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On the Composition and Decomposition of Positive Linear Operators (VI)

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Dedicated to academician Blagovest Sendov

We discuss several aspects in regard to decomposing the classical Bernstein operator. Piecewise linear interpolation at equidistant points and a classical Beta-type operator are in the focus.

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1. Introduction

In the present note we discuss the composition of a Beta-type operator introduced by Mühlbach and Lupaş in the early 70s of the last century and piecewise linear interpolation at equidistant points in $[0, 1]$. This produces a positive linear operator $\mathbb{G}_n : C[0, 1] \to C[0, 1]$ reproducing linear functions and retaining monotonicity and convexity.

We give a quantitative result involving the second order modulus of continuity, a quantitative Voronovskaya-type result and an estimate for the difference $B_n - \mathbb{G}_n$ where B_n is the classical Bernstein operator. Since not very much is known about $\mathbb{G}_n$, we will use lower and upper estimates for several quantities with the help of the Beta-type operators mentioned and a special case of operators introduced by Stancu and further investigated by Lupaş and Lupaş. Details will be given below.

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2. On Piecewise Linear Interpolation

In this section we collect some facts about the piecewise linear interpolation operator $S_{\Delta_n} : C[0,1] \to C[0,1]$ at the points

$$\Delta_n : \Big\{0, \frac{1}{n}, \dots, \frac{k}{n}, \dots, 1 - \frac{1}{n}, 1\Big\}$$

given by

$$(S_{\Delta_n} f)(x) = \frac{1}{n} \sum_{k=0}^{n} \Big[\frac{k-1}{n}, \frac{k}{n}, \frac{k+1}{n}; [\alpha - x]\Big]_\alpha f\Big(\frac{k}{n}\Big), \tag{1}$$

where $[a,b,c;f] = [a,b,c;f(\alpha)]_\alpha$ denotes the divided difference of a function $f : D \to \mathbb{R}$ on the (distinct knots) $\{a,b,c\} \subset D$, with respect to α. We will write

$$|\alpha - t|_+ = \max\{0, \alpha - t\} = \frac{\alpha - t + |\alpha - t|}{2},$$

$\lfloor \alpha \rfloor = \max\{z \in \mathbb{Z} : z \le \alpha\}$ is the floor function (integer part of α), and $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$ is the fractional part of α.

Theorem 1. *For $f : [0,1] \to \mathbb{R}$ and $t \in [0,1]$ the following representations hold:*

$$(S_{\Delta_n})(t) = (1 - \{nt\}) f\Big(\frac{\lfloor nt \rfloor}{n}\Big) + \{nt\} f\Big(\frac{1 + \lfloor nt \rfloor}{n}\Big), \tag{2}$$

$$\begin{aligned}(S_{\Delta_n} f)(t) &= \frac{1}{n} \sum_{k=0}^{n} \Big[\frac{k-1}{n}, \frac{k}{n}, \frac{k+1}{n}; |\alpha - t|\Big]_\alpha f\Big(\frac{k}{n}\Big) \\ &= |1 - nt|_+ f(0) + |nt - n + 1|_+ f(1) \\ &\quad + \frac{1}{n} \sum_{k=1}^{n-1} \Big[\frac{k-1}{n}, \frac{k}{n}, \frac{k+1}{n}; |\alpha - t|\Big]_\alpha f\Big(\frac{k}{n}\Big) \\ &= |1 - nt|_+ f(0) + |nt - n + 1|_+ f(1) \\ &\quad + \sum_{k=1}^{n-1} \big(|nt - k + 1|_+ - 2|nt - k|_+ + |nt - k - 1|_+\big) f\Big(\frac{k}{n}\Big),\end{aligned} \tag{3}$$

$$\begin{aligned}(S_{\Delta_n} f)(t) &= \frac{(1 - nt) f(0) + nt f\big(\frac{1}{n}\big)}{2} + \frac{n(1-t) f\big(1 - \frac{k}{n}\big) + (nt - n + 1) f(1)}{2} \\ &\quad + \frac{1}{n} \sum_{k=1}^{n-1} \Big[\frac{k-1}{n}, \frac{k}{n}, \frac{k+1}{n}; f\Big] \Big|t - \frac{k}{n}\Big|.\end{aligned} \tag{4}$$

If $t \in (0,1)$, $\{nt\} \neq 0$, then

$$(S_{\Delta_n} f)(t) = f(t) + \frac{\{nt\}(1 - \{nt\})}{n^2} \Big[\frac{\lfloor nt \rfloor}{n}, t, \frac{\lfloor nt + 1 \rfloor}{n}; f\Big]. \tag{5}$$

Proof. In order to show that (2) is the same with (3) it is sufficient to use the equalities

$$\frac{1}{n}\Big[\frac{k-1}{n},\frac{k}{n},\frac{k+1}{n};|\alpha-t|\Big]_\alpha=\begin{cases}0, & k\le\lfloor nt\rfloor-1 \quad\text{or}\quad k\ge\lfloor nt\rfloor+2,\\ 1-\{nt\}, & k=\lfloor nt\rfloor,\\ \{nt\}, & k=1+\lfloor nt\rfloor.\end{cases}$$

Observe that if $(A_n)_{n\ge0}$ is a real sequence, then

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n-1}&\Big[\frac{k-1}{n},\frac{k}{n},\frac{k+1}{n};f\Big]A_k\\ &=\frac{n}{2}\sum_{k=1}^{n-1}(A_{k+1}-2A_k+A_{k-1})f\Big(\frac{k}{n}\Big)\\ &\quad+\frac{n}{2}\Big[A_1f(0)-A_0f\Big(\frac{1}{n}\Big)+f(1)A_{n-1}-A_nf\Big(1-\frac{1}{n}\Big)\Big].\end{aligned}$$

By using this formula it is easy to prove that the right-hand member of (4) (which is a linear function on each interval $\big[\frac{j-1}{n},\frac{j}{n}\big]$, $j=1,\dots,n$) takes the value $f\big(\frac{j}{n}\big)$ at the point $\frac{j}{n}$, $j=0,1,\dots,n$. This proves (4), and (5) follows from (2). □

Concerning the degree of approximation by S_{Δ_n} one has

Theorem 2. *For $n\ge1$, $f\in C[0,1]$ and $x\in[0,1]$ there holds*

$$|S_{\Delta_n}(f;x)-f(x)|\le 1\cdot\omega_2\Big(f;\frac{1}{2n}\Big).$$

Proof. A natural way to find an estimate of the claimed type is to use a theorem by Păltănea [20, Cor. 3.1], saying that for a positive linear operator L reproducing linear functions one has for $0<h\le\frac{1}{2}$, $f\in C[0,1]$ and $x\in[0,1]$ the inequality

$$|L(f;x)-f(x)|\le\Big[1+\frac{1}{2h^2}L\big((e_1-x)^2;x\big)\Big]\omega_2(f;h).$$

For the second moments $S_{\Delta_n}\big((e_1-x)^2;x\big)$ we have

$$S_{\Delta_n}\big((e_1-x)^2;x\big)=\Big(x-\frac{\ell}{n}\Big)\Big(\frac{\ell+1}{n}-x\Big)\le\frac{1}{4n^2},\qquad x\in\Big[\frac{\ell}{n},\frac{\ell+1}{n}\Big].$$

But this implies the inequality with the constant $\frac{3}{2}$ only in front of the modulus.

For the constant 1 see Gonska and Kovacheva [6, Lemma 2.3]. □

Remark 1. It is also known that

$$|S_{\Delta_n}(f;x)-f(x)|\le 1\cdot\omega_1\Big(f;\frac{1}{n}\Big).$$

None of the two inequalities in terms of moduli implies the other one.

3. The Quadratic Splines of Sendov

In 1987 Bl. Sendov presented a piecewise quadratic spline $S_2 : C[0,1] \to W^2_\infty[0,1]$ which may be viewed as a "differentiable brother" of piecewise linear interpolation. We describe it next. Let $S_1(f,\cdot)$ denote the linear interpolation spline on equidistant knots with step size $h = \frac{1}{m}$, satisfying the conditions

$$S_1(f;ih) = f(ih), \qquad i = 0,1,\dots,m.$$

So this is what was denoted as S_{Δ_m} in the previous section.

$S_1(f;\cdot)$ is linear on every interval $[ih,(i+1)h]$, $i = 0,\dots,m-1$. The quadratic spline $S_2(f;x) \in C^1[0,1]$ is then defined by the conditions

$$S_2\Big(f;ih+\frac{h}{2}\Big) = \frac{1}{2}\big(f(ih)+f(ih+h)\big) = S_1\Big(f;ih+\frac{h}{2}\Big), \quad 0 \le i \le m-1,$$
$$S_2(f;x) = S_1(f;x) \quad \text{for} \quad x \in \Big[0,\frac{h}{2}\Big] \cup \Big[1-\frac{h}{2},1\Big].$$

The analytic representation of $S_2(f;x)$ for other values of x was given by Bl. Sendov as

$$S_2(f;x) = \frac{(x-ih)^2}{2h^2}\,\Delta_h^2 f(ih-h) + \frac{x-ih}{2h}\big(f(ih+h)-f(ih-h)\big) + f(ih) + \frac{1}{8}\,\Delta_h^2 f(ih-h),$$

for $x \in \big[ih-\frac{h}{2}, ih+\frac{h}{2}\big]$, $i = 1,\dots,m-1$.

However, $S_2(f;x)$, $x \in \big[ih-\frac{h}{2}, ih+\frac{h}{2}\big]$, is more easily understood if one thinks of it as being the second degree Bernstein polynomial over the interval $\big[ih-\frac{h}{2}, ih+\frac{h}{2}\big]$ determined by the ordinates $S_1\big(f;ih-\frac{h}{2}\big)$, $f(ih)$, and $S_1\big(f;ih+\frac{h}{2}\big)$.

For the spline operator S_2 the following are known

Theorem 3. *Let $m \in \mathbb{N}$, $m \ge 2$, $h = \frac{1}{m}$, and for $f \in C[0,1]$ let $S_2(f;\cdot)$ be the spline defined above. Then for $x \in [0,1]$ one has*

$$|f(x) - S_2(f;x)| \le \omega_2\Big(f;\frac{h}{2}\Big), \qquad 0 \le x \le \frac{h}{2}, \quad \text{or} \quad 1-\frac{h}{2} \le x \le 1,$$
$$|f(x) - S_2(f;x)| \le \omega_2(f;h), \qquad h \le x \le 1-\frac{h}{2},$$
$$|S_2(f;x)| \le \|f\|_\infty,$$
$$|(S_2f)'(x)| \le \frac{1}{h}\,\omega_1(f;h),$$
$$\|(S_2f)''\|_\infty \le \frac{1}{h^2}\,\omega_2(f;h).$$

Proof. See [21] and [6]. □

Remark 2. The reader should keep in mind that the above inequalities hold for $h = \frac{1}{m}$, $m \geq 2$, only. For other values of h it seems to be unknown if such good constants as 1 are valid.

4. Beta Operators of the Second Kind

The third class of operators which will be used in this note are certain Beta-type operators introduced by Mühlbach [18, 19] and Lupaş [16]. These mappings are given for $f \in C[0,1]$, $x \in [0,1]$ by

$$\overline{\mathbb{B}}_n(f;x) = \begin{cases} f(0), & x = 0, \\ \dfrac{1}{B(nx, n(1-x))} \displaystyle\int_0^1 t^{nx-1}(1-t)^{n(1-x)-1} f(t)\, dt, & x \in (0,1), \\ f(1), & x = 1. \end{cases}$$

Here $B(\,\cdot\,, *)$ is the Beta function. The $\overline{\mathbb{B}}_n$ are positive endomorphisms of $C[0,1]$; they reproduce linear functions and have second moments smaller than those of the Bernstein operators. More precisely, see [16],

$$\overline{\mathbb{B}}_n((e_1 - x)^2; x) = \frac{x(1-x)}{n+1} \leq \frac{x(1-x)}{n} = B_n((e_1 - x)^2; x).$$

Moreover, it is known from [1] and [2] that $\overline{\mathbb{B}}_n$ preserves monotonicity and (ordinary) convexity.

The general result of Păltănea mentioned already above entails

Theorem 4. *For $f \in C[0,1], x \in [0,1]$ and $n \in \mathbb{N}$ there holds*

$$|\overline{\mathbb{B}}_n(f;x) - f(x)| \leq \frac{3}{2}\,\omega_2\left(f; \sqrt{\frac{x(1-x)}{n+1}}\right).$$

5. The Operators $\mathbb{G}_n = \overline{\mathbb{B}}_n \circ S_{\Delta_n}$

The original question leading to this article was if it is possible to decompose the classical Bernstein operator B_n into non-trivial building blocks P and Q, i.e., $B_n = P \circ Q$.

Recall that for functions $f : [0,1] \to \mathbb{R}$, B_n is given by

$$B_n(f;x) = \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big) p_{n,k}(x) = \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big)\binom{n}{k} x^k (1-x)^{n-k}, \qquad x \in [0,1].$$

It is known that if one composes two positive linear operators P and Q, both reproducing linear functions, then for the second moment of the product operator one has

$$(P \circ Q)((e_1 - x)^2; x) = P^u\big(Q((e_1 - u)^2; u); x\big) + P((e_1 - x)^2; x).$$

Here the superscript in P^u indicates that the operator P is applied to functions in the variable u. Other results concerning the moments of the product operator can be found in [9].

Putting $P = \overline{\mathbb{B}}_n$ the question then was if there is another positive linear operator Q such that $\overline{\mathbb{B}}_n \circ Q = B_n$ and, in particular,

$$\begin{aligned}(\overline{\mathbb{B}}_n \circ Q)((e_1 - x)^2; x) = B_n((e_1 - x)^2; x) &= \frac{x(1-x)}{n}\\ &= \overline{\mathbb{B}}_n^u(Q((e_1 - u)^2; u)x) + \overline{\mathbb{B}}((e_1 - x)^2; x)\\ &= \overline{\mathbb{B}}_n^u(Q((e_1 - u)^2; x) + \frac{x(1-x)}{n+1}.\end{aligned}$$

Natural candidates for Q are operators of the form

$$Q(f; x) = \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big) r_{n,k}(x),$$

with $r_{n,k} \geq 0$, $x \in [0,1]$, $0 \leq k \leq n$, so that

$$(\overline{\mathbb{B}}_n \circ Q)(f; x) = \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big) \overline{\mathbb{B}}_n(r_{n,k}, x)$$

would become the Bernstein operator if $r_{n,k}$ could be chosen in a way such that

$$\overline{\mathbb{B}}_n(r_{n,k}; x) = p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad x \in [0,1], \quad 0 \leq k \leq n.$$

However, it was shown in [5, Section 5] that there is **no** positive linear operator $Q : C[0,1] \to \pi_n$ such that $B_n = \overline{\mathbb{B}}_n \circ Q$. But if one gives up the requirement of positivity, then an operator $F_n : C[0,1] \to \pi_n$ exists such that $B_n = \overline{\mathbb{B}}_n \circ F_n$ holds. For much more on these mappings see [5, 14, 13].

The first approach to find a decomposition of B_n used $Q = S_{\Delta_n}$. Here S_{Δ_n} is also a positive linear operator reproducing linear functions and preserving monotonicity and convexity/concavity. Moreover, it is of the appropriate form and hence it made sense to consider $\mathbb{G}_n := \overline{\mathbb{B}}_n \circ S_{\Delta_n}$, i.e., $\mathbb{G}_n : C[0,1] \to C[0,1]$, where

$$\mathbb{G}_n(f; 0) = S_{\Delta_n}(f; 0) = f(0), \qquad \mathbb{G}_n(f; 1) = S_{\Delta_n}(f; 1) = f(1),$$

and, for $x \in (0,1)$,

$$\mathbb{G}_n(f; x) = \frac{1}{B(nx, n(1-x))} \int_0^1 t^{nx-1}(1-t)^{n(1-x)-1} S_{\Delta_n}(f; t)\, dt.$$

The operator $\mathbb{G}_n$ is again positive and linear. As the composition of two operators preserving monotonicity and convexity, $\mathbb{G}_n$ also has these properties.

For a convex function g it is well-known that $g \le B_n g$. Now if $f \in C[0,1]$ is convex, then this is also true for $S_{\Delta_n} f$, so that

$$f \le S_{\Delta_n} f \le B_n(S_{\Delta_n} f) = B_n f,$$

implying

$$\overline{\mathbb{B}}_n f \le (\overline{\mathbb{B}}_n \circ S_{\Delta_n}) f = \mathbb{G}_n f \le (\overline{\mathbb{B}}_n \circ B_n) f = L_n f,$$

where L_n is a special case of the Stancu operator introduced in [22], namely for the case $\alpha = \frac{1}{n}$. It is given by (see [17])

$$L_n(f;x) = \frac{2(n!)}{(2n)!} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} (nx)_k (n-nx)_{n-k},$$

where

$$(a)_0 = 1, \qquad (a)_b = \prod_{k=0}^{b-1} (a-k), \qquad a \in \mathbb{R}, \quad b \in \mathbb{N}.$$

In particular,

$$\overline{\mathbb{B}}_n((e_1-x)^2;x) = \frac{x(1-x)}{n+1} \le \mathbb{G}_n((e_1-x)^2;x) \le L_n((e_1-x)^2;x) = \frac{2x(1-x)}{n+1}.$$

More generally, for $j \in \mathbb{N}_0$,

$$\mathbb{B}_n(|e_1-x|^j;x) \le \mathbb{G}_n(|e_1-x|^j;x) \le L_n(|e_1-x|^j;x).$$

Since the second moments of both $\mathbb{G}_n$ and B_n lie between $\frac{x(1-x)}{n+1}$ and $\frac{2x(1-x)}{n+1}$, there still is a chance that $\mathbb{G}_n = B_n$. However, in the next section we will show that $\mathbb{G}_2 \ne B_2$. We will also demonstrate that it is impossible to write $B_n = L \circ S_{\Delta_n}$ for a large class of positive integral operators.

6. Two Negative Results

Proposition 1. *If $\mathbb{G}_2$ and the Bernstein operator B_2 are given as above, then $\mathbb{G}_2 \ne B_2$.*

Proof. Indeed,

$$\mathbb{G}_2 f = \sum_{i=0}^{2} f\left(\frac{i}{2}\right) \overline{\mathbb{B}}_2 u_i, \qquad f \in C[0,1],$$

where $u_i \in C[0,1]$ is the piecewise linear function with $u_i\left(\frac{j}{2}\right) = \delta_{ij}$ for $i, j \in \{0,1,2\}$.

Suppose that $\mathbb{G}_2 = B_2$. Then $\overline{\mathbb{B}}_2 u_i = p_{2,i}$, $i = 0,1,2$. In particular, $\overline{\mathbb{B}}_2 u_2(x) = x^2$, $x \in [0,1]$, which leads to

$$\frac{\int_{\frac{1}{2}}^1 t^{2x-1}(1-t)^{1-2x}(2t-1)\,dt}{B(2x, 2(1-x))} = x^2, \qquad x \in (0,1).$$

For $x = \frac{1}{4}$ we get

$$\frac{\int_{\frac{1}{2}}^1 t^{-1/2}(1-t)^{1/2}(2t-1)\,dt}{\int_0^1 t^{-1/2}(1-t)^{1/2}\,dt} = \frac{1}{16}. \tag{6}$$

On $(0,1)$,

$$\int t^{-1/2}(1-t)^{1/2}(2t-1)\,dt = -\frac{1}{4}\big((6-4t)\sqrt{t(1-t)} + \arcsin(2t-1)\big)$$

and

$$\int t^{-1/2}(1-t)^{1/2} dt = \sqrt{t(1-t)} + \frac{1}{2}\,\arcsin(2t-1).$$

Now (6) becomes

$$\frac{\frac{1}{2} - \frac{\pi}{8}}{\frac{\pi}{2}} = \frac{1}{16},$$

i.e., $\pi = \frac{16}{5}$, a contradiction. This proves $\mathbb{G}_2 \neq B_2$. □

Proposition 2. *It is not possible to write* $B_n = L \circ S_{\Delta_n}$, $n \geq 2$, *for a large class of integral operators* L *.*

Proof. The operator $S_{\Delta_n} : C[0,1] \to C[0,1]$ can be described as in Section 2, or as

$$S_{\Delta_n} f(x) = \sum_{i=0}^n f\Big(\frac{i}{n}\Big) u_{n,i}(x), \qquad f \in C[0,1], \quad x \in [0,1],$$

where $u_{n,i} \in C[0,1]$ are piecewise linear functions such that $u_{n,i}\big(\frac{j}{n}\big) = \delta_{i,j}$, $i,j = 0,\ldots,n$.

Let $L : C[0,1] \to C[0,1]$ be an integral operator,

$$L(f;x) := \int_0^1 K(x,t) f(t)\,dt, \qquad f \in C[0,1], \quad x \in [0,1],$$

where the kernel K is non-negative on $[0,1]^2$ and $K(x,\cdot) \in L_1[0,1]$ for all $x \in [0,1]$. We shall prove that $L \circ S_{\Delta_n} \neq B_n$, $n \geq 2$.

Suppose that for a given $n \geq 2$ we have $L \circ S_{\Delta_n} = B_n$. Then

$$\sum_{i=0}^n L(u_{n,i};x) f\Big(\frac{i}{n}\Big) = \sum_{i=0}^n p_{n,i}(x) f\Big(\frac{i}{n}\Big), \qquad f \in C[0,1],$$

which entails

$$L(u_{n,i};x) = p_{n,i}(x), \qquad x \in [0,1], \quad i = 1, \dots, n.$$

In particular, $L(u_{n,i};0) = 0$, $i = 1, \dots, n$, and so we get

$$\int_0^1 K(0,t) u_{n,i}(t)\, dt = L(u_{n,i};0) = 0, \qquad i = 1, \dots, n.$$

It follows that

$$\int_0^1 K(0,t) \left(\sum_{i=1}^n u_{n,i}(t) \right) dt = 0.$$

But $\sum_{i=1}^n u_{n,i}(t) = 1 - u_{n,0}(t) > 0$, for all $t \in (0,1]$. We deduce that $K(0,\,\cdot\,) = 0$ a.e. on $[0,1]$, and so

$$L(e_0;0) = \int_0^1 K(0,t)\, dt = 0. \tag{7}$$

On the other hand,

$$L(e_0;0) = (L \circ S_{\Delta_n})(e_0;0) = B_n(e_0;0) = 1,$$

which contradicts (7). Thus, in fact, $L \circ S_{\Delta_n} \neq B_n$. □

Remark 3. Further compositions of S_{Δ_n} with other positive linear operators can be found in [15] and some of the papers cited there.

7. More on $\mathbb{G}_n$

Some notation/facts used below will be

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \qquad p, q \in \mathbb{R}, \quad p, q > 0,$$

$$b_{p,q}(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \dfrac{t^{p-1}(1-t)^{q-1}}{B(p,q)}, & t \in (0,1), \\ 0, & t \in [1, \infty). \end{cases}$$

$$I_\lambda(a,b) = \frac{1}{B(a,b)} \int_0^\lambda t^{a-1}(1-t)^{b-1}\, dt$$

is the regularized Beta function and $\lambda \mapsto \int_0^\lambda t^{a-1}(1-t)^{b-1}\, dt$ is the incomplete Beta function of argument λ and parameters a, b.

Let

$$I_{k,n}(x) := \int_0^1 b_{nx,n-nx}(t)|nt-k|\,dt, \qquad k = 1,\dots,n-1.$$

By using (4) we get

$$\begin{aligned}(\mathbb{G}_n f)(x) = {} & \frac{(1-nx)f(0)+nxf\big(\frac{1}{n}\big)}{2} + \frac{n(1-x)f\big(1-\frac{1}{n}\big)+(nx-n+1)f(1)}{2} \\ & + \frac{1}{n^2}\sum_{k=1}^{n-1}\Big[\frac{k-1}{n},\frac{k}{n},\frac{k+1}{n};f\Big]I_{k,n}(x), \qquad x\in(0,1), \\ & (\mathbb{G}_n f)(0) = f(0), \qquad\qquad (\mathbb{G}_n f)(1) = f(1).\end{aligned}$$

The next result provides a representation of $I_{k,n}(x)$, and hence of $(\mathbb{G}_n f)(x)$, in terms of the regularized Beta function.

Corollary 1. *For $x \in (0,1)$ we find*

$$\begin{aligned}I_{k,n}(x) &:= \int_0^1 b_{nx,n-nx}(t)|nt-k|\,dt \\ &= nx-k+2\,\frac{k}{n}\Big(1-\frac{k}{n}\Big)b_{nx+1,n-nx}\Big(\frac{k}{n}\Big) \\ &\quad -2(nx-k)I_{\frac{k}{n}}(nx+1,n-nx).\end{aligned}$$

Proof. Since

$$\int_0^1 b_{nx,n-nx}(t)\,dt = 1 \quad \text{and} \quad \int_0^1 tb_{nx,n-nx}(t)\,dt = x,$$

we get

$$I_{k,n}(x) = nx-k+2k\int_0^{\frac{k}{n}} b_{nx,n-nx}(t)\,dt - 2n\int_0^{\frac{k}{n}} tb_{nx,n-nx}(t)\,dt. \tag{8}$$

Let us remark that

$$\frac{d}{dt}(t^{nx}(1-t)^{n-nx}) = n(x-t)t^{nx-1}(1-t)^{n-nx-1}.$$

This implies

$$\begin{aligned}x\int_0^{\frac{k}{n}} t^{nx-1}(1-t)^{n-nx-1}dt = {} & \int_0^{\frac{k}{n}} t^{nx}(1-t)^{n-nx-1}\,dt \\ & + \frac{1}{n}\Big(\frac{k}{n}\Big)^{nx}\Big(1-\frac{k}{n}\Big)^{n-nx}.\end{aligned} \tag{9}$$

We have also

$$B(nx+1,n-nx) = xB(nx,n-nx). \tag{10}$$

From (8), (9) and (10) we derive

$$\begin{aligned} I_{k,n}(x) &= nx - k + \frac{2k}{B(nx+1,n-nx)} \\ &\quad \times \left(\int_0^{\frac{k}{n}} t^{nx}(1-t)^{n-nx-1}\,dt + \frac{1}{n}\left(\frac{k}{n}\right)^{nx}\left(1-\frac{k}{n}\right)^{n-nx} \right) \\ &\quad - \frac{2nx}{B(nx+1,n-nx)} \int_0^{\frac{k}{n}} t^{nx}(1-t)^{n-nx-1}\,dt \\ &= nx - k + 2\frac{k-nx}{B(nx+1,n-nx)} \int_0^{\frac{k}{n}} t^{nx}(1-t)^{n-nx-1}\,dt \\ &\quad + \frac{2k}{n}\left(\frac{k}{n}\right)^{nx}\left(1-\frac{k}{n}\right)^{n-nx} \frac{1}{B(nx+1,n-nx)} \\ &= nx - k - 2(nx-k) I_{\frac{k}{n}}(nx+1,n-nx) \\ &\quad + 2\,\frac{k}{n}\left(1-\frac{k}{n}\right) b_{nx+1,n-nx}\left(\frac{k}{n}\right), \end{aligned}$$

and this concludes the proof. □

8. Degree of Approximation by $\mathbb{G}_n$

As mentioned above we have

$$\overline{\mathbb{B}}_n((e_1-x)^2;x) = \frac{x(1-x)}{n+1} \le \mathbb{G}_n((e_1-x)^2;x) \le L_n((e_1-x)^2;x) = \frac{2x(1-x)}{n+1}.$$

For the second moment of B_n a similar inequality holds for $n \ge 1$:

$$\overline{\mathbb{B}}_n((e_1-x)^2;x) = \frac{x(1-x)}{n+1} \le \frac{x(1-x)}{n} = B_n((e_1-x)^2;x) \le \frac{2x(1-x)}{n+1}.$$

So at this moment we know that

$$\frac{x(1-x)}{n+1} \le j_n(x)\,\frac{x(1-x)}{n+1} = \mathbb{G}_n((e_1-x)^2;x) \le 2\,\frac{x(1-x)}{n+1}$$

with functions j_n such that $1 \le j_n(x) \le 2$, $x \in [0,1]$.

Păltănea's result from above implies

Theorem 5. *For $f \in C[0,1]$, $x \in [0,1]$, $n \in \mathbb{N}$ we have the inequality*

$$|\mathbb{G}_n(f;x) - f(x)| \le 2\omega_2\left(f; \sqrt{\frac{x(1-x)}{n+1}}\right).$$

We now give a Voronovskaya-type result using the least concave majorant of the first order modulus of continuity. Example 4.3 in [4] implies

$$\begin{aligned}
&\Big|\mathbb{G}_n(f;x) - f(x) - \frac{1}{2}\, j_n(x)\, \frac{x(1-x)}{n} f''(x)\Big| \\
&\le \frac{1}{2}\, \mathbb{G}_n((e_1-x)^2;x)\, \tilde{\omega}\Big(f'', \frac{1}{3}\, \frac{\mathbb{G}_n(|e_1-x|^3;x)}{\mathbb{G}_n((e_1-x)^2;x)}\Big) \\
&\le \frac{1}{2}\, \frac{2x(1-x)}{n+1}\, \tilde{\omega}\Big(f''; \frac{1}{3}\, \frac{L_n(|e_1-x|^3;x)}{\overline{\mathbb{B}}_n((e_1-x)^2;)}\Big) \\
&\le \frac{x(1-x)}{n+1}\, \tilde{\omega}\left(f''; \frac{1}{3}\, \frac{\sqrt{L_n((e_1-x)^2;x)\, L_n((e_1-x)^4;x)}}{x(1-x)/(n+1)}\right) \\
&= \frac{x(1-x)}{n+1}\, \tilde{\omega}\Bigg(f''; \frac{n+1}{3x(1-x)}\, \sqrt{2}\, \sqrt{\frac{x(1-x)}{n+1}} \\
&\qquad \times \sqrt{\frac{1}{(n+1)^4}\big[12(n^2-7n)x^2(1-x)^2 + (26n-2)x(1-x)\big]}\Bigg) \\
&\le \frac{x(1-x)}{n+1}\, \tilde{\omega}\left(f''; \frac{\sqrt{2}}{3}\, \sqrt{\frac{1}{(n+1)^3}\, 5n(n+1)}\right) \\
&\le \frac{x(1-x)}{n+1}\, \tilde{\omega}\left(f''; \frac{\sqrt{2}}{3}\, \sqrt{\frac{5}{n+1}}\right) \\
&\le \frac{x(1-x)}{n+1}\, \tilde{\omega}\Big(f''; \frac{2}{\sqrt{n+1}}\Big).
\end{aligned}$$

We thus have

Proposition 3. *For $f \in C^2[0,1]$, $x \in [0,1]$ and $n \in \mathbb{N}$ there holds:*

$$\Big|\mathbb{G}_n(f;x) - f(x) - \frac{1}{2}\, j_n(x)\, \frac{x(1-x)}{n}\, f''(x)\Big| \le \frac{x(1-x)}{n+1}\, \tilde{\omega}\Big(f'', \frac{2}{\sqrt{n+1}}\Big).$$

9. The Difference $B_n - \mathbb{G}_n$

As mentioned earlier, we were interested in decomposing the Bernstein operator B_n and tried to write it as $\overline{\mathbb{B}}_n \circ S_{\Delta_n} = \mathbb{G}_n$. But this is not possible as demonstrated above. Here we investigate the difference $B_n - \mathbb{G}_n$. To this end we proceed as in [10].

Putting $L := B_n - \mathbb{G}_n$ we consider, for $f \in C[0,1]$, $x \in [0,1]$, $L(f;x) = L(f-g;x) + L(g;x)$, $g \in C^2[0,1]$ arbitrary.

Here, $|L(f-g;x)| \le \|B_n - \mathbb{G}_n\|\, \|f-g\|_\infty \le 2\|f-g\|_\infty$.

Moreover,

$$\begin{aligned}
|L(g;x)| &\le |B_n(g;x)-g(x)| + |\mathbb{G}_n(g;x)-g(x)|\\
&\le \Big[\frac{1}{2}\,B_n((e_1-x)^2;x) + \frac{1}{2}\,\mathbb{G}_n((e_1-x)^2;x)\Big]\|g''\|_\infty\\
&\le \frac{1}{2}\Big[\frac{x(1-x)}{n} + \frac{2x(1-x)}{n+1}\Big]\|g''\|_\infty\\
&\le \frac{3}{2}\,\frac{x(1-x)}{n}\,\|g''\|_\infty .
\end{aligned}$$

Choosing g such that (see [6])

$$\begin{aligned}
\|f-g\|_\infty &\le \frac{3}{4}\,\omega_2(f;h),\\
\|g''\|_\infty &\le \frac{3}{2}\,h^{-2}\,\omega_2(f;h), \qquad 0<h\le\frac{1}{2},
\end{aligned}$$

and putting $h=\sqrt{\frac{x(1-x)}{n}}$, $x\in(0,1)$, gives

$$\begin{aligned}
|L(f;x)| &\le 2\left(\|f-g\|_\infty + \frac{3}{4}\,\frac{x(1-x)}{n}\,\|g''\|_\infty\right)\\
&\le 2\left(\frac{3}{4}\,\omega_2(f;h) + \frac{9}{8}\,\frac{x(1-x)}{n}\,\frac{1}{h^2}\,\omega_2(f;h)\right)\\
&= \frac{15}{4}\,\omega_2\left(f;\sqrt{\frac{x(1-x)}{n}}\right).
\end{aligned}$$

We thus know

Proposition 4. *For $f\in C[0,1], x\in[0,1]$ and $n\in\mathbb{N}$ there holds*

$$|B_n(f;x)-\mathbb{G}_n(f;x)| \le \frac{15}{4}\,\omega_2\left(f;\sqrt{\frac{x(1-x)}{n}}\right).$$

Proof. For $x\in(0,1)$ the inequality was shown above. For $x\in\{0,1\}$ it is trivial. □

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