Moduli of Roots of Hyperbolic Polynomials and Descartes' Rule of Signs

Vladimir Petrov Kostov

A real univariate polynomial with all roots real is called hyperbolic. By Descartes' rule of signs for hyperbolic polynomials (HPs) with all coefficients nonvanishing, a HP with c sign changes and p sign preservations in the sequence of its coefficients has exactly c positive and p negative roots. For $c = 2$ and for degree 6 HPs, we discuss the question: When the moduli of the 6 roots of a HP are arranged in the increasing order on the real half-line, at which positions can the moduli of its two positive roots be depending on the positions of the two sign changes in the sequence of coefficients?

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1. Introduction

The classical Descartes' rule of signs states that given a real univariate degree d polynomial P, the number *pos* of its positive roots is not larger than the number c of the sign changes in the sequence of its coefficients. In the present paper we consider only polynomials with all coefficients nonvanishing and with positive leading coefficients. Thus the number of sign changes for the polynomial $P(-x)$ is equal to the number p of sign preservations for the polynomial $P(x)$ hence one has $neg \leq p$, where neg stands for the number of negative roots of P. If P is *hyperbolic*, i.e. with all roots real, then the conditions

$$c \ + \ p \ = \ d \ = \ pos \ + \ neg\,, \quad pos \ \leq \ c \quad \text{and} \quad neg \ \leq \ p$$

imply $pos = c$ and $neg = p$. We consider only the *generic case* when all moduli of roots are distinct. Suppose that the d moduli of roots are arranged in the increasing order on the positive half-line. Then one can formulate the following problem:

Problem 1. *Knowing the positions of the sign changes and sign preserva-tions in the sequence of coefficients of a given hyperbolic polynomial (HP), what positions can occupy the moduli of its positive roots in this arrangement?*

For $c = 1$, the exhaustive answer to this problem (for any degree d) is given in [11]. Also in [11] one can find the answer to Problem 1 for $d \leq 5$ and $c = 2$. In the present paper we give the answer to it for $d = 6$ and $c = 2$.

Remark 1. For $d \leq 5$, it is sufficient to study Problem 1 for $c \leq 2$, because the polynomial $(-1)^d P(-x)$ has p sign changes and c sign preservations in the sequence of its coefficients. Thus if $P(x)$ has more than 2 sign changes, then $(-1)^d P(-x)$ has not more than 2 sign changes and one can consider $(-1)^d P(-x)$ instead of $P(x)$. For $d \geq 6$ this is not so. In particular, for $d = 6$, HPs with $c = 3$ remain such when $P(x)$ is replaced by $(-1)^d P(-x)$.

The paper is structured as follows. In Section 2 we recall some definitions and results of [11] and we mention some problems related to Problem 1. In Section 3 we first remind which cases have to be considered for $d = 6$ according to the signs of the coefficients of the HP, and then we resolve Problem 1 in each of these cases.

2. Definitions and Known Results

Definition 1. A *sign pattern (SP)* of length $d + 1$ is a sequence of $d + 1$ signs $+$ or $-$. We say that the polynomial $P := x^d + \sum_{j=0}^{d-1} a_j x^j$ defines the SP $(+, \text{sgn}(a_{d-1}), \ldots, \text{sgn}(a_0))$. We consider polynomials with positive leading coefficients, so the SPs we deal with begin with $+$. In the proofs we use also SPs some of whose components are zeros.

Notation 1. We denote the two positive roots of a HP (with two sign changes in the sequence of its coefficients) by $0 < \beta < \alpha$. By $0 < \gamma_1 < \cdots < \gamma_{d-2}$ we denote the moduli of its negative roots. By (a, b, w), a, b, $w \in \mathbb{N} \cup 0$, $a + b + w = d - 2$ we denote the case when $\gamma_a < \beta < \gamma_{a+1}$ and $\gamma_{a+b} < \alpha < \gamma_{a+b+1}$ setting $\gamma_0 := 0$ and $\gamma_{d-1} := \infty$. By $\Sigma_{m,n,q}$, for $c = 2$ (resp. by $\Sigma_{m,n}$, for $c = 1$) we denote the SP consisting of m pluses followed by n minuses followed by q pluses, where $m + n + q = d + 1$ (resp. consisting of m pluses followed by n minuses, where $m + n = d + 1$). For $c = 1$, the moduli of the positive and negative roots are denoted by α and $\gamma_1 < \cdots < \gamma_{d-1}$; the case (a, b), $a + b = d - 1$, means $\gamma_a < \alpha < \gamma_{a+1}$. The notation (a, b, w) and (a, b) is different from the one used in [11].

Definition 2. (i) Given a SP one defines its corresponding *canonical ar-rangement* as follows. The increasing order of moduli of positive and negative roots on the real half-line coincides with the order of sign changes and sign

preservations respectively when the SP is read backward. Example: for $d = 6$ and $c = 2$, the SP $\Sigma_{2,4,1} = (+, +, -, -, -, -, +)$ when read backward gives the following order of sign changes and sign preservations: (c, p, p, p, c, p). Hence the canonical arrangement defined by this SP is $\beta < \gamma_1 < \gamma_2 < \gamma_3 < \alpha < \gamma_4$ which is the case $(0, 3, 1)$. More generally, for the SP $\Sigma_{m,n,q}$, the corresponding canonical arrangement defines the case $(q - 1, n - 1, m - 1)$.

(ii) We say that a SP $\Sigma_{m,n,q}$ (resp. $\Sigma_{m,n}$) is *realizable* in the case (a, b, w) (resp. (a, b)) if there exists a degree d HP with $c = 2$ and $p = d - 2$ (resp. with $c = 1$ and $p = d - 1$) defining this SP, with distinct moduli of its roots which define the case (a, b, w) (resp. the case (a, b)). We can also say that the case (a, b, w) is realizable with the SP $\Sigma_{m,n,q}$.

The following result can be found in [11]:

Theorem 1. *(i) Every SP has a canonical realization, i.e. is realizable in the case defined by its canonical arrangement. The SPs $\Sigma_{1,d}$, $\Sigma_{d,1}$, $\Sigma_{1,d-1,1}$ and $\Sigma_{m,1,d-m}$ are realizable only in this case.*

(ii) The SPs $\Sigma_{d-2,2,1}$ and $\Sigma_{d-3,3,1}$ are realizable in the case $(1, 0, d - 3)$. They are realizable only in this case and in cases of the kind $(0, b, w)$, $b + w = d - 2$. For $n \geq 4$, the SP $\Sigma_{m,n,1}$ is realizable only in cases of the kind $(0, b, w)$, $b + w = d - 2$.

Remark 2. Given a degree d polynomial $P(x)$ we define its *reverted* polynomial P^R as $P^R(x) := x^d P(1/x)$. It is clear that if the SP $\Sigma_{m,n,q}$ (resp. $\Sigma_{m,n}$) is realizable in the case (a, b, w) (resp. (a, b)) by the HP $P(x)$, then the SP $\Sigma_{q,n,m}$ (resp. $\Sigma_{n,m}$) is realizable in the case (w, b, a) (resp. (b, a)) by the polynomial $\text{sgn}(P(0))P^R(x)$.

Another problem about real univariate polynomials (not necessarily hyperbolic) inspired by Descartes' rule of signs is the following one:

Problem 2. *Suppose that a SP with c sign changes and p sign preservations is given. For which pairs of nonzero integers (pos, neg) satisfying the conditions $pos \leq c$, $neg \leq p$ and $c - pos \in 2\mathbb{N} \cup 0 \ni p - neg$ do there exist such polynomials defining the given SP and having exactly pos positive and neg negative roots, all distinct?*

The problem seems to have been formulated for the first time in [2]. Its exhaustive answer for $d \leq 8$ is to be found in [7, 1, 5, 8] (and this answer is not trivial). An interesting particular case for $d = 11$ is considered in [9]. In [6] a tropical analog of Descartes' rule of signs is formulated. Different aspects of the theory of HPs are exposed in [10]. For metric inequalities involving moduli of roots of polynomials see [3] and [4].

3. Resolution of Problem 1 for $d = 6$

For $d = 6$, the SPs with $c = 2$ which need to be considered are the following ones: $\Sigma_{1,5,1}$, $\Sigma_{m,1,q}$ $(m + q = 6)$, $\Sigma_{k,6-k,1}$ ($k = 2$, 3 and 4), $\Sigma_{2,3,2}$ and $\Sigma_{3,2,2}$. The remaining SPs are obtained from these ones by reversion, see Remark 2. We consider different SPs in the subsequent subsections. In Subsection 3.7 we summarize the results.

In the proofs we use the following notation:

Notation 2. We remind that we consider HPs having four negative and two positive roots. These roots are denoted by

$$\xi_1 < \xi_2 < \xi_3 < \xi_4 < 0 < \xi_5 < \xi_6 \,.$$

For the roots of the derivatives we use the notation

$$\zeta_1 < \zeta_2 < \zeta_3 < \zeta_4 < \zeta_5 \,, \qquad \xi_j < \zeta_j < \xi_{j+1}$$

(the latter inequalities result from Rolle's theorem). One has $\zeta_3 < 0 < \zeta_4$, if the last but first sign of the SP is $+$, and $\zeta_4 < 0 < \zeta_5$, if it is $-$.

Remark 3. In the proofs we use one-parameter deformations of given HPs in which the deformation parameter is considered only for these nonnegative values for which the deformed polynomial is hyperbolic.

Remark 4. For n fixed, one could ask the question what the possible values of the quantity b can be. As we shall see, for $d = 6$ and $n = 2$, one has $b \leq 4$. The following example (with $d = 7$ and $n = 2$) shows that b can attain the value 5:

$$
\begin{aligned}
D := & (x - 0.9)(x + 0.98)(x + 0.99)(x + 1)(x + 1.01)(x + 1.02)(x - 1.1) \\
= & x^7 + 3x^6 + 0.9895x^5 - 5.0505x^4 - 5.09899496x^3 \\
& + 0.90101496x^2 + 2.94951496x + 0.9895050396 \,.
\end{aligned}
$$

The polynomial D defines the SP $\Sigma_{3,2,3}$ and realizes the case $(0, 5, 0)$. To obtain such examples with arbitrarily large values of $d \geq 7$ (and with $n = 2$) it suffices to multiply D by polynomials of the form $\prod_{\mu=1}^{\mu^*}(1 + \varepsilon_\mu x) \prod_{\nu=\mu^*+1}^{\mu^*+\nu^*}(x + \varepsilon_\nu)$, where ε_i are small positive quantities. Indeed, for $1 \leq \mu \leq \mu^*$, the moduli of the negative roots $-1/\varepsilon_\mu$ are large whereas for $\mu^* + 1 \leq \mu \leq \mu^* + \nu^*$, the moduli of the negative roots $-\varepsilon_\mu$ are small; the polynomial then defines the SP $\Sigma_{3+\mu^*,2,3+\nu^*}$ and realizes the case $(\nu^*, 5, \mu^*)$.

3.1. The SPs $\Sigma_{1,5,1}$ and $\Sigma_{m,1,q}$, $m + q = 6$

These two cases have only canonical realizations, see Theorem 1. This means that for the HPs realizing the SP $\Sigma_{1,5,1}$ one has $\beta < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4 < \alpha$ $(*)$. This is the case $(0, 4, 0)$.

Example 1. Consider the polynomial $P^* := (x - 0.01)(x + 0.25)^4(x - 1)$, i.e.

$$P^* = x^6 - 0.01x^5 - 0.625x^4 - 0.30625x^3$$
$$- 0.05546875x^2 - 0.0033203125x + 0.0000390625 \,.$$

This polynomial defines the SP $\Sigma_{1,5,1}$. Hence for $\varepsilon > 0$ small enough, the polynomial

$$(x - 0.01)(x + 0.25 - 2\varepsilon)(x + 0.25 - \varepsilon)(x + 0.25 + \varepsilon)(x + 0.25 + 2\varepsilon)(x - 1)$$

also defines the SP $\Sigma_{1,5,1}$ and has six distinct real roots whose moduli satisfy conditions $(*)$.

Example 2. For $m = 1$, 2 and 3, the following HPs define the corresponding SPs $\Sigma_{m,1,q}$:

$$(x + 0.01)^4(x - 1)^2 \qquad = x^6 - 1.96x^5 + 0.9206x^4 + 0.038804x^3$$
$$+ 0.00059201x^2 + 0.00000398x + 10^{-8} \,,$$

$$(x + 0.01)^3(x - 1)^2(x + 4) = x^6 + 2.03x^5 - 6.9397x^4 + 3.790601x^3$$
$$+ 0.117902x^2 + 0.001193x + 0.000004 \,,$$

$$(x + 0.01)^2(x - 1)^2(x + 4)^2 = x^6 + 6.02x^5 + 1.1201x^4 - 23.9794x^3$$
$$+ 15.5201x^2 + 0.3176x + 0.0016 \,.$$

The reverted of these HPs define these SPs with $m = 5$, 4 and 3, respectively. One can define one-parameter deformations of these HPs in which the multiple roots split into simple real ones while preserving the signs of the roots and of the coefficients of the corresponding polynomial (as this is done in Example 1). For small nonzero values of the deformation parameter, the deformations are canonical realizations of the corresponding SPs.

3.2. The SP $\Sigma_{2,4,1}$

For any HP realizing this SP, one has $\beta < \gamma_1$, see part (ii) of Theorem 1. Hence a priori the realizable cases are of the form $(0, b, 4 - b)$, $0 \le b \le 4$. The ones with $b = 2$, 3 and 4 are realizable:

$$(0, 2, 2) \ : \ (x - 0.001)(x + 0.3)(x + 0.4)(x - 1)(x + 1.01)(x + 1.02)$$
$$= x^6 + 1.729x^5 - 0.16053x^4 - 1.6063012x^3$$
$$- 0.83950954x^2 - 0.122782884x + 0.000123624 \,,$$

$$(0, 3, 1) \ : \ (x - 0.001)(x + 0.3)(x + 0.4)(x + 1)(x - 1.01)(x + 1.02)$$
$$= x^6 + 1.709x^5 - 0.19491x^4 - 1.6229468x^3$$
$$- 0.84194086x^2 - 0.122780436x + 0.000123624 \,,$$

$$(0,4,0) \ : \ (x - 0.001)(x + 0.3)(x + 0.4)(x + 1)(x + 1.01)(x - 1.02)$$
$$= x^6 + 1.689x^5 - 0.22889x^4 - 1.6393128x^3$$
$$- 0.84432446x^2 - 0.122778036x + 0.000123624 \,.$$

Proposition 1. *The case* $(0, 0, 4)$ *is not realizable with the SP* $\Sigma_{2,4,1}$.

Proof. Indeed, suppose that the polynomial P realizes the SP $\Sigma_{2,4,1}$ in the case $(0, 0, 4)$. Hence $\zeta_4 < 0 < \zeta_5$ and $-\xi_4 = \gamma_1 > \alpha = \xi_6$, see Notation 2. This means that

$$-\zeta_1 > -\zeta_2 > -\zeta_3 > -\xi_4 > \xi_6 > \zeta_5 \,. \tag{1}$$

The HP P' defines the SP $\Sigma_{2,4}$. It follows from [11, Corollary 1] that for $d = 5$, the cases $(3, 1)$ and $(2, 2)$ are realizable with the SP $\Sigma_{2,4}$, but the cases $(1, 3)$ and $(0, 4)$ are not. This is a contradiction with (1). $\qquad\square$

Proposition 2. *The case* $(0, 1, 3)$ *is not realizable with the SP* $\Sigma_{2,4,1}$.

Proof. Suppose that the HP P realizes the SP $\Sigma_{2,4,1}$ in the case $(0, 1, 3)$. We use Notation 2. Consider for $t \geq 0$ the one-parameter deformation $P_t := P + tx^4(x - \xi_6)$. As t increases, the root ξ_6 does not change, ξ_1, ξ_3 and ξ_5 decrease while ξ_2 and ξ_4 increase; the SP does not change. For some value $t_0 > 0$ of t, at least one of the two things happens:

A) the roots ξ_2 and ξ_3 coalesce;

B) one has $|\xi_4| = |\xi_5|$, i.e. $-\xi_4 = \xi_5$.

Set $Q := P_{t_0}$. If A) takes place, then we consider the one-parameter deformation $Q_s := Q - s(x - \xi_2)^2 x^2$, $s \geq 0$. As s increases, the double root $\xi_2 = \xi_3$ does not change, ξ_1 and ξ_5 decrease while ξ_4 and ξ_6 increase; the SP does not change. Then for some $s = s_0 > 0$, either B) or C) takes place, with

C) one has $|\xi_6| = |\xi_2|$, i.e. $-\xi_2 = \xi_6$.

We denote by AB) "A) followed by B)". Thus if A) takes place, then there exists a HP defining the SP $\Sigma_{2,4,1}$ for which either AB) or AC) takes place.

Suppose that B) takes place. Consider the one-parameter deformation $Q_u := Q + uQ^*$, $Q^* := (x^2 - \xi_4^2)(x - \xi_6)$, $u \geq 0$. The roots ξ_4, ξ_5 and ξ_6 do not change, ξ_1 and ξ_3 decrease while ξ_2 increases. One has

$$Q^* = x^3 - \xi_6 x^2 - (\xi_4)^2 x + \xi_4^2 \xi_6 = x^3 - U_2 x^2 - U_1 x + U_0 \,, \qquad U_j > 0 \,.$$

Hence as u increases, the signs of the last three coefficients of Q_u do not change and the first three coefficients do not change at all. The coefficient of x^3 cannot become negative. Indeed, Q_u is hyperbolic and four sign changes in the sequence of its coefficients means four positive roots which is not the case.

Thus u can be increased only until for some value u_0, A) takes place, so one can assume that either AB) (or BA) which is the same) or AC) takes place.

Set $R := Q_{u_0}$. Suppose that AB) takes place. Consider the one-parameter deformation $R_v := R - vR^*$, $R^* := (x - \xi_2)^2(x^2 - \xi_4^2)$, $v \geq 0$. One has

$$R^* = x^4 - 2\xi_2 x^3 + (\xi_2^2 - \xi_4^2)x^2 + 2\xi_2\xi_4^2 x - \xi_2^2\xi_4^2$$
$$= x^4 + R_3 x^3 + R_2 x^2 - R_1 x - R_0 , \qquad R_j > 0 .$$

Thus in the deformation R_v only the sign of the linear term can change. The roots ξ_i, $2 \leq i \leq 5$, do not change, ξ_1 decreases and ξ_6 increases, so for some value $v_0 > 0$ either C) or D) takes place, with

D) the coefficient of x equals 0.

Hence if AB) takes place, then it suffices to consider the possibility ABC) or ABD) to take place.

Suppose that AC) takes place. Consider the deformation $R_r := R + rR^\dagger$, $r \geq 0$,
$$R^\dagger := (x + \xi_6)^2(x - \xi_6) = x^3 + \xi_6 x^2 - \xi_6^2 x - \xi_6^3 .$$

The roots $\xi_2 = \xi_3$ and ξ_6 do not change, ξ_1 and ξ_5 decrease while ξ_4 increases. In the deformation R_r the coefficient of x^3 or the one of x^2 or both of them cannot become positive, because this would mean at least three sign changes, i.e. at least three positive roots which is impossible. The constant term of R_r cannot vanish. Indeed, in this case after rescaling one can set $\xi_6 = 1$ hence $R_r = xT$, where

$$T = (x + 1)^2(x - 1)(x + g)(x + h) , \qquad g = -\xi_1 > 1 , \qquad h = -\xi_4 \in (0, 1) .$$

Observe that the SP $\Sigma_{2,4,1}$ implies that $R_r'(0) < 0$ hence it is ξ_5 and not ξ_4 that vanishes. The coefficient of x^3 in T equals $-1 + g + h + gh > 0$ whereas it must be negative. However, as $R^\dagger(0) < 0$, for $r = -R(0)/R^\dagger(0) > 0$, one has $R_r(0) = 0$, i.e. for $r = r_0$, the constant term does vanish. This contradiction shows that one cannot have AC). So one cannot have ABC) either and only the possibility ABD) remains.

We rescale the variable x so that $\xi_4 = -1 = -\xi_5$. We set $\xi_2 = \xi_3 = -g$, $g > 1$, $\xi_6 = A > 1$, $\xi_1 = -B$, $B > 1$. So we consider the polynomial

$$F := (x + g)^2(x^2 - 1)(x + B)(x - A) = x^6 + F_5 x^5 + \cdots + F_0 .$$

Since $F_1 = g(-gB + gA + 2AB)$, the condition $F_1 = 0$ yields $g = g_* := 2AB/(B - A)$. Hence $B > A$, because $g > 0$. One gets

$$F_4\big|_{g=g_*} = (-2A^2B^2 - B^2 + 2AB - A^2 + 3AB^3 + 3A^3B)/(-B + A)^2 .$$

However the inequalities $B > A > 1$ imply $2AB^3 > 2A^2B^2$, $AB^3 > B^2$ and $AB > A^2$, i.e. $F_4\big|_{g=g_*} > 0$ which is in contradiction with the SP $\Sigma_{2,4,1}$. This contradiction proves the proposition. $\qquad\square$

3.3. The SP $\Sigma_{3,3,1}$

According to part (ii) of Theorem 1, the HPs realizing this SP either satisfy the inequalities $\gamma_1 < \beta < \alpha < \gamma_2 < \gamma_3 < \gamma_4$, which is the case $(1,0,3)$ realizable by the HP

$$(x + 0.98)(x - 0.99)(x - 1)(x + 2.05)(x + 2.1)(x + 40)$$
$$= x^6 + 43.14x^5 + 124.7533x^4 - 41.23068x^3$$
$$- 294.614531x^2 - 0.116529x + 167.06844 ,$$

or they realize one of the cases $(0, b, 4 - b)$, $0 \leq b \leq 4$. These cases are also realizable:

$(0,0,4)$: $(x - 0.1)(x - 9)(x + 9.6)(x + 9.7)(x + 9.8)(x + 9.9)$
$$= x^6 + 29.9x^5 + 216.35x^4 - 1448.135x^3$$
$$- 24185.4276x^2 - 78877.71684x + 8131.05216 ,$$

$(0,1,3)$: $(x - 0.1)(x + 0.99)(x - 1)(x + 1.01)(x + 1.02)(x + 40)$
$$= x^6 + 41.92x^5 + 76.6179x^4 - 9.305992x^3$$
$$- 81.6975778x^2 - 32.6139222x + 4.079592 ,$$

$(0,2,2)$: $(x - 0.1)(x + 0.99)(x + 0.995)(x - 1)(x + 1.02)(x + 40)$
$$= x^6 + 41.905x^5 + 76.00425x^4 - 9.835474x^3$$
$$- 81.0232111x^2 - 32.0695689x + 4.019004 ,$$

$(0,3,1)$: $(x - 0.1)(x + 0.99)(x + 0.995)(x + 0.999)(x - 1)(x + 40)$
$$= x^6 + 41.884x^5 + 75.145665x^4 - 10.55580655x^3$$
$$- 80.08192694x^2 - 31.3281913x + 3.9362598 ,$$

$(0,4,0)$: $(x - 0.1)(x + 9.6)(x + 9.7)(x + 9.8)(x + 9.9)(x - 10)$
$$= x^6 + 28.9x^5 + 177.45x^4 - 2014.585x^3$$
$$- 27835.3426x^2 - 87541.52424x + 9034.5024 .$$

3.4. The SP $\Sigma_{4,2,1}$

For the HPs realizing this SP there exist two possibilities, see Theorem 1. The first of them is to satisfy the inequalities $\gamma_1 < \beta < \alpha < \gamma_2 < \gamma_3 < \gamma_4$, and this is the case $(1,0,3)$ realizable by the polynomial

$$(x + 1)(x - 1.5)(x - 1.6)(x + 10)(x + 11)(x + 12)$$
$$= x^6 + 30.9x^5 + 292x^4 + 539.1x^3 - 2946.2x^2 - 55.2x + 3168 .$$

The second is one of the cases $(0, b, 4 - b)$, $0 \le b \le 4$, to be realizable. For $b = 0$, 1 and 2 we provide examples of HPs realizing the corresponding cases:

$$
\begin{aligned}
(0,0,4) \ : \ & (x - 1)(x - 4)(x + 5)(x + 6)(x + 100)(x + 101) \\
= \ & x^6 + 207x^5 + 11285x^4 + 56273x^3 \\
& - 233286x^2 - 1046480x + 1212000 \,,
\end{aligned}
$$

$$
\begin{aligned}
(0,1,3) \ : \ & (x - 1)(x + 2)(x - 4)(x + 5)(x + 100)(x + 101) \\
= \ & x^6 + 203x^5 + 10481x^4 + 15957x^3 \\
& - 216482x^2 - 214160x + 404000 \,,
\end{aligned}
$$

$$
\begin{aligned}
(0,2,2) \ : \ & (x - 1)(x + 2.1)(x + 3)(x - 4)(x + 1000)(x + 1001) \\
= \ & x^6 + 2001.1x^5 + 1.0011849 \times 10^6 x^4 + 69673.7x^3 \\
& - 10^7 \big(1.52373859x^2 - 1.10606748x + 2.52252 \big) \,.
\end{aligned}
$$

Proposition 3. *The case $(0, 4, 0)$ is not realizable with the SP $\Sigma_{4,2,1}$.*

Proof. Suppose that the SP $\Sigma_{4,2,1}$ is realizable in the case $(0, 4, 0)$ by the HP P. Then the SP $\Sigma_{1,2,4}$ is realizable in the case $(0, 4, 0)$ by the HP $Q := P^R$. For the roots of Q and Q' we use Notation 2. The HP Q' defines the SP $\Sigma_{1,2,3}$ and as $|\xi_\nu| \in (\xi_5, \xi_6)$, $1 \le \nu \le 4$, one has $|\zeta_k| > \zeta_4$, $k = 1$, 2 and 3. One has $\zeta_5 > \zeta_4$; the position of ζ_5 w.r.t. $|\zeta_k|$, $1 \le k \le 3$, cannot be specified.

The HP $(Q')^R$ defines the SP $\Sigma_{3,2,1}$ and has roots $\eta_j = 1/\zeta_j$ for which one has $|\eta_k| < \eta_4$ and $\eta_k < 0$, $k = 1$, 2 and 3. Notice that η_4 is the largest root η_j. Indeed, there are two positive roots η_4 and η_5, and as $\zeta_5 > \zeta_4$, this implies $\eta_5 < \eta_4$. From part (ii) of Theorem 1 and from [11, Proposition 1] (which claims that for $d = 5$, the case $(0, 3, 0)$ is not realizable with the SP $\Sigma_{3,2,1}$) one deduces that this is impossible. $\qquad\square$

Proposition 4. *The case $(0, 3, 1)$ is not realizable with the SP $\Sigma_{4,2,1}$.*

Proof. Suppose that the HP P realizes the SP $\Sigma_{4,2,1}$ in the case $(0, 3, 1)$. We use Notation 2. Consider the polynomial $V := x^3(x - \xi_2)(x - \xi_3)(x - \xi_4)$. It defines the SP $(+, +, +, +, 0, 0, 0)$. Hence throughout the deformation $P_t := P + tV$, $t \ge 0$, the SP does not change, the roots ξ_2, ξ_3 and ξ_4 remain the same, ξ_1 and ξ_5 increase while ξ_6 decreases. Hence for some $t = t_0 > 0$, at least one of the following things happens:

A) one has $|\xi_4| = |\xi_5|$, i.e. $-\xi_4 = \xi_5$;

B) one has $|\xi_1| = |\xi_6|$, i.e. $-\xi_1 = \xi_6$;

C) one has $|\xi_2| = |\xi_6|$, i.e. $-\xi_2 = \xi_6$.

Suppose that A) takes place. Set $Q := P_{t_0}$ and consider the deformation $Q_s := Q + sQ^\triangle$, $Q^\triangle := (x^2 - \xi_5^2)(x^2 - \xi_6^2)$, $s \ge 0$. The SP defined by the

polynomial $Q^\triangle$ is $(0, 0, +, 0, -, 0, +)$, so the SP does not change throughout the deformation Q_s. The roots ξ_4, ξ_5 and ξ_6 do not change; ξ_1 increases without reaching $-\xi_6$; ξ_2 increases and ξ_3 decreases. Hence for some $s = s_0 > 0$, one has

D) the roots ξ_2 and ξ_3 coalesce.

By rescaling the x-axis one obtains the condition $\xi_2 = \xi_3 = -1$. Hence one can set $-\xi_4 = \xi_5 = g \in (0, 1)$, $\xi_6 = A > 1$ and $-\xi_1 = B > A$. The corresponding polynomial equals $K := (x + 1)^2(x^2 - g^2)(x + B)(x - A)$. Its coefficient of x^3 is

$$-2g^2 + g^2 A - A - Bg^2 + B - 2AB = -2g^2 - (1 - g^2)A - Bg^2 + (1 - A)B - AB < 0$$

which contradicts the SP $\Sigma_{4,2,1}$.

Suppose that B) takes place. Consider the deformation $Q_v := Q + vQ^\dagger$, $Q^\dagger := (x^2 - \xi_6^2)x^2$, $v \geq 0$. The polynomial $Q^\dagger$ defines the SP $(0, 0, +, 0, -, 0, 0)$, so the SP does not change throughout the deformation Q_v. The roots ξ_1 and ξ_6 do not change. The root ξ_4 increases while ξ_5 decreases, both of them keeping away from 0 (because $Q_v(0)$ does not change). As ξ_2 increases and ξ_3 decreases, for some $v = v_0 > 0$, one has either A) (a case already considered) or D).

If B) and D) take place, then by rescaling the x-axis so that $\xi_2 = \xi_3 = -1$, one sets $-\xi_1 = \xi_6 = A > 1$ and $-\xi_4 = g \in (0, 1) \ni h = \xi_5$. The corresponding polynomial equals $(x + 1)^2(x^2 - A^2)(x - h)(x + g)$, with coefficient of x^3 equal to

$$-2A^2 - A^2 g + g + hA^2 - h - 2gh = (h - 1)A^2 + (g - A^2) - A^2 g - h - 2gh < 0$$

which contradicts the SP $\Sigma_{4,2,1}$.

Suppose that C) takes place. Consider the family of polynomials $Q + \delta Q^\bullet$, where $\delta \geq 0$ and $Q^\bullet := (x^2 - \xi_6^2)(x^2 - \xi_5^2)$. The polynomial $Q^\bullet$ defines the SP $(0, 0, +, 0, -, 0, +)$, so the SP does not change throughout the deformation. The roots ξ_2, ξ_5 and ξ_6 do not change, ξ_1 and ξ_4 increase (ξ_4 never becomes equal to $-\xi_5$) while ξ_3 decreases. For some $\delta = \delta_0 > 0$, takes place either D) or

E) the roots ξ_1 and ξ_2 coalesce.

If E) takes place, then after rescaling one obtains for $\xi_1 = -1 = \xi_2 = -\xi_6$ the polynomial

$$G := (x + 1)^2(x - 1)(x + g)(x + h)(x - k), \quad k = \xi_5, \quad g = -\xi_3, \quad h = -\xi_4,$$

hence $0 < k < h < g < 1$. The coefficient of x^3 equals

$$-1 - g - h + gh + k - kg - kh - ghk = -1 - g(1 - h) - (h - k) - kg - kh - ghk < 0$$

which contradicts the SP $\Sigma_{4,2,1}$. If D) takes place, then we rescale the x-axis to obtain the polynomial G with the equalities $\xi_2 = -1 = \xi_3 = -\xi_6$, $k = \xi_5$, $g = -\xi_1$, $h = -\xi_4$, hence $0 < k < h < 1 < g$. With the same form of the coefficient of x^3 one concludes that the negative sign of this coefficient contradicts the SP $\Sigma_{4,2,1}$.

The proposition is proved. $\square$

3.5. The SP $\Sigma_{2,3,2}$

Proposition 5. *All cases* (a, b, w), $a + b + w = 4$, $0 \leq a$, b, $w \leq 4$, *are realizable with the SP* $\Sigma_{2,3,2}$.

Proof. The SP $\Sigma_{2,3,2}$ is center-symmetric, this is why if this SP is realizable in the case (a, b, w) by a HP P, then it is realizable in the case (w, b, a) by the HP P^R. We prove the proposition by exhibiting examples of HPs which realize $\Sigma_{2,3,2}$ in the indicated cases. The above observations allow to skip some of the cases.

$$(1, 0, 3) \; : \; (x + 0.01)(x - 0.1)(x - 1)(x + 1.01)(x + 1.02)(x + 1.03)$$
$$= x^6 + 1.97x^5 - 0.1253x^4 - 2.067553x^3$$
$$- 0.87576764x^2 + 0.097559534x + 0.001061106 \,,$$

$$(1, 1, 2) \; : \; (x + 0.01)(x - 0.1)(x + 0.99)(x - 1)(x + 1.02)(x + 1.03)$$
$$= x^6 + 1.95x^5 - 0.1445x^4 - 2.045655x^3$$
$$- 0.85653356x^2 + 0.095648466x + 0.001040094 \,,$$

$$(1, 2, 1) \; : \; (x + 0.01)(x - 0.1)(x + 0.98)(x + 0.99)(x - 1)(x + 1.03)$$
$$= x^6 + 1.91x^5 - 0.1817x^4 - 2.001931x^3$$
$$- 0.81930584x^2 + 0.091937534x + 0.000999306 \,,$$

$$(1, 3, 0) \; : \; (x + 0.01)(x - 0.1)(x + 0.97)(x + 0.98)(x + 0.99)(x - 1)$$
$$= x^6 + 1.85x^5 - 0.2345x^4 - 1.936645x^3$$
$$- 0.76643456x^2 + 0.086638466x + 0.000941094 \,,$$

$$(0, 2, 2) \; : \; (x - 1)(x + 1.1)(x + 2)(x - 2.1)(x + 2.2)(x + 2.3)$$
$$= x^6 + 4.5x^5 - 0.25x^4 - 24.205x^3$$
$$- 23.6436x^2 + 19.2214x + 23.3772 \,,$$

$$(0, 1, 3) \; : \; (x - 1)(x + 1.1)(x - 2)(x + 2.05)(x + 2.1)(x + 2.15)$$
$$= x^6 + 4.4x^5 - 0.0425x^4 - 21.8665x^3$$
$$- 20.921675x^2 + 17.068025x + 20.36265 \,,$$

$$(0, 4, 0) \; : \; (x - 1)(x + 2.9)(x + 3)(x + 3.1)(x + 3.2)(x - 8)$$
$$= x^6 + 3.2x^5 - 46.01x^4 - 291.172x^3$$
$$- 487.418x^2 + 129.968x + 690.432 \,,$$

$$(2, 0, 2) \; : \; (x + 0.8)(x + 0.9)(x - 1)(x - 5)(x + 5.1)(x + 5.2)$$
$$= x^6 + 6x^5 - 22.25x^4 - 156x^3$$
$$- 72.1556x^2 + 147.9336x + 95.472 \,,$$

$$(0, 0, 4) \; : \; (x - 1)(x - 1.001)(x + 1.002)(x + 1.01)(x + 1.02)(x + 1.1)$$
$$= x^6 + 2.131x^5 - 0.867672x^4 - 4.26624106x^3$$
$$- 1.268949846x^2 + 2.135240980x + 1.136621926 \,.$$

$\square$

3.6. The SP $\Sigma_{3,2,2}$

Proposition 6. *The cases (a, b, w) with $w \geq 1$ are realizable with the SP $\Sigma_{3,2,2}$.*

Proof. For $d = 5$ and for the SP $\Sigma_{2,2,2}$, all cases (a, b, w) with $a + b + w = 3$ are realizable, see [11]. Hence for $d = 6$, all corresponding cases $(a, b, w + 1)$ are also realizable. Indeed, if the degree 5 HP $P(x)$ defines the SP $\Sigma_{2,2,2}$ and realizes the case (a, b, w), then for $\varepsilon > 0$ small enough, the HP $(1 + \varepsilon x)P(x)$ defines the SP $\Sigma_{3,2,2}$ and realizes the case $(a, b, w + 1)$. Indeed, the root $-1/\varepsilon$ has the largest of the moduli of its roots. $\square$

Thus it remains to consider the cases $(a, b, 0)$ with $a + b = 4$. The case $(0, 4, 0)$ is realizable by the HP

$$(x - 1)(x + 1.9)(x + 1.91)(x + 1.92)(x + 2)(x - 2.1)$$
$$= x^6 + 4.63x^5 + 0.5412x^4 - 24.36394x^3$$
$$- 28.469668x^2 + 17.398152x + 29.264256 \,.$$

Proposition 7. *The case $(4, 0, 0)$ is not realizable with the SP $\Sigma_{3,2,2}$.*

Proof. Suppose that the HP P realizes the case $(4, 0, 0)$ with the SP $\Sigma_{3,2,2}$. For the roots of P and P' we use Notation 2; in particular, $\xi_4 < 0 < \xi_5$ and $\zeta_3 < 0 < \zeta_4$. The polynomial P' defines the SP $\Sigma_{3,2,1}$ and for its roots one has $\zeta_5 > |\zeta_k|$, $k = 1$, 2 and 3. It follows from part (ii) of Theorem 1 and [11, Proposition 1] (by which for $d = 5$, the case $(0, 3, 0)$ is not realizable with the SP $\Sigma_{3,2,1}$) that this is impossible. $\square$

Proposition 8. *The cases $(1, 3, 0)$, $(2, 2, 0)$ and $(3, 1, 0)$ are not realizable with the SP $\Sigma_{3,2,2}$.*

Proof. Suppose that the HP P realizes the case $(1, 3, 0)$ or $(2, 2, 0)$ or $(3, 1, 0)$ with the SP $\Sigma_{3,2,2}$. We use Notation 2. Consider the one-parameter deformation

$$P_t := P + t(x - \xi_1)(x - \xi_6)x^2, \qquad t \geq 0.$$

The polynomial $(x - \xi_1)(x - \xi_6)x^2$ defines the SP $(0, 0, +, -, -, 0, 0)$, because $|\xi_1| < \xi_6$; therefore the SP does not change throughout the deformation P_t. The roots ξ_1 and ξ_6 do not change, ξ_2 and ξ_4 increase while ξ_3 and ξ_5 decrease. Hence there exists $t_0 > 0$ such that for P_{t_0} at least one of the two things holds true:

A) in case $(1, 3, 0)$, one has $|\xi_4| = |\xi_5|$, i.e. $-\xi_4 = \xi_5$;
   in case $(2, 2, 0)$, one has $|\xi_3| = |\xi_5|$, i.e. $-\xi_3 = \xi_5$;
      (the possibility $|\xi_2| = |\xi_5|$ also exists, but we consider it
      only in the case $(3, 1, 0)$, because in both cases $(2, 2, 0)$ and $(3, 1, 0)$
      it gives $\xi_2 = -\xi_5$, $-\xi_1 < \xi_6$);
   in case $(3, 1, 0)$, one has $|\xi_2| = |\xi_5|$, i.e. $-\xi_2 = \xi_5$;

B) the roots ξ_2 and ξ_3 coalesce.

Consider the case $(1, 3, 0)$. Suppose that A) takes place. Set $Q := P_{t_0}$. Then throughout the deformation

$$Q_s := Q + s(x^2 - \xi_1^2)(x^2 - \xi_5^2), \qquad s \geq 0,, \tag{2}$$

the SP does not change. Indeed, the polynomial $(x^2 - \xi_1^2)(x^2 - \xi_5^2)$ defines the SP $(0, 0, +, 0, -, 0, +)$. The roots ξ_1, ξ_4 and ξ_5 do not change while ξ_6 decreases (but never becomes equal to $-\xi_1$). The root ξ_2 increases while ξ_3 decreases.

Therefore there exists $s_0 > 0$ for which both A) and B) take place. In this case after rescaling of x one can have

$$\xi_2 = \xi_3 = -1, \quad -\xi_4 = \xi_5 = g \in (0, 1), \quad -\xi_1 = B > 1 \quad \text{and} \quad \xi_6 = A > B.$$

The corresponding polynomial equals $(x + B)(x + 1)^2(x^2 - g^2)(x - A)$ and its coefficient of x^4 is $2(B - A) + (1 - AB) - g^2 < 0$ which contradicts the SP $\Sigma_{3,2,2}$. Hence A) does not take place.

Suppose that B) takes place. We use again the rescaling of x leading to $\xi_2 = \xi_3 = -1$. Consider the polynomial $S := (x + 1)^2(x - 1/2)$. It defines the SP $(0, 0, 0, +, +, 0, -)$. Then throughout the deformation

$$Q_v := Q - vS, \qquad v \geq 0, \tag{3}$$

the SP does not change. The roots ξ_2 and ξ_3 do not change while ξ_1 and ξ_6 increase; ξ_6 and does not coalesce with ξ_5. The root ξ_5 increases, if $\xi_5 < 1/2$, decreases, if $\xi_5 > 1/2$, and remains fixed, if $\xi_5 = 1/2$. As A) does not take place, for some $v_0 > 0$,

C) the root ξ_1 coalesces with $\xi_2 = \xi_3$.

In this case $Q_{v_0} = (x+1)^3(x+g)(x-h)(x-A)$, where $-\xi_4 = g < h = \xi_5 < 1$ and $A = \xi_6 > 1$. The coefficient of x^4 in Q_{v_0} equals

$$3+3g-3h-gh-(3+g-h)A \leq 3+3g-3h-gh-3-g+h = 2(g-h)-gh < 0,$$

which contradicts the SP $\Sigma_{3,2,2}$. Thus the case $(1,3,0)$ is not realizable with the SP $\Sigma_{3,2,2}$.

Consider the case $(2,2,0)$. Suppose that A) takes place. Throughout the deformation Q_s (see (2)) the roots ξ_1, ξ_3 and ξ_5 do not change; ξ_6 decreases without becoming equal to $-\xi_1$; ξ_4 decreases and ξ_2 increases. Hence, for some $s = s_0 > 0$, there are two possibilities. The first is to have A) and B), i.e. one can set

$$\xi_2 = \xi_3 = -1 = -\xi_5, \quad -\xi_4 = g \in (0,1), \quad -\xi_1 = B > 1 \quad \text{and} \quad \xi_6 = A > B.$$

The corresponding polynomial is $W := (x + 1)^2(x - 1)(x + g)(x + B)(x - A)$ whose coefficient of x^4 equals

$$W_4 := -1+g+B+gB-A-Ag-AB = (B-A)(g+1)+(g-AB)-1 < 0,$$

which contradicts the SP $\Sigma_{3,2,2}$. The second is to have A) and

D) the root ξ_4 coalesces with ξ_3.

In this case one sets

$$\xi_3 = \xi_4 = -1 = -\xi_5, \quad -\xi_2 = g > 1, \quad -\xi_1 = B > g \quad \text{and} \quad \xi_6 = A > B.$$

The corresponding polynomial equals W and its coefficient of x^4 is $W_4 < 0$. So suppose that B) takes place. Then A) takes place as well, and this possibility was already rejected.

Consider the case $(3,1,0)$. Suppose that A) takes place. Throughout the deformation Q_s (see (2)) the roots ξ_1, ξ_2 and ξ_5 do not change, ξ_6 decreases, but remains larger than $|\xi_1|$, ξ_3 increases and ξ_4 decreases. Hence for some $s = s_0 > 0$, one has D). Consider the polynomial $(x^2-1)(x+g)^2(x+B)(x-A)$, where

$$-\xi_2 = \xi_5 = 1, \quad -\xi_3 = -\xi_4 = g \in (0,1), \quad -\xi_1 = B > 1 \quad \text{and} \quad \xi_6 = A > B.$$

The coefficient of x^4 equals

$$-1+g^2+2gB-2Ag-BA = -(1-g^2)-2g(A-B)-BA < 0,$$

which contradicts the SP $\Sigma_{3,2,2}$.

If B) (but not A)) takes place, then in the deformation Q_v (see (3)) the roots $\xi_2 = \xi_3$ do not change, ξ_1 and ξ_6 increase while ξ_4 decreases. We admit that ξ_5 might increase or decrease or remain fixed. Hence for some $v = v^*$, either D) takes place or

E) one has $|\xi_5| = |\xi_1|$, i.e. $\xi_5 = -\xi_1$.

If B) and D) take place, then one considers the polynomial $(x + 1)^3(x - g)(x + B)(x - A)$, where

$$1 = -\xi_2 = -\xi_3 = -\xi_4\,, \quad 1 < g = \xi_5 < B = -\xi_1 < A = \xi_6\,.$$

The coefficient of x^4 equals

$$3 - 3g + 3B - gB - 3A + Ag - AB = -3(g-1) - 3(A-B) - A(B-g) - gB < 0\,,$$

which contradicts the SP $\Sigma_{3,2,2}$.

If B) and E) take place, then one considers the polynomial $(x + 1)^2(x^2 - B^2)(x + h)(x - A)$ with

$$-\xi_4 = h < 1 = -\xi_2 = -\xi_3 < B = \xi_5 = -\xi_1 < A = \xi_6\,.$$

The coefficient of x^4 equals

$$1 - B^2 + 2h - 2A - Ah = -(B^2 - 1) - 2(A - h) - Ah < 0\,,$$

which is a contradiction with the SP $\Sigma_{3,2,2}$.

The proposition is proved. $\qquad\qquad\square$

3.7. Summarization of the Results

In this subsection we summarize the results of the section. For each SP of the left column we give in the column Y the cases (a, b, w), $a + b + w = 4$, which are realizable, and in the column N the ones that are not realizable. We list only cases which are allowed by Theorem 1. The results concerning SPs which are not in the list are obtained by reversion, see Remark 2.

SP	Y	N
$(1, 5, 1)$	$(0, 4, 0)$	
$(m, 1, q)$, $m + q = 6$	$(q - 1, 0, m - 1)$	
$(2, 4, 1)$	$(0, 2, 2)$, $(0, 3, 1)$, $(0, 4, 0)$	$(0, 0, 4)$, $(0, 1, 3)$
$(3, 3, 1)$	$(1, 0, 3)$, $(0, 0, 4)$, $(0, 1, 3)$, $(0, 2, 2)$, $(0, 3, 1)$, $(0, 4, 0)$	
$(4, 2, 1)$	$(1, 0, 3)$, $(0, 0, 4)$, $(0, 1, 3)$, $(0, 2, 2)$	$(0, 3, 1)$, $(0, 4, 0)$
$(2, 3, 2)$	all possible cases	
$(3, 2, 2)$	all other cases	$(4, 0, 0)$, $(3, 1, 0)$, $(2, 2, 0)$, $(1, 3, 0)$

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Vladimir Petrov Kostov

Université Côte dAzur
CNRS, LJAD
FRANCE
E-mail: vladimir.kostov@unice.fr