k\, d\mu_{sN}(\lambda)$$

(by Hölder's inequality)

$$\begin{aligned}
&\le \left(\int_{[m,M]} 1\, d\mu_{sN}(\lambda)\right)^{1-k/n} \left(\int_{[m,M]} |\lambda - s|^n\, d\mu_{sN}(\lambda)\right)^{k/n} \\
&= \big((L_N 1)(s)\big)^{1-k/n} \big(\big(L_N|\cdot - s|^n\big)(s)\big)^{k/n} \\
&\le \mu^{1-k/n} \big(\big(L_N|\cdot - s|^n\big)(s)\big)^{k/n}.
\end{aligned}$$

That is

$$\big|\big(L_N(\cdot - s)^k\big)(s)\big| \le \mu^{1-k/n}\big(\big(L_N|\cdot - s|^n\big)(s)\big)^{k/n}, \qquad k = 1, \ldots, n-1.$$

Of course it holds

$$\big|\big(L_N(\cdot - s)^n\big)(s)\big| \le \big(L_N|\cdot - s|^n\big)(s).$$

By property (P) we obtain

$$\big|\big(L_N(\cdot - A)^k\big)(A)\big| \le \mu^{1-k/n}\big(\big(L_N|\cdot - A|^n\big)(A)\big)^{k/n}, \qquad k = 1, \ldots, n-1,$$

and

$$\big|\big(L_N(\cdot - A)^n\big)(A)\big| \le \big(L_N|\cdot - A|^n\big)(A).$$

Therefore

$$\begin{aligned}
\big\|\big(L_N(\cdot - A)^k\big)(A)\big\| &\le \mu^{1-k/n}\big\|\big(\big(L_N|\cdot - A|^n\big)(A)\big)^{k/n}\big\| \\
&= \mu^{1-k/n} \sup_{x\in H,\ \|x\|=1} \big\langle\big(\big(L_N|\cdot - A|^n\big)(A)\big)^{k/n} x,\, x\big\rangle
\end{aligned}$$

(by Hölder's-McCarthy inequality)

$$\begin{aligned}
&\le \mu^{1-k/n} \sup_{x\in H,\ \|x\|=1} \big\langle\big(L_N|\cdot - A|^n\big)(A)x,\, x\big\rangle^{k/n} \\
&= \mu^{1-k/n}\Big(\sup_{x\in H,\ \|x\|=1} \big\langle\big(L_N|\cdot - A|^n\big)(A)x,\, x\big\rangle\Big)^{k/n} \\
&= \mu^{1-k/n}\big\|\big(L_N|\cdot - A|^n\big)(A)\big\|^{k/n}.
\end{aligned}$$

Therefore it holds

$$\big\|\big(L_N(t-A)^k\big)(A)\big\| \leq \mu^{1-k/n}\big\|\big(L_N|t-A|^n\big)(A)\big\|^{k/n}, \tag{8}$$

for $k = 1, \ldots, n-1$, and of course

$$\big\|\big(L_N(t-A)^n\big)(A)\big\| \leq \big\|\big(L_N|t-A|^n\big)(A)\big\|. \tag{9}$$

Based on (8) and (9) and by assuming that $(L_n 1)(A) \to 1_H$ and $\big(L_N|t-A|^n\big)(A) \to 0_H$, uniformly, as $N \to +\infty$, we obtain by (7) that $(L_N f)(A) \to f(A)$, uniformly as $N \to +\infty$.

We mention

Theorem 7 ([1, p. 230]). *For any $f \in C^1([0,1])$ consider the Bernstein polynomials*

$$\big(\beta_n f\big)(t) := \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big)\binom{n}{k} t^k (1-t)^{n-k}, \qquad t \in [0,1].$$

Then

$$\|\beta_n f - f\|_\infty \leq \frac{0.78125}{\sqrt{n}}\,\omega_1\Big(f', \frac{1}{4\sqrt{n}}\Big). \tag{10}$$

We make

Remark 4. The map φ,

$$s = \varphi(t) = (M-m)t + m, \qquad t \in [0,1],$$

maps $(1-1)$ and onto, $[0,1]$ onto $[m, M]$.

Let $f \in C^1([m,M])$, then

$$f(s) = f(\varphi(t)) = f((M-m)t + m),$$

and

$$\frac{df(s)}{dt} = \big(f(\varphi(t))\big)' = (M-m)f'(\varphi(t)) = (M-m)f'(s).$$

By (10) we get that

$$\big\|\beta_n f((M-m)t+m) - f((M-m)t+m)\big\|_{\infty,[0,1]} \leq \frac{0.78125}{\sqrt{n}}\,(M-m)\omega_1\Big(f'(s), \frac{1}{4\sqrt{n}}\Big). \tag{11}$$

However we have

$$\begin{aligned}\omega_1\Big(f'(s), \frac{1}{4\sqrt{n}}\Big) &= \omega_1\Big(f'((M-m)t+m), \frac{1}{4\sqrt{n}}\Big)\\ &= \sup_{\substack{t_1,t_2\in[0,1]\\ |t_1-t_2|\le \frac{1}{4\sqrt{n}}}} \big|f'((M-m)t_1+m) - f'((M-m)t_2+m)\big|\\ &= \sup_{\substack{s_1,s_2\in[m,M]\\ |s_1-s_2|\le \frac{M-m}{4\sqrt{n}}}} \big|f'(s_1)-f'(s_2)\big| = \omega_1\Big(f', \frac{M-m}{4\sqrt{n}}\Big),\end{aligned}$$

above notice that

$$|s_1-s_2| = \big|((M-m)t_1+m)-((M-m)t_2+m)\big| = (M-m)|t_1-t_2| \le \frac{M-m}{4\sqrt{n}}.$$

So we have proved that

$$\omega_1\Big(f'(s), \frac{1}{4\sqrt{n}}\Big) = \omega_1\Big(f', \frac{M-m}{4\sqrt{n}}\Big). \tag{12}$$

Finally, we observe that

$$\begin{aligned}\big(\beta_n f((M-m)t+m)\big)(t) &= \sum_{k=0}^{n} f\Big((M-m)\frac{k}{n}+m\Big)\binom{n}{k} t^k(1-t)^{n-k}\\ &= \sum_{k=0}^{n} f\Big((M-m)\frac{k}{n}+m\Big)\binom{n}{k}\Big(\frac{s-m}{M-m}\Big)^k\Big(\frac{M-s}{M-m}\Big)^{n-k}\\ &:= \big(B_n f\big)(s),\end{aligned}$$

$s \in [m, M]$.

The operators $(B_n f)(s)$ are the general Bernstein polynomials.

From (11) and (12), we derive that

$$\|(B_n f)(s) - f(s)\|_{\infty,[m,M]} \le \frac{0.78125}{\sqrt{n}}\,(M-m)\omega_1\Big(f', \frac{M-m}{4\sqrt{n}}\Big).$$

Based on the above and the property (iii), we can give

Theorem 8. *Let $f' \in [m, M]$. Then*

$$\|(B_n f)(A) - f(A)\| \le \frac{0.78125}{\sqrt{n}}\,(M-m)\omega_1\Big(f', \frac{M-m}{4\sqrt{n}}\Big),$$

i.e. $(B_n f)(A) \to f(A)$, uniformly, with rates as $n \to +\infty$.

We make

Remark 5. Let $f \in C([m,M])$, then the function $f((M-m)t+m)$ is a continuous function in $t \in [0,1]$.

Let $r \in \mathbb{N}$, we evaluate the modulus of smoothness ($\delta > 0$)

$$\begin{aligned}
&\omega_r\big(f((M-m)t+m),\delta\big) \\
&\qquad = \sup_{0\le h\le\delta} \Big\| \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} f((M-m)(t+kh)+m) \Big\|_{\infty,[0,1-rh]} \\
&\qquad = \sup_{0\le h^*\le\delta(M-m)} \Big\| \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} f(s+kh^*) \Big\|_{\infty,[0,M-rh^*]}
\end{aligned}$$

(with $h^* = (M-m)h$)

$$= \omega_r(f,(M-m)\delta),$$

proving that

$$\omega_r\big(f((M-m)t+m),\delta\big) = \omega_r(f,(M-m)\delta),$$

for any $r \in \mathbb{N}$ and $\delta > 0$.

We need

Theorem 9 ([11, p. 97]). *For $f \in C([0,1])$, $n \in \mathbb{N}$, we have*

$$\|\beta_n f - f\| \le \omega_2\Big(f, \frac{1}{\sqrt{n}}\Big),$$

a sharp inequality.

We get

Theorem 10. *Let $f \in C([m,M])$, $n \in \mathbb{N}$. Then*

$$\big\|(B_n f)(A) - f(A)\big\| = \|B_n f - f\|_\infty \le \omega_2\Big(f, \frac{M-m}{\sqrt{n}}\Big).$$

We need

Definition 1 ([11, p. 151]). Let $f \in C([0,1])$, $n \in \mathbb{N}$. We define the Durrmeyer type operators (the genuine Bernstein-Durrmeyer operators)

$$\begin{aligned}
(M_n^{-1,-1} f)(x) &= f(0)(1-x)^n + f(1)x^n \\
&\quad + (n-1)\sum_{k=1}^{n-1} p_{n,k}(x) \int_0^1 f(t) p_{n-2,k-1}(t)\,dt, \qquad (13)
\end{aligned}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad n \in \mathbb{N}, \quad x \in [0,1].$$

We will use

Theorem 11 ([11, p. 155]). *For $f \in C([0,1])$, $n \in \mathbb{N}$, we have*

$$\|M_n^{-1,-1} f - f\|_\infty \le \frac{5}{4}\,\omega_2\Big(f, \frac{1}{\sqrt{n+1}}\Big).$$

We make

Remark 6. Let $f \in C([m,M])$, then $f((M-m)t+m) \in C([0,1])$. Hence by (13) for $s \in [m,M]$, $t \in [0,1]$,

$$\begin{aligned}
\big(\overline{M}_n^{-1,-1} f\big)(s) &:= M_n^{-1,-1}\big(f((M-m)t+m)\big)(t) \\
&= f(m)\Big(\frac{M-s}{M-m}\Big)^n + f(M)\Big(\frac{s-m}{M-m}\Big)^n \\
&\quad + (n-1)\sum_{k=1}^{n-1}\binom{n}{k}\Big(\frac{s-m}{M-m}\Big)^k\Big(\frac{M-s}{M-m}\Big)^{n-k}\binom{n-2}{k-1} \\
&\quad\quad \times \int_m^M f(\overline{s})\Big(\frac{\overline{s}-m}{M-m}\Big)^{k-1}\Big(\frac{M-\overline{s}}{M-m}\Big)^{n-k-1} d\overline{s}.
\end{aligned}$$

We give

Theorem 12. *Let $f \in C([m,M])$, $n \in \mathbb{N}$. Then*

$$\big\|\big(\overline{M}_n^{-1,-1} f\big)(A) - f(A)\big\| = \big\|\big(\overline{M}_n^{-1,-1} f\big) - f\big\|_\infty \le \frac{5}{4}\,\omega_2\Big(f, \frac{M-m}{\sqrt{n+1}}\Big).$$

We need

Definition 2 ([7]). *For $f \in C([0,1])$, $w \in \mathbb{N}$, and $0 \le \beta \le \gamma$, we define the Stancu-type positive linear operators*

$$\big(L_{w_0}^{\langle 0\beta\gamma\rangle} f\big)(x) = \sum_{k=0}^{w} f\Big(\frac{k+\beta}{w+\gamma}\Big) p_{w,k}(x), \qquad x \in [0,1], \tag{14}$$

$$p_{w,k}(x) = \binom{w}{k} x^k (1-x)^{w-k}.$$

We need

Theorem 13 ([2, p. 516], [7]). *For $w \in \mathbb{N}$, $w > \lceil \gamma^2 \rceil$ ($\lceil \cdot \rceil$ is the ceiling), $f \in C([0,1])$ we have:*

$$\big\|L_{w_0}^{\langle 0\beta\gamma\rangle} f - f\big\|_\infty \le \Big[3 + \frac{w^3 + 4w\beta^2(w-\gamma^2)}{4(w-\gamma^2)(w+\gamma)^2}\Big]\omega_2\Big(f, \frac{1}{\sqrt{w}}\Big) + \frac{2(\beta+\gamma)\sqrt{w}}{w+\gamma}\,\omega_1\Big(f, \frac{1}{\sqrt{w}}\Big).$$

We make

Remark 7. Let $f \in C([m,M])$, then $f((M-m)t+m) \in C([0,1])$. Hence by (14) for $s \in [m,M]$, $t \in [0,1]$,

$$\begin{aligned}\big(\overline{L}_{w_0}^{\langle 0\beta\gamma\rangle} f\big)(s) &:= \big(L_{w_0}^{\langle 0\beta\gamma\rangle} f((M-m)t+m)\big)(t)\\ &= \sum_{k=0}^{w} f\Big((M-m)\frac{k+\beta}{w+\gamma}+m\Big)\binom{w}{k}\Big(\frac{s-m}{M-m}\Big)^k\Big(\frac{M-s}{M-m}\Big)^{w-k}.\end{aligned}$$

We give

Theorem 14. *Let $f \in C([m,M])$, $w \in \mathbb{N}$, $0 \le \beta \le \gamma$. We take $w > \lceil \gamma^2 \rceil$. Then*

$$\begin{aligned}&\big\|\big(\overline{L}_{w_0}^{\langle 0\beta\gamma\rangle} f\big)(A) - f(A)\big\| = \big\|\overline{L}_{w_0}^{\langle 0\beta\gamma\rangle} f - f\big\|_\infty\\ &\le \Big[3+\frac{w^3+4w\beta^2(w-\gamma^2)}{4(w-\gamma^2)(w+\gamma)^2}\Big]\omega_2\Big(f,\frac{M-m}{\sqrt{w}}\Big)+\frac{2(\beta+\gamma)\sqrt{w}}{w+\gamma}\,\omega_1\Big(f,\frac{M-m}{\sqrt{w}}\Big).\end{aligned}$$

We make

Remark 8. Next we assume that the spectrum of A is $[0,1]$. For example, it could be $Af = xf(x)$ on $L^2([0,1])$ which is a self adjoint operator and it has spectrum $[0,1]$.

We need

Definition 3 ([14]). Let $f \in C([0,1])$, we define the special Stancu operator

$$S_n(f,x) = \frac{2(n!)}{(2n)!}\sum_{k=0}^{n} f\Big(\frac{k}{n}\Big)\binom{n}{k}(nx)_k\,(n-nx)_{n-k},$$

where $(a)_0 = 1$, $(a)_b = \sum_{k=0}^{b-1}(a-k)$, $a \in \mathbb{R}$, $b \in \mathbb{N}$, $n \in \mathbb{N}$, $x \in [0,1]$.

Theorem 15 ([8, p. 75]). *Let $f \in C([0,1])$, $n \in \mathbb{N}$. Then*

$$\big|((S_n - M_n^{-1,-1})(f)(x)\big| \le c_1\omega_4\left(f,\sqrt[4]{\frac{3x(1-x)}{n(n+1)}}\right), \qquad \forall x \in [0,1],$$

where $c_1 > 0$ is an absolute constant independent of n, f and x.

We obtain

Theorem 16. *Let $f \in C([0,1])$, $n \in \mathbb{N}$. Then*

$$\|(S_n - M_n^{-1,-1})(A)\| = \|S_n - M_n^{-1,-1}\|_\infty \le c_1\omega_4\left(f,\sqrt[4]{\frac{3}{4n(n+1)}}\right).$$

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