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Two Sequences of Regular Three-row Almost Hermitian Incidence Matrices

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We find two sequences of regular three-row almost Hermitian incidence matrices. Our proof is based on investigation of certain quadratic forms of a specific Jacobi polynomial and its derivative.

Keywords and Phrases: Birkhoff interpolation problem, almost Hermitian incidence matrices, Jacobi polynomials.

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1. Introduction and Statement of the Results

In 1906 Birkhoff [2] proposed a general interpolation problem for algebraic polynomials, which contains the Lagrange and Hermite interpolation problems as particular cases. The Birkhoff interpolation problem is formulated with an *incidence matrix*.

Definition 1. An *incidence matrix* $E = \{e_{ij}\}_{i=1,\, j=0}^{n}{}^{r}$ is a matrix with elements $e_{ij} \in \{0, 1\}$. The number of 1-entries in E is denoted by $|E|$.

Throughout this paper, π_m will stand for the set of algebraic polynomials of degree at most m.

Birkhoff interpolation problem (BIP). Given an incidence matrix $E = \{e_{ij}\}_{i=1,\, j=0}^{n}{}^{r}$, a vector $\mathbf{X} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $x_1 < x_2 < \cdots < x_n$, and a data set $\{\gamma_{ij} \in \mathbb{C} \,:\, e_{ij} = 1\}$, find a polynomial $p \in \pi_{|E|-1}$ such that

$$p^{(j)}(x_i) = \gamma_{ij} \quad \text{for all} \quad \{i, j\} \,:\, e_{ij} = 1\,.$$

In contrast with the Lagrange and Hermite interpolation problems, which are known to have a unique solution, the general BIP is not always solvable.

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Definition 2. An incidence matrix $E = \{e_{ij}\}_{i=1,\, j=0}^{n\ \ \, r}$ is said to be (order) *regular*, if for every vector of interpolation nodes $\mathbf{X} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, satisfying $x_1 < x_2 < \cdots < x_n$, and data set $\{\gamma_{ij} \in \mathbb{C} : e_{ij} = 1\}$, the associated Birkhoff interpolation problem has a unique solution.

Despite studied by many mathematicians, the problem of complete characterization of the regular incidence matrices remains open. A simple necessary condition for regularity was found by Pólya.

Pólya condition. *A necessary condition for* $E = \{e_{ij}\}_{i=1,\, j=0}^{n\ \ \, r}$ *to be regular is*

$$\sum_{i=1}^{n} \sum_{j=0}^{k} e_{ij} \geq k+1\,, \qquad k = 0, \ldots, |E|-1\,. \tag{1}$$

In 1969 Atkinson and Sharma [1] found a simple sufficient condition for regularity. We need another definition before formulating their result.

Definition 3. A *block* is any maximal sequence of 1-entries in a row of E. The block $e_{ij} = e_{i,j+1} = \cdots = e_{i,j+\ell-1} = 1$ is even, resp. odd, if its length ℓ is even, resp. odd number. The smallest column index j of 1-entry in a block defines its level, and a block with level 0 is called Hermitian. A row $\mathbf{e}_i = (e_{i,0}, e_{i,1}, \ldots, e_{ir})$ of E is called Hermitian row of length k if it contains a single block which is Hermitian with length k.

A block $e_{ij} = e_{i,j+1} = \cdots = e_{i,j+\ell-1} = 1$ in an interior row $\mathbf{e}_i$, $1 < i < n$, is called *supported*, if there are 1-entries in rows i_1 and i_2, $i_1 < i < i_2$ with column indices $j_1, j_2 < j$.

Atkinson–Sharma Theorem. *Every incidence matrix* $E = \{e_{ij}\}_{i=1,\, j=0}^{n\ \ \, r}$ *which satisfies the Pólya condition* (1) *and does not contain supported odd blocks is regular.*

Atkinson and Sharma also conjectured that every incidence matrix which contains a supported odd block is singular. Lorentz and Zeller [11] disproved this conjecture, showing that the matrix

$$E = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

is regular, despite having two supported odd blocks.

The complete characterization of regular incidence matrices turned out to be extremely difficult, perhaps this was the main reason some authors [3, 4, 5, 6] to examine special cases, the so-called almost Hermitian matrices, which are matrices having only one non-Hermitian interior row. Another reason is that singularity of such matrices would imply, through the technique of decoalescence of rows [7], singularity in more general cases.

Our concern in this paper is a special class of almost Hermitian three-row matrices.

The notation $E(p,q;k_1,k_2)$ stands for the incidence matrix with three rows, the first and the third of them being Hermitian of lengths p and q, respectively, while the second row contains only two 1-entries which are in columns k_1 and k_2, $k_1 < k_2$. For instance, the matrix in (2) is $E(2,2;1,4)$.

It is known (see [12, Chapter 8]) that $E(p,q;k_1,k_2)$ is not regular except possibly when one of the following conditions is satisfied:

$$p \le k_1 < k_2 - 1 \le q, \tag{3}$$

$$q+1 < k_2 \quad \text{and} \quad k_1 + k_2 = p + q + 1. \tag{4}$$

For (4) (the so-called exterior symmetric case) a complete characterization of regularity is known: a necessary and sufficient condition for regularity is E to be symmetrical matrix, i.e. $p = q$ (see [12, Chapter 8], and [14] for another proof of the sufficiency part).

Regarding (3), let us point out that: 1) the case $k_2 - 1 = k_1$ is settled by the Atkinson-Sharma Theorem, and 2) the assumption $p < q$ when $k_1 < k_2 - 1$ is not a restriction when studying regularity, as the case $p > q$ can by obtained by reflection of the interpolation nodes. A complete characterization of the regular incidence matrices in the case (3) is known only when $p = 1$, a result due to DeVore, Meir and Sharma [3]:

DeVore-Meir-Sharma Theorem. *The matrix $E(1,q;k_1,k_2)$ is regular if and only if $\varrho(k_1,k_2) < 0$, where*

$$\varrho(k_1,k_2) = (q+2)(k_1+k_2-1)^2 - 4(q+1)k_1k_2\,.$$

The following conjecture was posed in [15]:

Conjecture 1. *For every p, q, $k \in \mathbb{N}$ such that $q-p \ge 2$ and $p \le k \le q-1$ the incidence matrix $E(p,q;k,k+2)$ is regular.*

Remark 1. The case $q-p=1$ is excluded since matrices $E(p,p+1;p,p+2)$ belong to the exterior symmetric case (4). These matrices are weakly singular: $\Delta(t)$ vanishes in $(0,1)$ though without changing its sign (see Proposition 1 below).

Conjecture 1 is based on some experiments carried out in [15] with *Wolfram Mathematica* software. For $p = 1$ Conjecture 1 is verified to be true with DeVore-Meir-Sharma Theorem: in this case $\varrho(k,k+2) = 4k^2 - 4qk + q + 2$ is negative only when $q \ge 3$ and $1 \le k \le q-1$.

In this paper we give proof of Conjecture 1 in the case $q-p=2$. This case includes two sequences of three-row almost Hermitian incidence matrices. For the sake of notational convenience, we set $p = m \in \mathbb{N}$, then $q = m+2$ and $k = m$ or $k = m+1$. Our result is formulated in the following two theorems.

Theorem 1. *For every $m \in \mathbb{N}$, the three-row almost Hermitian incidence matrix $E(m, m+2; m, m+2)$ is regular.*

Theorem 2. *For every $m \in \mathbb{N}$, the three-row almost Hermitian incidence matrix $E(m, m+2; m+1, m+3)$ is regular.*

The rest of the paper is organized as follows. In Section 2 we provide some necessary background concerning BIP induced by an incidence matrix $E(p, q, k_1, k_2)$, as well as some properties of Jacobi polynomials, in particular, we give bounds for the extreme zeros of Jacobi polynomials. The proofs of Theorem 1 and Theorem 2 are given in Section 3 and Section 4, respectively.

2. Preliminaries

Without loss of generality we may assume that the set of interpolation nodes is $\mathbf{X} = (-1, t, 1)$, where $t \in (-1, 1)$. The incidence matrix $E(p, q; k_1, k_2)$ is regular if and only if the only polynomial $P \in \pi_n$, $n = p + q + 1$, which has zero at -1 of multiplicity p, zero at $x = 1$ of multiplicity q and satisfies

$$P^{(k_1)}(t) = P^{(k_2)}(t) = 0 \quad \text{for some} \quad t \in (-1, 1)$$

is $P \equiv 0$. Equivalently, the following necessary and sufficient condition for regularity of $E(p, q; k_1, k_2)$ holds true (see [12, Theorem 8.3]):

Proposition 1. *Let $\omega(x) = (1+x)^p(1-x)^q$. The matrix $E(p, q; k_1, k_2)$ is regular if and only if*

$$\Delta(t) := k_2\, \omega^{(k_1)}(t)\, \omega^{(k_2-1)}(t) - k_1\, \omega^{(k_1-1)}(t)\, \omega^{(k_2)}(t) \neq 0, \quad t \in (-1, 1)\,.$$

When $k_2 = k_1 + 2$, the condition in Proposition 1 is verified by study of some quadratic forms of specific Jacobi polynomial and its derivative. For easy reference, we collect in the next lemma some well-known properties of Jacobi polynomials $P_n^{(\alpha,\beta)}$, the orthogonal in $[-1, 1]$ polynomials with respect to Jacobi weight function $w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$ (see, e.g. [16, Chapter 4]):

Lemma 1. *The following are properties of Jacobi polynomials:*

(i) $P_n^{(\alpha,\beta)}$ *is defined by the Rodrigues' formula*

$$(-1)^n 2^n n!\, (1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{d^n}{dx^n}\big\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\big\}\,;$$

(ii) $y = P_n^{(\alpha,\beta)}$ *satisfies the ordinary differential equation*

$$(1-x^2)y''(x) + \big(\beta - \alpha - (\alpha+\beta+2)x\big)y'(x) + n(n+\alpha+\beta+1)y(x) = 0\,;$$

(iii) The derivative of Jacobi polynomial is Jacobi polynomial as well, namely,

$$\frac{d}{dx}\,P_n^{(\alpha,\beta)}(x)=\frac{1}{2}\,(n+\alpha+\beta+1)\,P_{n-1}^{(\alpha+1,\beta+1)}(x)\,;$$

(iv) The following identity holds true:

$$(x+1)\frac{d}{dx}\,P_n^{(\alpha,\beta)}(x)=(n+\beta)P_n^{(\alpha+1,\beta-1)}(x)-\beta\,P_n^{(\alpha,\beta)}(x)\,.$$

We shall need the following bounds for the zeros of $P_n^{(\alpha,\beta)}$, proved in [13]:

Lemma 2. *For every $n\geq 2$ and $\alpha,\beta>-1$, the zeros $\{x_{k,n}(\alpha,\beta)\}_{k=1}^n$ of $P_n^{(\alpha,\beta)}$ satisfy the inequalities*

$$-1+\frac{2(\beta+1)}{n(n+\alpha)+\beta+1}<x_{k,n}(\alpha,\beta)<1-\frac{2(\alpha+1)}{n(n+\beta)+\alpha+1}\,.$$

Lemma 2 improves upon the bounds implied by the Newton-Raphston iteration method:

$$x_{k,n}(\alpha,\beta)\in\Big(-1+\frac{2(\beta+1)}{n(n+\alpha+\beta+1)},\,1-\frac{2(\alpha+1)}{n(n+\alpha+\beta+1)}\Big)\,.$$

Although there are more precise estimates for the extreme zeroes of Jacobi polynomials, an advantage of the bounds given above is their simple form, which makes them easy to work with. These bounds represent correctly the behavior of the extreme zeros of $P_n^{(\alpha,\beta)}$ when either one of parameters α and β tends to -1 or to infinity, or n grows.

3. Proof of Theorem 1

We set

$$\omega(x)=(1-x)^{m+2}(1+x)^m\,, \tag{5}$$

then, according to Proposition 1, we need to show that for every $t\in(-1,1)$,

$$\Delta(t)=(m+2)\,\omega^{(m)}(t)\,\omega^{(m+1)}(t)-m\,\omega^{(m-1)}(t)\,\omega^{(m+2)}(t)\neq 0. \tag{6}$$

From Lemma 1 (i) with $n=m-1$, $\alpha=3$ and $\beta=1$ we deduce that

$$\omega^{(m-1)}(t)=(-1)^{m-1}2^{m-1}(m-1)!\,(1-t)^3(1+t)P_{m-1}^{(3,1)}(t)\,.$$

Using Lemma 1 (iii) with $n=m$, $\alpha=2$ and $\beta=0$, we obtain

$$P_{m-1}^{(3,1)}(t)=\frac{2}{m+3}\,\frac{d}{dt}\,P_m^{(2,0)}(t)\,,$$

whence

$$\omega^{(m-1)}(t) = c(1-t)^3(1+t)y'(t)\,, \tag{7}$$

where

$$y = P_m^{(2,0)}, \qquad c = (-1)^{m+1}2^m\frac{(m-1)!}{m+3}\,.$$

After differentiation of (7) and replacement of y'' using the differential equation (see Lemma 1 (ii))

$$(1-t^2)y''(t) - (2+4t)y'(t) + m(m+3)y(t) = 0\,, \tag{8}$$

we obtain

$$\omega^{(m)}(t) = -c\,m(m+3)(1-t)^2y(t)\,. \tag{9}$$

Two further differentiations of (9) and use of the differential equation for $y(t)$ yield

$$\omega^{(m+1)}(t) = c\,m(m+3)(1-t)\big[2y(t) - (1-t)y'(t)\big]\,,$$

$$\omega^{(m+2)}(t) = c\,\frac{m(m+3)}{1+t}\Big[\big((m+1)(m+2)(1-t)-4\big)y(t) + 2(1-t)y'(t)\Big]\,.$$

By replacing the derivatives of ω occurring in (6) we find

$$\Delta(t) = -2c^2m^2(m+3)(1-t)^3\,K(t) = -\frac{2^{2m+1}(m!)^2}{m+3}\,K(t)\,(1-t)^3\,,$$

where $K(t)$ is the following quadratic form of $y(t)$ and $y'(t)$:

$$K(t) = (m+2)(m+3)y^2(t) - \big[2+(m+2)(1-t)\big]y(t)\,y'(t) + (1-t)\big[y'(t)\big]^2.$$

Let us assume that $\Delta(t)$ has a zero in $(-1,1)$, then $K(t)$ will vanish in $(-1,1)$, too. Therefore the discriminant $D(t)$ of $K(t)$ must be non-negative therein. From

$$D(t) = (m+2)^2(1-t)^2 - 4(m+2)^2(1-t) + 4 \tag{10}$$

we conclude that the requirement $D(t) \geq 0$ in $(-1,1)$ is equivalent to

$$t \in \Big[\frac{2\sqrt{(m+1)(m+3)}}{m+2} - 1, 1\Big), \tag{11}$$

which in particular implies $t \in (0,1)$. According to Lemma 2, the largest zero τ of $y = P_m^{(2,0)}$ satisfies $\tau < 1 - \frac{6}{m^2+3}$, and it is easily verified that

$$\frac{2\sqrt{(m+1)(m+3)}}{m+2} - 1 > 1 - \frac{6}{m^2+3}\,.$$

This means that if (11) is satisfied, then t is located to the right of the largest zero of y. As a consequence, under the assumption (11) we have $y(t) > 0$ and $y'(t) > 0$.

We rewrite $K(t)$ as

$$K(t) = \big[(m+2)^2 y(t) - 2y'(t)\big]y(t) \\ + (m+2)y^2(t) - (m+2)(1-t)y(t)y'(t) + (1-t)[y'(t)]^2$$

and prove that, under the assumption (11), the expressions in the two lines on the right-hand side are both positive. Firstly, let us show that the quadratic form in the second line has negative discriminant $\widetilde{D}(t)$,

$$\widetilde{D}(t) = (m+2)^2(1-t)\Big(1-t-\frac{4}{m+2}\Big).$$

Since $t \in (0,1)$, we deduce from (10)

$$\begin{aligned} 0 \le D(t) &= (m+2)^2(1-t)^2 - 4(m+2)^2(1-t) + 4 \\ &< (m+2)^2(1-t) - 4(m+2)^2(1-t) + 4 \\ &= 4 - 3(m+2)^2(1-t). \end{aligned}$$

Hence

$$1-t < \frac{4}{3(m+2)^2} \tag{12}$$

and consequently

$$\widetilde{D}(t) < (m+2)^2(1-t)\Big(\frac{4}{3(m+2)^2} - \frac{4}{m+2}\Big) < 0.$$

Secondly, we prove that the expression in the first line is positive, i.e.,

$$\big[(m+2)^2 y(t) - 2y'(t)\big]y(t) > 0.$$

Since $y(t) > 0$ and $y'(t)$ is monotonically increasing to the right of τ, the largest zero of $y = P_m^{(2,0)}$, it suffices to show that for t satisfying (11) there holds

$$(m+2)^2 y(t) \ge 2y'(1). \tag{13}$$

From

$$y(t) \ge y(1) - y'(1)(1-t), \qquad t > \tau,$$

we conclude that a sufficient condition for (13) to hold true is

$$(m+2)^2\big[y(1) - y'(1)(1-t)\big] \ge 2y'(1),$$

or, equivalently,

$$(m+2)^2\big[1 - z(1-t)\big] \ge 2z, \qquad z = \frac{y'(1)}{y(1)}.$$

From the differential equation (8) we find $z = m(m+3)/6$, and the last inequality becomes

$$(m+2)^2\Big[1 - \frac{m(m+3)}{6}(1-t)\Big] \geq \frac{m(m+3)}{3}\,,$$

which easily follows from (12). Indeed, we have

$$\begin{aligned}(m+2)^2\Big[1 - \frac{m(m+3)}{6}(1-t)\Big] &> (m+2)^2\Big[1 - \frac{2m(m+3)}{9(m+2)^2}\Big]\\ &= (m+2)^2 - \frac{2}{9}m(m+3)\\ &> \frac{1}{3}m(m+3)\,.\end{aligned}$$

Thus, (13) holds true, and consequently $K(t) > 0$ provided (11) is satisfied. On the other hand, if $t \in (-1,1)$ and (11) is violated, then again $K(t) > 0$, since the discriminant of $K(t)$ is negative. Thus, we showed that $K(t) \neq 0$ in $(-1,1)$, and consequently $\Delta(t) \neq 0$ in $(-1,1)$. This completes the proof of Theorem 1. □

4. Proof of Theorem 2

According to Proposition 1, with ω as defined in (5) we need to show that for every $t \in (-1,1)$,

$$\Delta(t) = (m+3)\omega^{(m+1)}(t)\,\omega^{(m+2)}(t) - (m+1)\omega^{(m)}(t)\omega^{(m+3)}(t) \neq 0.$$

Lemma 1 (i) with $n = m$, $\alpha = 2$ and $\beta = 0$ yields

$$\omega^{(m)}(t) = c\,(1-t)^2 P_m^{(2,0)}(t)\,, \qquad c = (-1)^m 2^m m!\,.$$

From Lemma 1 (iv) with $n = m$ and $\alpha = \beta = 1$ we find

$$(t+1)\frac{d}{dt}\,P_m^{(1,1)}(t) = (m+1)P_m^{(2,0)}(t) - P_m^{(1,1)}(t)\,,$$

whence

$$(m+1)\omega^{(m)}(t) = c\,(1-t)^2\big[y(t) + (1+t)y'(t)\big] \tag{14}$$

with

$$y(t) = P_m^{(1,1)}(t)\,.$$

After differentiation of (14) and replacement of $y''(t)$ from the differential equation of Lemma 1 (ii) for $y = P_m^{(1,1)}$,

$$(1-t^2)y''(t) - 4t\,y'(t) + m(m+3)y(t) = 0\,, \tag{15}$$

we find

$$\omega^{(m+1)}(t) = -c\,(m+2)(1-t)y(t)\,. \tag{16}$$

Two further differentiations of (16) yield

$$\omega^{(m+2)}(t) = -c\,(m+2)\big[(1-t)y'(t) - y(t)\big]\,, \tag{17}$$

$$\omega^{(m+3)}(t) = -c(m+2)\big[(1-t)y''(t) - 2y'(t)\big]\,. \tag{18}$$

We replace y'' in (18) using differential equation (15) to obtain

$$\omega^{(m+3)}(t) = \frac{c(m+2)}{1+t}\,\big[m(m+3)y(t) + 2(1-t)y'(t)\big]. \tag{19}$$

By substituting in $\Delta(t)$ the expressions for $\omega^{(j)}(t)$, $m \le j \le m+3$ from (14), (16), (17) and (19), we arrive at the following representation of $\Delta(t)$:

$$\Delta(t) = \frac{-2c^2\,(m+2)(1-t)}{1+t}\,K(t), \tag{20}$$

where

$$\begin{aligned} K(t) = (m+3)(m+1+t)y(t)^2 - (1-t)\big[(m+4)(1+t) - 2\big]y(t)\,y'(t) \\ + (1-t)(1-t^2)\big[y'(t)\big]^2. \end{aligned} \tag{21}$$

The discriminant D of the quadratic form $K(t)$ is non-negative in $(-1,1)$ if and only if

$$t \in \Big(-1, 1 - \frac{2\sqrt{(m+1)(m+3)}}{m+2}\Big]\,, \tag{22}$$

which in particular implies that $t < 0$. On the other hand, according to Lemma 2, the smallest zero τ of $y = P_m^{(1,1)}$ satisfies

$$\tau \ge -1 + \frac{4}{m^2+m+2}.$$

Comparison with (22) shows that t is located to the left from τ, and as a consequence, $y(t)y'(t) < 0$.

We rewrite (21) in the form

$$\begin{aligned} K(t) = {} & \big[m(m+3)y(t) + 2(1-t)y'(t)\big]y(t) \\ & + (1+t)\big[(m+3)y^2(t) - (m+4)(1-t)y(t)y'(t) + (1-t)^2\big[y'(t)\big]^2\big]\,. \end{aligned}$$

By using (15), we substitute in the first line

$$m(m+3)y(t) + 2(1-t)y'(t) = (1+t)\big[2y'(t) - (1-t)y''(t)\big]$$

to obtain $K(t) = (1+t)\widetilde{K}(t)$, where

$$\begin{aligned} \widetilde{K}(t) = (m+3)y^2(t) - \big[(m+4)(1-t) - 2\big]y(t)y'(t) \\ + (1-t)\big[(1-t)\big[y'(t)\big]^2 - y(t)y''(t)\big]. \end{aligned} \tag{23}$$

The last summand in $\widetilde{K}(t)$ is positive under the assumption (22), since then $t < 0$ and

$$(1-t)\big[y'(t)\big]^2 - y(t)y''(t) > \big[y'(t)\big]^2 - y(t)y''(t) > 0\,,$$

by the Laguerre inequality for real-root polynomials.

It is easily verified that under the assumption (22) we also have

$$(m+4)(1-t) - 2 > 2\Big(\frac{(m+4)\sqrt{(m+1)(m+3)}}{m+2} - 1\Big) > 0,$$

and since $y(t)y'(t) < 0$, it follows from (23) that $\widetilde{K}(t) > 0$ and consequently $K(t) > 0$ when (22) is assumed. On the other hand, if $t \in (-1,1)$ and (22) is violated, then $K(t) > 0$ as its discriminant is negative. Thus, $K(t) > 0$ and therefore $\Delta(t) < 0$ in $(-1,1)$. Theorem 2 is proved. □

Remark 2. The above proof of Theorem 2 follows the approach applied to the proof of Theorem 1. After the proof of Theorem 2 was accomplished, we found the representation for $\Delta(t)$ in (20):

$$\begin{aligned}-\frac{\Delta(t)}{c^2(m+2)(1-t)} &= \big[(m+2)y(t) - (1-t)y'(t)\big]^2 + (m+2)y^2(t)\\ &\quad + (1-t)^2\big[\big[y'(t)\big]^2 - y(t)y''(t)\big]\,,\end{aligned}$$

whence the conclusion $\Delta(t) < 0$ in $(-1,1)$ follows immediately without any reference to discriminants and bounds for the extreme zeros of $y(t)$.

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