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An Exact Pointwise Markov Inequality on the Standard Triangle*

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An actual field of research in the theory of approximations is to extend the classical polynomial inequalities to multivariate setting. This question concerning Markov inequality is satisfactorily settled for centrally symmetric bodies. On the other side, the problem to find exact pointwise Markov type inequalities for multivariate polynomials is studied only for homogeneous polynomials of degree two.

In the present paper we prove a sharp pointwise estimate of Markov type for a set of strictly definite polynomials on the standard simplex Δ in $\mathbb{R}^2$. Our approach is based on the explicit description of the stricly definite extreme points of the unit ball of the space of second degree polynomials, endowed with the supremum norm on Δ, which was given recently by the authors.

Keywords and Phrases: Markov type inequalities, polynomials, convexity, extreme points.

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1. Introduction

Denote by π_n^d the set of all real algebraic polynomials of d variables and of total degree not exceeding n. In what follows, $|X|_2$ stands for the Euclidean norm of $X \in \mathbb{R}^d$. Let K be a compact set in $\mathbb{R}^d$ and $\|f\|_{C(K)} := \max_{X\in K} |f(X)|$ be the uniform norm on K. We use the notation $B_n(K)$ for the unit ball of π_n^d with respect to $\|\cdot\|_{C(K)}$, i.e. $B_n(K) = \{p \in \pi_n^d \,:\, \|p\|_{C(K)} \le 1\}$.

The classical inequality of A. Markov [4] states

$$\|p'\|_{C[a,b]} \le \frac{2n^2}{b-a}\,\|p\|_{C[a,b]}, \quad \text{for every} \;\; p \in \pi_n^1. \tag{1}$$

It has numerous applications in the theory of approximations.

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A challenging problem is to extend Markov's inequality to convex bodies in $\mathbb{R}^d$. The first exact result in this direction belongs to Kellogg [2] and concerns the case $K = B^d := \{X \in \mathbb{R}^d : |X|_2 \leq 1\}$. A more general inequality was established by Sarantopoulos [9]. It has the form

$$\| |\mathrm{grad}\, p|_2 \|_{C(K)} \leq \frac{2n^2}{w(K)} \|p\|_{C(K)}, \quad \text{for every} \quad p \in \pi_n^d,$$

where K is an arbitrary convex and central symmetric body in $\mathbb{R}^d$ and $w(K)$ denotes the width of K, i.e. the minimal distance between two parallel supporting hyperplanes for K. A similar inequality, which is sharp, in general, for the class of all convex bodies in $\mathbb{R}^d$ was proved by Skalyga [10]. The reader can find more details and additional references in the paper of Kroó [3].

Note that A. Markov proved (1) by a careful study of the pointwise upper bound

$$M_n(x) := \max_{p \in B_n([a,b])} |p'(x)|, \qquad x \in [a,b].$$

Moreover, A. Markov gave explicit expressions for $M_n(x)$, $n = 2, 3$. This motivates the problem to find exact pointwise Markov type inequalities for multivariate polynomials. To the best of our knowledge such inequalities are established only for homogeneous polynomials of degree two, see [8, 1].

The goal of the present paper is to prove a sharp pointwise estimate of Markov type for a set of bivariate polynomials on the standard simplex.

We shall need an additional notation. Recall that a point p of a convex set B is said to be *extreme* if the equality $p = \lambda p_1 + (1-\lambda)p_2$ for some $p_1, p_2 \in B$ and $\lambda \in (0,1)$ implies $p = p_1 = p_2$. The set of all extreme points of $B_n(K)$ will be denoted by $E_n(K)$. By the theorem of Krein-Milman, $B_n(K)$ is the convex hull of $E_n(K)$. As a consequence we have the equality

$$\max_{p \in B_n(K)} f(p) = \max_{p \in E_n(K)} f(p),$$

provided f is a convex function defined on $B_n(K)$.

Let Δ be the standard simplex in $\mathbb{R}^2$, i.e.

$$\Delta := \big\{(x,y) \in \mathbb{R}^2 \,:\, x \geq 0,\ y \geq 0,\ x + y \leq 1\big\}.$$

In [5, 6, 7] we described the set $E_2(\Delta)$. It consists of three parts E_I, E_{II} and E_{III}, which contain the strictly definite, indefinite and semi-definite elements, respectively. Explicit formulas for the elements of E_I are given in the beginning of Section 2.

Next we formulate our main result, which is an exact pointwise inequality of Markov type for $\mathrm{conv}\, E_I$, the convex hull of E_I.

Theorem 1. *For every $p \in \mathrm{conv}\, E_I$ and $(x,y) \in \Delta$ the inequality*

$$|\mathrm{grad}\ p(x,y)|_2 \leq 16 \max\left\{ \left|x - \frac{1}{2}\right|,\ \left|y - \frac{1}{2}\right|,\ \sqrt{2}\left|x + y - \frac{1}{2}\right| \right\} \tag{2}$$

holds true. Moreover, for every $(x,y) \in \Delta$ the inequality (2) *is sharp.*

Remark 1. Let us consider the polynomials $p_1(x,y) = 1 - 8\big(x - \frac{1}{2}\big)^2$, $p_2(x,y) = 1 - 8\big(y - \frac{1}{2}\big)^2$, and $p_3(x,y) = 1 - 8\big(x + y - \frac{1}{2}\big)^2$, which belong to ∂E_I. It can be checked that $|\mathrm{grad}\; p_1(x,y)|_2 = 16\big|x - \frac{1}{2}\big|$, $|\mathrm{grad}\; p_2(x,y)|_2 = 16\big|y - \frac{1}{2}\big|$, and $|\mathrm{grad}\; p_3(x,y)|_2 = 16\sqrt{2}\,\big|x + y - \frac{1}{2}\big|$. This implies the sharpness of (2) for every $(x,y) \in \Delta$.

2. Proof of Theorem 1

We shall introduce some additional notations. Let

$$M(x,y) := \sup_{p \in E_I} |\mathrm{grad}\; p(x,y)|_2$$

be the Markov's majorant for E_I. According to [5] the strictly concave elements of E_I have the form

$$P(x,y) := P_{x_0,y_0}(x,y) = 1 + \alpha(x - x_0)^2 + 2\beta(x - x_0)(y - y_0) + \gamma(y - y_0)^2, \quad (3)$$

where

$$\alpha = \frac{2(2y_0 - 1)}{x_0(1 - x_0 - y_0)}, \quad \beta = -\frac{(1 - 2x_0)(1 - 2y_0)}{x_0 y_0 (1 - x_0 - y_0)}, \quad \gamma = \frac{2(2x_0 - 1)}{y_0(1 - x_0 - y_0)}, \quad (4)$$

and the point (x_0, y_0) belongs to the interior of the triangle Δ_1 with vertices $O_1 = \big(\frac{1}{2}, \frac{1}{2}\big)$, $A_1 = \big(0, \frac{1}{2}\big)$, and $B_1 = \big(\frac{1}{2}, 0\big)$. All strictly convex elements of $E_2(\Delta)$ have the form $Q = -P$, where P is given by (3) and (4).

The above representation gives a parametric expression for the Markov's majorant:

$$M(x,y) = \sup_{(x_0,y_0) \in \mathrm{int}\, \Delta_1} |\mathrm{grad}\; P_{x_0,y_0}(x,y)|_2.$$

We set for convenience

$$\begin{aligned} K(x_0, y_0; x, y) &:= \frac{1}{4}\, |\mathrm{grad}\; P_{x_0,y_0}(x,y)|_2^2 \\ &= \big[\alpha(x - x_0) + \beta(y - y_0)\big]^2 + \big[\beta(x - x_0) + \gamma(y - y_0)\big]^2. \end{aligned}$$

Given $(x,y) \in \Delta$, we set

$$B(x,y) := \max_{(x_0,y_0) \in \partial^\circ \Delta_1} K(x_0, y_0; x, y),$$

where $\partial^\circ \Delta_1 = \partial \Delta_1 \setminus \{O_1, A_1, B_1\}$. Note that $K(x_0, y_0; x, y)$ is not defined for the vertices of Δ_1.

The following lemma provides an explicit expression for $B(x,y)$ depending on the location of the point (x,y).

Lemma 1. *We have*

$$B(x,y)=\begin{cases}64\big(x-\frac12\big)^2, & (x,y)\in QRTS=:D_1;\\ 64\big(y-\frac12\big)^2, & (x,y)\in VRTU=:D_2;\\ 128\big(x+y-\frac12\big)^2, & (x,y)\in ABSTU\cup OVRQ=:D_3,\end{cases}$$

where $Q=\big(0,\frac{\sqrt2-1}{2\sqrt2}\big)$, $R=\big(\frac{3-\sqrt2}{14},\frac{3-\sqrt2}{14}\big)$, $T=\big(\frac{3+\sqrt2}{14},\frac{3+\sqrt2}{14}\big)$, $S=\big(0,\frac{\sqrt2+1}{2\sqrt2}\big)$, $V=\big(\frac{\sqrt2-1}{2\sqrt2},0\big)$, $U=\big(\frac{\sqrt2+1}{2\sqrt2},0\big)$.

Proof. We have $(A_1O_1)=\{\big(x_0,\frac12\big):x_0\in\big(0,\frac12\big)\}$. Then $\alpha=\beta=0$, $\gamma=-8$ and $K\big(x_0,\frac12;x,y\big)=64\big(y-\frac12\big)^2$. Similarly, $(B_1O_1)=\{\big(\frac12,y_0\big):y_0\in\big(0,\frac12\big)\}$, so that $\beta=\gamma=0$, $\alpha=-8$ and $K\big(\frac12,y_0;x,y\big)=64\big(x-\frac12\big)^2$. For $(x_0,y_0)\in(A_1B_1)$ we get $\alpha=\beta=\gamma=-8$ and $K(x_0,y_0;x,y)=128\big(x+y-\frac12\big)^2$. Therefore

$$B(x,y)=64\max\Big\{\Big(x-\frac12\Big)^2,\ \Big(y-\frac12\Big)^2,\ 2\Big(x+y-\frac12\Big)^2\Big\}.$$

Now a comparison between the functions $\big(x-\frac12\big)^2$, $\big(y-\frac12\big)^2$, and $2\big(x+y-\frac12\big)^2$ for $(x,y)\in\Delta$ gives the result. Lemma 1 is proved. $\square$

It follows from the proof of Lemma 1 that the extremal polynomial for D_i, $i=1,2,3$, is p_i, defined in Theorem 1 and Remark 1, see Figure 1.

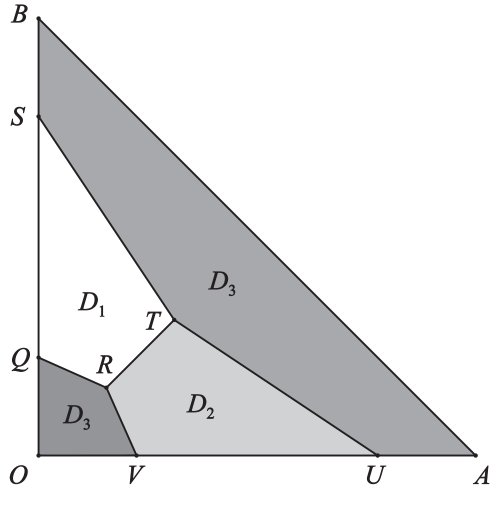

Figure 1. The domains where $p_i,\ i=1,2,3$ are extremal.

Lemma 2. *For every* $(x_0,y_0)\in\operatorname{int}\Delta_1$ *and* $(x,y)\in D_1$ *we have*

$$K(x_0,y_0;x,y)\le 64\Big(x-\frac12\Big)^2.$$

Proof. Substituting the values for α, β, γ from (4), it is sufficient to prove the inequality

$$\begin{aligned} f(x_0, y_0; x, y) := \big\{&(2y_0-1)^2[2y_0(x-x_0)+(1-2x_0)(y-y_0)]^2 \\ &+ (2x_0-1)^2[(1-2y_0)(x-x_0)+2x_0(y-y_0)]^2\big\} \\ &- 64x_0^2y_0^2(1-x_0-y_0)^2\Big(x-\frac{1}{2}\Big)^2 \le 0, \end{aligned} \tag{5}$$

for all $(x_0, y_0) \in \Delta_1$ and $(x, y) \in D_1$.

Note that f is a polynomial from π_2^1 with respect to y, which has a positive leading coefficient. Hence it is sufficient to prove that

$$f|_{\partial D_1} \le 0. \tag{6}$$

1. Let us consider first the segment

$$[QS] = \{(0, y) \,:\, y \in I_1\},$$

where $I_1 = [y_1, y_2]$, $y_1 = \frac{\sqrt{2}-1}{2\sqrt{2}}$, $y_2 = \frac{\sqrt{2}+1}{2\sqrt{2}}$. The corresponding restriction of f is

$$\begin{aligned} g(y) &:= f(x_0, y_0; 0, y) \\ &= (1-2x_0)^2[(1-2y_0)^2+4x_0^2](y-y_0)^2 \\ &\quad - 4(1-2x_0)(1-2y_0)[y_0(1-2y_0)+x_0(1-2x_0)]x_0(y-y_0) \\ &\quad + (1-2y_0)^2[4y_0^2+(1-2x_0)^2]x_0^2 - 16x_0^2y_0^2(1-x_0-y_0)^2. \end{aligned}$$

We have to prove that $g(y) \le 0$ for all $y \in I_1$ and $(x_0, y_0) \in \operatorname{int}\Delta_1$. In view of convexity of g it suffices to prove $g(y_1) \le 0$ and $g(y_2) \le 0$.

1.1. $g(y_2) \le 0$. We have

$$g(y_2) = \frac{1}{8}\,(1-2x_0)(1-2x_0-2y_0)\,q(x_0, y_0),$$

where

$$\begin{aligned} q(x_0, y_0) := 3 &+ 2\sqrt{2} + 4x_0^2 - 14y_0 - 8\sqrt{2}\,y_0 - 4x_0y_0 + 24y_0^2 \\ &+ 8\sqrt{2}\,y_0^2 + 32x_0y_0^2 - 32x_0^2y_0^2 - 16y_0^3 - 32x_0y_0^3. \end{aligned}$$

Since $1-2x_0 \ge 0$ and $1-2x_0-2y_0 \le 0$, we have to prove $q(x_0, y_0) \ge 0$ for every $(x_0, y_0) \in \Delta_1$.

The restriction $q|_{[A_1O_1]} = 2x_0(1-2x_0) \ge 0$. Therefore, it suffices to prove

$$\frac{\partial q}{\partial y_0} = -14-8\sqrt{2}-4x_0+48y_0+16\sqrt{2}\,y_0+64x_0y_0-64x_0^2y_0-48y_0^2-96x_0y_0^2 < 0, \tag{7}$$

for every $(x_0, y_0) \in \Delta_1$.

We set $\frac{\partial q}{\partial y_0} = a+b+c$, where $a = -14+48y_0-48y_0^2$, $b = -8\sqrt{2}+16\sqrt{2}\,y_0$, and $c = 64x_0y_0 - 64x_0^2y_0 - 96x_0y_0^2 - 4x_0$. It is easy to see that $a \le -2$ and $b \le 0$. Next we shall prove that $c \le 1$, which will finish the proof of (7).

The partial derivatives of c are:

$$\frac{\partial c}{\partial x_0} = 4(16y_0 - 32x_0y_0 - 24y_0^2 - 1), \qquad \frac{\partial c}{\partial y_0} = 64x_0(1 - x_0 - 3y_0).$$

The solutions of the system $\frac{\partial c}{\partial x_0} = 0$, $\frac{\partial c}{\partial y_0} = 0$ are

$$\Big(0, \frac{4 \pm \sqrt{10}}{12}\Big) \text{ and } \Big(\frac{8 \pm \sqrt{34}}{12}, \frac{4 \mp \sqrt{34}}{12}\Big).$$

None of them belongs to $\operatorname{int}\Delta_1$, hence

$$\max_{(x_0,y_0)\in\Delta_1} c(x_0,y_0) = \max_{(x_0,y_0)\in\partial\Delta_1} c(x_0,y_0).$$

We have $c|_{[A_1O_1]} = 4x_0(1-8x_0) < 1$, $c|_{[B_1O_1]} = -2(24y_0^2 - 8y_0 + 1) < 0$. Consider now the restriction $c|_{[A_1B_1]} = c\big(x_0, \frac{1}{2} - x_0\big) = -24x_0^3 + 4x_0$ for $x_0 \in \big[0, \frac{1}{2}\big]$. It is easily seen that

$$c|_{[A_1B_1]} \le (-24x_0^3 + 4x_0)|_{x_0 = \frac{1}{3\sqrt{2}}} = \frac{8}{9\sqrt{2}} < 1.$$

This finishes the proof of *1.1.*

1.2. $g(y_1) \le 0$. We have

$$g(y_1) = -\frac{1}{8}(1-2x_0)(1-2x_0-2y_0)\,q(x_0,y_0),$$

where

$$\begin{aligned} q(x_0,y_0) := {} & -3 + 2\sqrt{2} - 4x_0^2 + 14y_0 - 8\sqrt{2}\,y_0 + 4x_0y_0 - 24y_0^2 \\ & + 8\sqrt{2}\,y_0^2 - 32x_0y_0^2 + 32x_0^2y_0^2 + 16y_0^3 + 32x_0y_0^3. \end{aligned}$$

Arguing as above in *1.1*, we have to prove that $q(x_0,y_0) \le 0$ for every $(x_0,y_0) \in \Delta_1$. Note that q is a second degree polynomial with respect to x_0. More precisely, $q = a(y_0)x_0^2 + b(y_0)x_0 + c(y_0)$, where $a(y_0) = 4(8y_0^2 - 1)$ and $b(y_0) = 4y_0(1 - 8y_0 + 8y_0^2)$. (The explicit expression for $c(y_0)$ is not needed.) Clearly, for $y_0 \in \big[0, \frac{1}{2}\big]$, $a \ge 0$ if and only if $y_0 \in \big[\frac{1}{2\sqrt{2}}, \frac{1}{2}\big]$. Consequently, $q = q(x_0)$ is a convex function for every fixed $y_0 \in \big[\frac{1}{2\sqrt{2}}, \frac{1}{2}\big]$.

Suppose now that $y_0 \in \big[0, \frac{1}{2\sqrt{2}}\big)$. The only critical point of $q(x_0)$ is

$$x_0^* = \frac{y_0(1 - 8y_0 + 8y_0^2)}{2(1 - 8y_0^2)}.$$

We claim that $x_0^* < \frac{1}{2} - y_0$. Indeed, let us set

$$h(y_0) := \frac{1}{2} - y_0 - x_0^* = \frac{1 - 3y_0 + 8y_0^3}{2(1 - 8y_0^2)}.$$

It is easily seen that the denominator is positive, while the numerator is strictly decreasing function on $\bigl[0, \frac{1}{2\sqrt{2}}\bigr)$ and $1 - 3y_0 + 8y_0^3 \geq 1 - \frac{1}{\sqrt{2}} > 0$. The claim is proved. Hence, in view of $a(y_0) < 0$, we conclude that $q(x_0)$ is a strictly decreasing function on $\bigl[\frac{1}{2} - y_0, \frac{1}{2}\bigr]$.

Taking into account the above considerations, the problem to prove that $q(x_0, y_0) \leq 0$ for every $(x_0, y_0) \in \Delta_1$ is reduced to the following inequalities:

$$q\Bigl(\frac{1}{2} - y_0, y_0\Bigr) \leq 0, \qquad \forall y_0 \in \Bigl[0, \frac{1}{2}\Bigr], \tag{8}$$

and

$$q\Bigl(\frac{1}{2}, y_0\Bigr) \leq 0, \qquad \forall y_0 \in \Bigl[\frac{1}{2\sqrt{2}}, \frac{1}{2}\Bigr]. \tag{9}$$

It is easily seen that the last multiplier in the representation

$$q\Bigl(\frac{1}{2} - y_0, y_0\Bigr) = 2(-1 + 2y_0)\bigl[2 - \sqrt{2} + 2(-3 + \sqrt{2})y_0 + 8y_0^2\bigr].$$

is positive for every $y_0 \in \mathbb{R}$, which yields the validity of (8). The proof of (9) is analogous because of the representation

$$q\Bigl(\frac{1}{2}, y_0\Bigr) = 2(-1 + 2y_0)\bigl[2 - \sqrt{2} + 2(-2 + \sqrt{2})y_0 + 8y_0^2\bigr].$$

This completes *1.2* and the proof of (6) for the segment $[QS]$.

2. Now we consider the segment

$$[RT] = \{(x, x) \,:\, x \in I_2\},$$

where $I_2 = [x_1, x_2]$, $x_1 = \frac{3-\sqrt{2}}{14}$, $x_2 = \frac{3+\sqrt{2}}{14}$. The corresponding restriction of f is

$$\begin{aligned} g(x) := f(x_0, y_0; x, x) &= (1 - 2y_0)^2[2y_0(x - x_0) + (1 - 2x_0)(x - y_0)]^2 \\ &\quad + (1 - 2x_0)^2[(1 - 2y_0)(x - x_0) + 2x_0(x - y_0)]^2 \\ &\quad - 64x_0^2y_0^2(1 - x_0 - y_0)^2\Bigl(x - \frac{1}{2}\Bigr)^2. \end{aligned}$$

We have to prove that $g(x) \leq 0$ for all $x \in I_2$ and $(x_0, y_0) \in \operatorname{int}\Delta_1$.

It is sufficient to prove that $g(x_1) \leq 0$, $g(x_2) \leq 0$ and $g'(x_2) \geq 0$. Indeed, if the leading coefficient of $g \in \pi_2$ is nonnegative then the inequalities $g(x_1) \leq 0$ and $g(x_2) \leq 0$ imply $g(x) \leq 0$ for all $x \in I_2$. Otherwise, it follows from $g'(x_2) \geq 0$ that $g'(x) \geq 0$ for all $x \in I_2$, hence $g(x) \leq g(x_2) \leq 0$ on I_2.

2.1. $g'(x_2) \ge 0$, for every $(x_0, y_0) \in \Delta_1$.

We have

$$g'(x_2) = -\frac{2}{7}\,(1-2x_0)(1-2y_0)h(x_0,y_0),$$

where

$$\begin{aligned} h(x_0,y_0) &= -3-\sqrt{2}+7(x_0+y_0)+(6+2\sqrt{2})(x_0-y_0)^2 \\ &\qquad -(16-4\sqrt{2})(x_0-y_0)^2(x_0+y_0)-(32-8\sqrt{2})x_0y_0(x_0+y_0)^2. \end{aligned}$$

Since $1-2x_0>0$ and $1-2y_0>0$ in Δ_1, we have to prove that $h(x_0,y_0)\le 0$ in Δ_1. In fact, we shall prove that the inequality $h(x_0,y_0)\le 0$ holds in the set $\left\{(x_0,y_0)\,:\,0\le x_0,y_0\le \frac{1}{2}\right\}$. By the symmetry of h, it is sufficient to consider the subset $\left\{(x_0,y_0)\,:\,0\le y_0\le x_0\le \frac{1}{2}\right\}=\triangle OB_1O_1$.

After a change of the variables $y_0=cx_0$, the problem is equivalent to

$$s_c(x_0) := h(x_0,cx_0)\le 0, \quad \text{for all } c\in[0,1] \text{ and } x_0\in\left[0,\frac{1}{2}\right]. \tag{10}$$

In explicit form,

$$\begin{aligned} s_c(x_0) &= 8(-4+\sqrt{2})c(1+c)^2x_0^4+4(-4+\sqrt{2})(c-1)^2(1+c)x_0^3 \\ &\qquad +2(3+\sqrt{2})(c-1)^2x_0^2+7(1+c)x_0-3-\sqrt{2}. \end{aligned}$$

Next we consider two cases.

(a) $c\in\left[0,\frac{2}{5}\right]$. We claim that $p_c(x_0):=\frac{\partial s_c}{\partial x_0}>0$ for every $x_0\in\left[0,\frac{1}{2}\right]$. We have

$$\begin{aligned} p_c(x_0) &= 32(-4+\sqrt{2})c(1+c)^2x_0^3+12(-4+\sqrt{2})(-1+c)^2(1+c)x_0^2 \\ &\qquad +4(3+\sqrt{2})(-1+c)^2x_0+7(1+c). \end{aligned}$$

Since

$$p_c''(x_0) = -24(4-\sqrt{2})(1+c)[8c(c+1)x_0+(1-c)^2]<0,$$

it suffices to prove that $p_c(0)>0$ and $p_c\left(\frac{1}{2}\right)>0$. Clearly, $p_c(0)=7(1+c)>0$. Furthermore, $r(c):=p_c\left(\frac{1}{2}\right)=7(-4+\sqrt{2})c^3+7(-2+\sqrt{2})c^2-3(3+\sqrt{2})c+5\sqrt{2}+1$. Clearly, $r'(c)=21(-4+\sqrt{2})c^2+14(-2+\sqrt{2})c-3(3+\sqrt{2})<0$ for every $c\in\mathbb{R}$, hence $r(c)\ge r\left(\frac{2}{5}\right)=\frac{671\sqrt{2}-829}{125}>0.9>0$ for all $c\in\left[0,\frac{2}{5}\right]$.

Hence (10) is reduced to $s_c\left(\frac{1}{2}\right)\le 0$, for every $c\in\left[0,\frac{2}{5}\right]$. This follows from $s_c\left(\frac{1}{2}\right)=(-4+\sqrt{2})c^3+\left(\frac{1}{2}-\sqrt{2}\right)c(1-c)\le\left(\frac{1}{2}-\sqrt{2}\right)c(1-c)\le 0$ for all $c\in[0,1]$.

(b) $c\in\left[\frac{2}{5},1\right]$. The Descarte's rule implies that $p_c(x_0)$ has a unique positive root $x_0^*(c)$. We shall prove that

$$x_0^*(c)\in\left[\frac{0.69}{1+c},\frac{0.73}{1+c}\right], \quad \text{for all } c\in\left[\frac{2}{5},1\right].$$

For every $a \in \mathbb{R}$ we set

$$\begin{aligned} q_a(c) &:= (1+c)\, p_c\Big(\frac{a}{1+c}\Big) \\ &= 4a(3+\sqrt{2}+3(\sqrt{2}-4)a)(c-1)^2 + 7(c+1)^2 + 32(-4+\sqrt{2})a^3 c. \end{aligned}$$

It is easily seen that the leading coefficient of q_a is positive for every $a \in [0.69, 0.73]$. A direct computation shows that $q_{0.73}(0.4) = -0.468\ldots < 0$ and $q_{0.73}(1) = -4.189\ldots < 0$, hence $q_{0.73}(c) < 0$ for every $c \in [0.4, 1]$, which establishes the upper bound for $x_0^*(c)$.

Similarly, the quadratic function $q_{0.69}(c)$ attains its minimum at the point $c_0 = 0.907\ldots$ and the minimal value is $q_{0.69}(c_0) = 0.779\ldots > 0$. This implies $q_{0.69}(c) > 0$ for every $c \in [0.4, 1]$ and finishes the proof for the lower estimate of $x_0^*(c)$.

Since $p_c(x_0) > 0$ for $x_0 < x_0^*(c)$ and $p_c(x_0) < 0$ for $x_0 > x_0^*(c)$, the function $s_c(x_0)$ attains its maximum on $[0, \infty)$ at $x_0^*(c)$. Therefore, it suffices to prove that

$$s_c(x_0^*(c)) \le 0, \quad \text{for every } c \in \Big[\frac{2}{5}, 1\Big].$$

Let us set $t_0(c) := \frac{0.71}{1+c}$. By Taylor's formula,

$$\begin{aligned} r(c) &:= s_c(t_0(c)) \\ &= s_c(x_0^*(c)) + s_c'(x_0^*(c))(t_0(c) - x_0^*(c)) + \frac{1}{2}\, s_c''(\xi)(t_0(c) - x_0^*(c))^2, \qquad (11) \end{aligned}$$

where $\xi \in \big[\frac{0.69}{1+c}, \frac{0.73}{1+c}\big] \subset [0,1]$. We have $r(c) = \alpha - \frac{\beta c}{(c+1)^2}$, where $\alpha = 1.304\ldots$, $\beta = 8.250\ldots$. Since $r'(c) = \frac{\beta(c-1)}{(c+1)^3} \le 0$ for every $c \le 1$, it follows that $r(c) \le r(0.4) < -0.3$ for every $c \in [0.4, 1]$.

Next we obtain an upper estimate of $|s_c''(x_0)|$ for $x_0 \in [0,1]$ and $c \in [0.4, 1]$. Clearly,

$$s_c''(x_0) = 96(-4+\sqrt{2})c(1+c)^2 x_0^2 + 24(-4+\sqrt{2})(c-1)^2(1+c)x_0 + 4(3+\sqrt{2})(c-1)^2,$$

hence $|s_c''(x_0)| \le 96(4-\sqrt{2})\cdot 4 + 24(4-\sqrt{2})\cdot 2 + 4(3+\sqrt{2}) < 1135$ in the above domain for (x_0, c).

Finally, by (11) we get

$$\begin{aligned} s_c(x_0^*(c)) &= s_c(t_0(c)) - \frac{1}{2}\, s_c''(\xi)(t_0(c) - x_0^*(c))^2 \\ &\le -0.3 + 0.5 \cdot 1135 \cdot 0.02^2 = -0.073 < 0. \end{aligned}$$

This completes the proof of *2.1.*

2.2. $g(x_2) \le 0$*, for every* $(x_0, y_0) \in \Delta_1$*.*

We have

$$g(x_2) = \frac{1}{98}(-1+2x_0)(-1+2y_0)(-1+2x_0+2y_0)h(x_0, y_0),$$

where

$$\begin{aligned}h(x_0,y_0) &= -11-6\sqrt{2}+2(10+\sqrt{2})(x_0+y_0)\\ &\qquad +4(-9+4\sqrt{2})(x_0^2+y_0^2-x_0y_0)+8(-9+4\sqrt{2})(x_0^2y_0+x_0y_0^2).\end{aligned}$$

As in *2.1*, it is sufficient to prove that $h(x_0,y_0)\le 0$ for every $(x_0,y_0)\in\Delta_1$. The gradient system $\frac{\partial h}{\partial x_0}=0$, $\frac{\partial h}{\partial y_0}=0$ has exactly four real solutions. Only

$$(x_0^*,y_0^*) := \frac{1}{12}\Big(-1+\sqrt{25+12\sqrt{2}},\ -1+\sqrt{25+12\sqrt{2}}\Big) = (0.456\ldots,\ 0.456\ldots)$$

belongs to the interior of Δ_1. Since $h(x_0^*,y_0^*)=-6.518\ldots<0$, it remains to prove that $h\le 0$ on $\partial\Delta_1$. The restrictions of h corresponding to the sides of Δ_1 are:

$$\begin{aligned}h|_{[A_1O_1]} &= -10-\sqrt{2}+2(10+\sqrt{2})x_0+8(-9+4\sqrt{2})x_0^2, \quad x_0\in\Big[0,\frac{1}{2}\Big],\\ h|_{[B_1O_1]} &= -10-\sqrt{2}+2(10+\sqrt{2})y_0+8(-9+4\sqrt{2})y_0^2, \quad y_0\in\Big[0,\frac{1}{2}\Big],\\ h|_{[A_1B_1]} &= -10-\sqrt{2}-4(-9+4\sqrt{2})t+8(-9+4\sqrt{2})t^2, \quad t\in\Big[0,\frac{1}{2}\Big].\end{aligned}$$

It is easy to see that they all are negative on their domains, which finishes the proof of *2.2*.

2.3. *$g(x_1)\le 0$, for every $(x_0,y_0)\in\Delta_1$.*
We have

$$g(x_1) = -\frac{1}{98}(-1+2x_0)(-1+2y_0)(-1+2x_0+2y_0)h(x_0,y_0),$$

where

$$\begin{aligned}h(x_0,y_0) &= 11-6\sqrt{2}-2(10-\sqrt{2})(x_0+y_0)\\ &\qquad +4(9+4\sqrt{2})(x_0^2+y_0^2-x_0y_0)+8(9+4\sqrt{2})(x_0^2y_0+x_0y_0^2).\end{aligned}$$

In this case we have to prove that $h(x_0,y_0)\ge 0$ for every $(x_0,y_0)\in\Delta_1$. The solutions of the gradient system are:

$$\frac{1}{12}\Big(-1\pm\sqrt{25-12\sqrt{2}},\ -1\pm\sqrt{25-12\sqrt{2}}\Big)$$

and

$$\frac{1}{4}\Big(3\pm\sqrt{25+4\sqrt{2}},\ 3\mp\sqrt{25+4\sqrt{2}}\Big).$$

They do not belong to $\operatorname{int}\Delta_1$ hence the problem reduces to $\partial\Delta_1$. As in *2.2*, the restrictions of h on the sides of Δ_1 are second degree polynomials and it is easy to show that they are positive.

This completes *2.3* and the proof of (6) for the segment $[RT]$.

3. The segment $[QR]$. It can be parameterized as follows:

$$[QR] = \Big\{ \Big(x, \frac{\sqrt{2}-1}{\sqrt{2}} \Big(\frac{1}{2} - x \Big) \Big) \,:\, x \in I_3 \Big\}, \qquad I_3 = \Big[0, \frac{3-\sqrt{2}}{14} \Big]$$

The restriction of f on $[QR]$ is $g(x) := f\big(x_0, y_0; x, \frac{\sqrt{2}-1}{\sqrt{2}}\big(\frac{1}{2} - x\big)\big)$, which is a second degree polynomial. Its derivative is

$$g'(x) = -\frac{1}{2}(-1 + 2x_0)(-1 + 2x_0 + 2y_0)h(x),$$

where

$$h(x) := h(x_0, y_0; x) := a(x_0, y_0)x + b(x_0, y_0),$$

$$\begin{aligned} a(x_0, y_0) &= -10 + 4\sqrt{2} + 8(2 - \sqrt{2})x_0 + 8(2\sqrt{2} - 3)x_0^2 + 4(9 - 4\sqrt{2})y_0 \\ &\quad + 16(\sqrt{2} - 3)y_0^2 - 8x_0y_0 + 32y_0^3 + 64x_0y_0^2(x_0 + y_0 - 1), \\ b(x_0, y_0) &= 3 - 2\sqrt{2} + 2\sqrt{2}\,x_0 + 4(1 - \sqrt{2})x_0^2 + 2(4\sqrt{2} - 7)y_0 \\ &\quad + 8(3 - \sqrt{2})y_0^2 - 4x_0y_0 - 16y_0^3 + 32x_0y_0^2(1 - x_0 - y_0). \end{aligned}$$

By **2.**, we have $g\big(\frac{3-\sqrt{2}}{14}\big) = f\big(x_0, y_0; \frac{3-\sqrt{2}}{14}, \frac{3-\sqrt{2}}{14}\big) \le 0$. Therefore, it is sufficient to prove that $g'(x) \ge 0$, for every $x \in I_3$. This is equivalent to $h(x) \ge 0$, $x \in I_3$. Since h is a linear function, it suffices to check the validity of the above inequality only for $x = 0$ and $x = \frac{3-\sqrt{2}}{14}$.

Let $h_1(x_0, y_0) := h(0) = b(x_0, y_0)$. Clearly, h_1 is a second degree polynomial with respect to x_0, which has negative leading coefficient $4(1 - \sqrt{2}) - 32y_0^2$. Therefore, we shall prove that the restrictions $h_1|_{x_0=0}$ and $h_1|_{x_0=\frac{1}{2}}$ are positive for $y_0 \in \big[0, \frac{1}{2}\big]$. This follows easily from the explicit expressions

$$\begin{aligned} h_1(0, y_0) &= -16\Big(y_0 - \frac{1}{2}\Big)\Big(y_0 - \frac{2-\sqrt{2}}{4}\Big)^2, \\ h_1\Big(\frac{1}{2}, y_0\Big) &= -2(-1 + 2y_0)\big(2 - \sqrt{2} + 2(\sqrt{2} - 2)y_0 + 8y_0^2\big). \end{aligned}$$

Next we consider $h_2(x_0, y_0) := h\big(\frac{3-\sqrt{2}}{14}\big)$. It is a second degree polynomial with respect to x_0 with leading coefficient $\frac{1}{7}\big(8(\sqrt{2} - 3) - 32(4 + \sqrt{2})y_0^2\big)$ which is negative. It remains to check that the restrictions $h_2|_{y_0=\frac{1}{2}-x_0}$ and $h_2|_{x_0=\frac{1}{2}}$ are positive for $x_0, y_0 \in \big[0, \frac{1}{2}\big]$. As above, this can be deduced from the formulas:

$$\begin{aligned} h_2\Big(x_0, \frac{1}{2} - x_0\Big) &= \frac{2}{7}\,x_0\big(3(4 + \sqrt{2}) - 2(16 + 11\sqrt{2})x_0 + 16(4 + \sqrt{2})x_0^2\big), \\ h_2\Big(\frac{1}{2}, y_0\Big) &= -\frac{4}{7}\,(-1 + 2y_0)\big(3 - \sqrt{2} + 2(\sqrt{2} - 3)y_0 + 4(4 + \sqrt{2})y_0^2\big). \end{aligned}$$

The proof of (6) for the segment $[QR]$ is completed.

4. The segment $[ST]$. We have

$$[ST] = \Big\{ \Big(x, \frac{\sqrt{2}+1}{\sqrt{2}} \Big(\frac{1}{2} - x \Big) \Big) \,:\, x \in I_4 \Big\}, \qquad I_4 = \Big[0, \frac{3+\sqrt{2}}{14} \Big].$$

As in the previous case, we set $g(x) := f|_{[ST]} = f\big(x_0, y_0; x, \frac{\sqrt{2}+1}{\sqrt{2}}(\frac{1}{2} - x)\big)$, We have to prove that $g(x) \le 0$ for every $x \in I_4$ and $(x_0, y_0) \in \operatorname{int} \Delta_1$. A computation gives

$$g'(x) = \frac{1}{2}\,(-1 + 2x_0)(-1 + 2x_0 + 2y_0)h(x),$$

where

$$h(x) := h(x_0, y_0; x) := a(x_0, y_0)x + b(x_0, y_0),$$

and

$$\begin{aligned} a(x_0, y_0) &= 10 + 4\sqrt{2} - 8(2 + \sqrt{2})x_0 + 8(2\sqrt{2} + 3)x_0^2 - 4(9 + 4\sqrt{2})y_0 \\ &\quad + 16(\sqrt{2} + 3)y_0^2 + 8x_0y_0 - 32y_0^3 - 64x_0y_0^2(x_0 + y_0 - 1), \\ b(x_0, y_0) &= -3 - 2\sqrt{2} + 2\sqrt{2}\,x_0 - 4(1 + \sqrt{2})x_0^2 + 2(4\sqrt{2} + 7)y_0 \\ &\quad - 8(3 + \sqrt{2})y_0^2 + 4x_0y_0 + 16y_0^3 - 32x_0y_0^2(1 - x_0 - y_0). \end{aligned}$$

We shall prove first that $g'\big(\frac{3+\sqrt{2}}{14}\big) \ge 0$ for every $(x_0, y_0) \in \operatorname{int} \Delta_1$. The last inequality is equivalent to $h\big(\frac{3+\sqrt{2}}{14}\big) \le 0$. Let $h_1(x_0, y_0) := h\big(\frac{3+\sqrt{2}}{14}\big)$. Note that h_1 is a second degree polynomial with respect to x_0, whose leading coefficient is $\frac{8}{7}\,[3 + \sqrt{2} + 4(4 - \sqrt{2})y_0^2] > 0$. Thus, h_1 is a convex function in x_0 and it is sufficient to prove that $h_1\big(t, \frac{1}{2} - t\big)$ and $h_1\big(\frac{1}{2}, t\big)$ are non-positive for all $t \in \big[0, \frac{1}{2}\big]$. This can be deduced from the representations

$$\begin{aligned} h_1\Big(t, \frac{1}{2} - t\Big) &= \frac{2}{7}\,t\big[3(-4 + \sqrt{2}) + 2(16 - 11\sqrt{2})t + 16(-4 + \sqrt{2})t^2\big], \\ h_1\Big(\frac{1}{2}, t\Big) &= -\frac{4}{7}\,(-1 + 2t)\big[-3 - \sqrt{2} + 2(\sqrt{2} + 3)t - 4(4 - \sqrt{2})t^2\big]. \end{aligned}$$

There are two cases for the sign of $g'(0)$. If $g'(0) \ge 0$ then $g(x)$ is a non-decreasing function on I_4, hence $g(x) \le g\big(\frac{3+\sqrt{2}}{14}\big) = f\big(x_0, y_0; \frac{3+\sqrt{2}}{14}, \frac{3+\sqrt{2}}{14}\big) \le 0$. The last inequality was proved in **2**. Suppose now that $g'(0) < 0$. Noticing that $g(x)$ is a second degree polynomial we conclude that g attains its minimum at a point from I_4. Consequently, the inequality $g(x) \le 0$ for all $x \in I_4$ is equivalent to $g(0) = f\big(x_0, y_0; 0, \frac{\sqrt{2}+1}{\sqrt{2}}\big) \le 0$ and $g\big(\frac{3+\sqrt{2}}{14}\big) = f(x_0, y_0; \frac{3+\sqrt{2}}{14}, \frac{3+\sqrt{2}}{14})) \le 0$, which were proved above.

The finishes the proof of (6) for the segment $[ST]$. Lemma 2 is proved. □

Corollary 1. *For every* $(x_0, y_0) \in \operatorname{int}\Delta_1$ *and* $(x, y) \in D_2$ *we have*

$$K(x_0, y_0; x, y) \le 64\Big(y - \frac{1}{2}\Big)^2.$$

Proof. As in Lemma 2, it is sufficient to prove the inequality

$$\begin{aligned}\bar f(x_0, y_0; x, y) := \big\{&(2y_0-1)^2[2y_0(x-x_0) + (1-2x_0)(y-y_0)]^2\\ &+ (2x_0-1)^2[(1-2y_0)(x-x_0) + 2x_0(y-y_0)]^2\big\}\\ &- 64x_0^2y_0^2(1-x_0-y_0)^2\Big(y-\frac{1}{2}\Big)^2 \le 0,\end{aligned} \tag{12}$$

for all $(x_0, y_0) \in \Delta_1$ and $(x, y) \in D_2$. Since $\bar f(x_0, y_0; x, y) = f(y_0, x_0; y, x)$ (see (5)), the inequality (12) follows from (5) and the fact that D_2 is symmetrical to D_1 with respect to the line $y = x$. ☐

Lemma 3. *For every* $(x_0, y_0) \in \operatorname{int}\Delta_1$ *and* $(x, y) \in OVRQ$ *we have*

$$K(x_0, y_0; x, y) \le 128\Big(x + y - \frac{1}{2}\Big)^2. \tag{13}$$

Proof. Now (13) will follow from the inequality

$$\begin{aligned}f(x_0, y_0; x, y) := \big\{&(2y_0-1)^2[2y_0(x-x_0) + (1-2x_0)(y-y_0)]^2\\ &+ (2x_0-1)^2[(1-2y_0)(x-x_0) + 2x_0(y-y_0)]^2\big\}\\ &- 128x_0^2y_0^2(1-x_0-y_0)^2\Big(x+y-\frac{1}{2}\Big)^2 \le 0,\end{aligned} \tag{14}$$

for all $(x_0, y_0) \in \Delta_1$ and $(x, y) \in OVRQ$.

Since f is a symmetric function of x and y, we shall prove (14) only for $(x, y) \in OQR$.

Note that the leading coefficients of f with respect to x and y are negative (see (16)), hence the method of the proof of Lemma 2 cannot be applied.

Now we define the auxiliary points $P_1 = (0.12, 0)$, $P_2 = (0.12, 0.15)$ and $P_3 = (0, 0.15)$. It is seen that $OQR \subset OP_1P_2P_3$.

1. We claim that

$$\frac{\partial f}{\partial x}\Big|_{(x,y)\in[P_1P_2]} \ge 0, \quad \text{for every } (x_0, y_0) \in \Delta_1. \tag{15}$$

We have

$$\frac{\partial f}{\partial x}\Big|_{x=0.12} = -\frac{2}{25}\,(2x_0 + 2y_0 - 1)h(y),$$

where

$$h(y) := h(x_0, y_0; y) := a(x_0, y_0)y + b(x_0, y_0),$$

$$\begin{aligned}a(x_0,y_0) &= 50x_0 - 100x_0^2 + 50y_0 + 200x_0^2y_0 - 100y_0^2\\ &\quad + 200x_0y_0^2 - 2400x_0^2y_0^2 + 1600x_0^3y_0^2 + 1600x_0^2y_0^3,\\ b(x_0,y_0) &= 3 - 31x_0 + 50x_0^2 - 6y_0 + 24x_0y_0 - 100x_0^2y_0 - 38y_0^2\\ &\quad - 76x_0y_0^2 + 912x_0^2y_0^2 - 608x_0^3y_0^2 + 76y_0^3 - 608x_0^2y_0^3.\end{aligned}$$

Since $2x_0 + 2y_0 - 1 \geq 0$ for every $(x_0, y_0) \in \Delta_1$, the inequality (15) will follow from $h(y) \leq 0$ for every $y \in [0, 0.15]$. But h is a linear function, hence it suffices to prove the last inequality only for $y = 0$ and $y = 0.15$.

We set $h_1(x_0, y_0) := h(0) = b(x_0, y_0)$. Note that h_1 is a third degree polynomial with respect to each of the variables x_0 and y_0.

We consider h_1 as a function of $x_0 \in \left[\frac{1}{2} - y_0, \frac{1}{2}\right]$ for a fixed $y_0 \in \left[0, \frac{1}{2}\right]$. The derivative $h_1'(x_0)$ is a second degree polynomial. It is easily seen that $h_1'(0) < 0$ and $h_1'\left(\frac{1}{2}\right) \geq 0$ for every $y_0 \in \left[0, \frac{1}{2}\right]$. Hence $h_1'(x_0)$ has a unique zero $x_0^*(y_0)$ in the interval $\left(0, \frac{1}{2}\right]$, h_1 is decreasing in $[0, x_0^*]$ and increasing in $\left[x_0^*, \frac{1}{2}\right]$. Therefore

$$\max_{x_0 \in [\frac{1}{2} - y_0, \frac{1}{2}]} h_1(x_0) = \max\Big\{h_1\Big(\frac{1}{2} - y_0\Big), h_1\Big(\frac{1}{2}\Big)\Big\}.$$

From the explicit expressions $h_1\left(\frac{1}{2} - y_0\right) = 2y_0(2y_0 - 1)(152y_0^2 - 63y_0 + 19)$ and $h_1\left(\frac{1}{2}\right) = -19y_0(2y_0 - 1)^2$ it follows that $h_1\left(\frac{1}{2} - y_0\right) \leq 0$ and $h_1\left(\frac{1}{2}\right) \leq 0$ for every $y_0 \in \left[0, \frac{1}{2}\right]$, i.e. $h(0) \leq 0$.

Next we consider $h_2(x_0, y_0) := h(0.15)$. In explicit form

$$\begin{aligned}h_2(x_0,y_0) &= 3 - \frac{47}{2}x_0 + 35x_0^2 + \frac{3}{2}y_0 + 24x_0y_0 - 70x_0^2y_0 - 53y_0^2\\ &\quad - 46x_0y_0^2 + 552x_0^2y_0^2 - 368x_0^3y_0^2 + 76y_0^3 - 368x_0^2y_0^3.\end{aligned}$$

As for h_1, the derivative $h_2'(x_0)$ has a unique zero x_0^* in the interval $\left(0, \frac{1}{2}\right]$, and h_2 attains its minimum at x_0^*. Hence, it is suffices to prove the inequalities $h_2\left(\frac{1}{2} - y_0\right) \leq 0$ and $h_2\left(\frac{1}{2}\right) \leq 0$ for every $y_0 \in \left[0, \frac{1}{2}\right]$. They can be deduced from $h_2\left(\frac{1}{2} - y_0\right) = \frac{1}{2}(736y_0^3 - 632y_0^2 + 194y_0 - 31)$ and $h_2\left(\frac{1}{2}\right) = -4y_0(2y_0 - 1)^2$.

This finishes the proof of (15).

2. Next we shall prove that

$$\frac{\partial^2 f}{\partial x^2} \leq 0, \quad \text{for every} \quad (x_0, y_0) \in \Delta_1. \tag{16}$$

Note that the above derivative does not depend on x and y. We have

$$\frac{\partial^2 f}{\partial x^2} = -2(2x_0 + 2y_0 - 1)r(x_0, y_0),$$

where

$$r(x_0,y_0) = 1 - 2x_0 - 2y_0 + 8x_0y_0 + 4y_0^2 + 8x_0y_0^2 - 96x_0^2y_0^2 + 64x_0^3y_0^2 - 8y_0^3 + 64x_0^2y_0^3.$$

Thus (16) is equivalent to $r(x_0,y_0)\ge 0$ for every $(x_0,y_0)\in\Delta_1$.

A small computation gives

$$\frac{\partial^2 r}{\partial x_0^2}=64y_0^2(6x_0+2y_0-3).$$

Let $T_1=\left(\frac{1}{3},\frac{1}{2}\right)$. Clearly, $\Delta_1=A_1B_1T_1\cup B_1O_1T_1$. Then $\frac{\partial^2 r}{\partial x_0^2}\le 0$ (≥ 0) provided $(x_0,y_0)\in A_1B_1T_1$ (resp., $(x_0,y_0)\in B_1O_1T_1$). Therefore,

$$\min_{(x_0,y_0)\in A_1B_1T_1} r(x_0,y_0)=\min_{(x_0,y_0)\in[A_1B_1]\cup[B_1T_1]} r(x_0,y_0). \tag{17}$$

Our next claim is

$$\frac{\partial r}{\partial x_0}\le 0 \quad\text{for every}\quad (x_0,y_0)\in B_1O_1T_1. \tag{18}$$

This follows from $\frac{\partial r}{\partial x_0}\Big|_{[B_1O_1]}=64y_0^3-40y_0^2+8y_0-2\le 0$ for every $y_0\in\left[0,\frac{1}{2}\right]$ and the fact (see above) that $\frac{\partial r}{\partial x_0}$ is a nondecreasing function with respect to x_0 in $B_1O_1T_1$.

Taking into account (17) and (18), the proof of (16) is reduced to the inequalities $r\big|_{[A_1B_1]}\ge 0$ and $r\big|_{[O_1B_1]}\ge 0$. These follow from the equalities

$$r\Big(x_0,\frac{1}{2}-x_0\Big)=-8x_0(2x_0-1)(4x_0^2-3x_0+1),\quad r\Big(\frac{1}{2},y_0\Big)=2y_0(2y_0-1)^2.$$

The proof of (16) is completed.

3. As a consequence of (15) and (16) we conclude that

$$\frac{\partial f}{\partial x}\ge 0,\quad\text{for all}\quad (x,y)\in OP_1P_2P_3 \text{ and } (x_0,y_0)\in\Delta_1. \tag{19}$$

Therefore,

$$\max_{(x,y)\in OQR} f(x_0,y_0;x,y)=\max_{(x,y)\in[OR]\cup[QR]} f(x_0,y_0;x,y).$$

In order to complete the proof of Lemma 3, it is sufficient to establish that $f\big|_{[OR]}\le 0$ and $f\big|_{[QR]}\le 0$.

Let

$$g(t):=f\big|_{[OR]}=f(x_0,y_0;t,t),\qquad t\in I=\Big[0,\frac{3-\sqrt{2}}{14}\Big].$$

We have

$$g'(t)=\frac{\partial f}{\partial x}(x_0,y_0;t,t)+\frac{\partial f}{\partial y}(x_0,y_0;t,t).$$

Using (19) and $[OR]\subset OP_1P_2P_3$, we conclude that the first summand is nonnegative for every $t\in I$. Furthermore, we introduce the function

$$\bar f(x_0,y_0;x,y):=f(y_0,x_0;x,y).$$

Then,

$$f(x_0,y_0;x,y)=f(y_0,x_0;y,x)=\bar{f}(x_0,y_0;y,x).$$

Consequently,

$$\frac{\partial f}{\partial y}(x_0,y_0;x,y)=\frac{\partial \bar{f}}{\partial x}(x_0,y_0;y,x).$$

By (19) we obtain

$$\frac{\partial f}{\partial y}(x_0,y_0;t,t)=\frac{\partial \bar{f}}{\partial x}(x_0,y_0;t,t)=\frac{\partial f}{\partial x}(y_0,x_0;t,t)\geq 0,$$

for every $t\in I$. Therefore, $g'(t)\geq 0$ for every $t\in I$ and

$$\max_{(x,y)\in[OR]} f(x_0,y_0;x,y)=f\Big(x_0,y_0;\frac{3-\sqrt{2}}{14},\frac{3-\sqrt{2}}{14}\Big).$$

Finally, note that the inequality $f\big|_{[QR]}\leq 0$ follows from Lemma 2 since the corresponding functions f coincide on the segment $[QR]$.

This completes the proof of Lemma 3. ☐

Lemma 4. *The inequality* (13) *holds true for every* $(x_0,y_0)\in \operatorname{int}\Delta_1$ *and* $(x,y)\in ABSTU$.

Proof. The inequality (13) is equivalent to

$$\Phi(x_0,y_0;x,y):=K(x_0,y_0;x,y)-128\Big(x+y-\frac{1}{2}\Big)^2\leq 0, \tag{20}$$

for all $(x_0,y_0)\in \operatorname{int}\Delta_1$ and $(x,y)\in ABSTU$. It is seen that Φ can be written in the form

$$\Phi(x,y)=(x-x_0,y-y_0)H\begin{pmatrix}x-x_0\\ y-y_0\end{pmatrix}-128\Big(x_0+y_0-\frac{1}{2}\Big)\Big(2x+2y-x_0-y_0-\frac{1}{2}\Big),$$

where

$$H=\begin{pmatrix}\alpha^2+\beta^2-128 & \beta(\alpha+\gamma)-128\\ \beta(\alpha+\gamma)-128 & \beta^2+\gamma^2-128\end{pmatrix},$$

and α,β,γ are given by (4).

The gradient system for Φ with respect to (x,y) is equivalent to

$$H\begin{pmatrix}x-x_0\\ y-y_0\end{pmatrix}=128\Big(x_0+y_0-\frac{1}{2}\Big)\begin{pmatrix}1\\ 1\end{pmatrix}. \tag{21}$$

Let (x^*,y^*) be a solution of (21). Then

$$\begin{aligned}\Phi(x^*,y^*)&=128\Big(x_0+y_0-\frac{1}{2}\Big)\\ &\quad\times\Big[(x^*-x_0,y^*-y_0)\begin{pmatrix}1\\ 1\end{pmatrix}-\Big(2x^*+2y^*-x_0-y_0-\frac{1}{2}\Big)\Big]\\ &=-128\Big(x_0+y_0-\frac{1}{2}\Big)\Big(x^*+y^*-\frac{1}{2}\Big).\end{aligned}$$

If $(x^*, y^*) \in \operatorname{int} ABSTU$ then $x^* + y^* - \frac{1}{2} > 0$, hence $\Phi(x^*, y^*) < 0$. Therefore, it remains to prove (20) for every $(x, y) \in \partial(ABSTU)$. For this purpose, it suffices to establish the inequality

$$f(x_0, y_0; x, y) \le 0, \tag{22}$$

for all $(x_0, y_0) \in \operatorname{int} \Delta_1$ and $(x, y) \in \partial(ABSTU)$, where f is defined in (14).

1. Let us consider the segment $[AB] := \{(x, 1-x) : x \in [0, 1]\}$. We have

$$\begin{aligned} g(x) := f|_{[AB]} &= (1-2y_0)^2 \big[(2x_0 + 2y_0 - 1)x + 1 - 2x_0 - y_0\big]^2 \\ &\quad + (1-2x_0)^2 \big[(2x_0 + 2y_0 - 1)x - x_0\big]^2 - 32x_0^2 y_0^2 (1 - x_0 - y_0)^2. \end{aligned}$$

Clearly, g is a second degree polynomial, which has a positive leading coefficient, provided $(x_0, y_0) \in \operatorname{int} \Delta_1$. In order to prove that $g(x) \le 0$ for every $x \in [0, 1]$, it is sufficient to prove the inequalities $g(0) \le 0$ and $g(1) \le 0$. These inequalities are equivalent to

$$(1-2y_0)^2 (1 - 2x_0 - y_0)^2 + (1-2x_0)^2 x_0^2 \le 32x_0^2 y_0^2 (1 - x_0 - y_0)^2, \tag{23}$$

$$(1-2y_0)^2 y_0^2 + (1-2x_0)^2 (x_0 + 2y_0 - 1)^2 \le 32x_0^2 y_0^2 (1 - x_0 - y_0)^2, \tag{24}$$

respectively. Observe that the left-hand sides of (23) and (24) do not exceed

$$(1-2y_0)^2 y_0^2 + (1-2x_0)^2 x_0^2.$$

This follows from the inequalities $|1 - 2x_0 - y_0| \le y_0$ and $|x_0 + 2y_0 - 1| \le x_0$, which hold true for every $(x_0, y_0) \in \Delta_1$.

Thus, we shall prove the stronger inequality

$$(1-2y_0)^2 y_0^2 + (1-2x_0)^2 x_0^2 \le 32x_0^2 y_0^2 (1 - x_0 - y_0)^2. \tag{25}$$

It will follow from the inequalities

$$\begin{aligned} (1-2x_0)^2 x_0^2 &\le 16x_0^2 y_0^2 (1 - x_0 - y_0)^2, \\ (1-2y_0)^2 y_0^2 &\le 16x_0^2 y_0^2 (1 - x_0 - y_0)^2. \end{aligned}$$

Because of the symmetry, it suffices to prove only the first one. It is equivalent to

$$x_0(1 - 2y_0)(1 - 2x_0 - 2y_0) \le 0, \tag{26}$$

which obviously holds for $(x_0, y_0) \in \Delta_1$. This completes the proof of (22) for the segment $[AB]$.

2. The segment $[SB] = \{(0, y) : y \in [s, 1]\}$, $s = \frac{\sqrt{2}+1}{2\sqrt{2}}$. We have

$$\begin{aligned} g(y) := f|_{[SB]} = \big\{&(2y_0 - 1)^2 [-2x_0 y_0 + (1 - 2x_0)(y - y_0)]^2 \\ &+ (2x_0 - 1)^2 [-x_0(1 - 2y_0) + 2x_0(y - y_0)]^2\big\} \\ &- 128x_0^2 y_0^2 (1 - x_0 - y_0)^2 \Big(y - \frac{1}{2}\Big)^2. \end{aligned}$$

The function g is a second degree polynomial whose leading coefficient is:

$$c=(1-2x_0)^2(1-2y_0)^2+4x_0^2(1-2x_0)^2-128x_0^2y_0^2(1-x_0-y_0)^2.$$

We claim that $c<0$ for every $(x_0,y_0)\in\operatorname{int}\Delta_1$. Since $0\le 1-2x_0\le 2y_0$ for $(x_0,y_0)\in\Delta_1$, it is sufficient to prove the inequality

$$4y_0^2(1-2y_0)^2+4x_0^2(1-2x_0)^2-128x_0^2y_0^2(1-x_0-y_0)^2<0.$$

It follows from (25) noticing that (26) is fulfilled as a strict inequality for $(x_0,y_0)\in\operatorname{int}\Delta_1$.

A direct substitution yields $g\big(\frac{1}{2}\big)\ge 0$. By the result for the domain D_1, we have $g(s)=f\big(x_0,y_0;0,\frac{\sqrt{2}+1}{2\sqrt{2}}\big)\le 0$. Now the only way for the concave quadratic function $g(y)$ to attain a nonnegative value at $\frac{1}{2}$ and a nonpositive one at s is that it keeps to be negative and decreasing to the right of s. In particular $g(y)\le 0$ for every $y\in[s,1]$, which finishes the proof for $[SB]$.

3. Inequality (22) for $(x,y)\in[ST]\cup[TU]$ follows by the continuity of f with respect to (x,y) and the results for the domains D_1 and D_2 (Lemma 2 and Corollary 1).

Since f has the symmetric property $f(x_0,y_0;x,y)=f(y_0,x_0;y,x)$, the proof of (22) for $[UA]$ is reduced to that for $[SB]$. This completes the proof of Lemma 4. □

Now we are ready to finish the proof of Theorem 1. It follows from Lemmas 1–4 that

$$K(x_0,y_0;x,y)\le B(x,y),$$

for all $(x_0,y_0)\in\operatorname{int}\Delta_1$ and $(x,y)\in\Delta$. Then, by the parametric representation of the Markov's majorant we get

$$M(x,y)\le 2\sqrt{B(x,y)},\tag{27}$$

which implies (2) for all $p\in E_I$. The general case $p\in\operatorname{conv}E_I$ follows from the linearity of the gradient and the convexity of the norm.

Theorem 1 is proved. □

Remark 2. In fact, for every $(x,y)\in\Delta$ the inequality (27) is fulfilled as equality. Indeed, if $(x,y)\in D_i$, $i=1,2,3$, we can take a sequence from E_I tending to p_i and in view of Lemma 1 and Remark 1, we obtain

$$M(x,y)\ge 2\sqrt{B(x,y)}.$$

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