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Inequalities for Polynomials on Simplexes [*]

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Denote by π_n^d the set of all algebraic polynomials of d variables and of total degree not exceeding n. Let $\|\cdot\|_K$ be the uniform norm on a compact set $K \subset \mathbb{R}^d$. We use the notation Δ^d for the standard simplex in $\mathbb{R}^d$. We also set $\pi_2 := \pi_2^2$ and $\Delta := \Delta^2$.

In the papers [11, 12, 13] we described the set E_Δ of all extreme points of $B_\Delta := \{f \in \pi_2 : \|f\|_\Delta \le 1\}$.

Recently in [16] we found a characterization and explicit formulas for the strictly definite extreme points of $B_{\partial\Delta} := \{f \in \pi_2 : \|f\|_{\partial\Delta} \le 1\}$. As an application we proved some inequalities connecting $\|f\|_{\Delta^d}$ and $\|f\|_{\partial\Delta^d}$, for $f \in \pi_2^d$ and $d \ge 2$. In particular, for $d = 2$ we have the exact inequality

$$\|f\|_\Delta \le \frac{5}{3}\,\|f\|_{\partial\Delta}, \quad \text{for every } f \in \pi_2. \qquad (*)$$

Here we present the main results from [16] and prove a generalization for arbitrary simplex in $\mathbb{R}^d$. We also give a graphical illustration of the extremal polynomial in the inequality $(*)$.

Keywords and Phrases: Polynomial inequalities, extreme points, simplex.

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1. Introduction

Denote by π_n the set of all real algebraic polynomials of two variables and of total degree not exceeding n. Let

$$\Delta := \{(x, y) \in \mathbb{R}^2 : x \ge 0,\ y \ge 0,\ x + y \le 1\}$$

be the standard simplex in $\mathbb{R}^2$ and $\partial\Delta$ be the boundary of Δ.

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We use the notations $\|f\|_\Delta$ and $\|f\|_{\partial\Delta}$ for the uniform norm of f on Δ and $\partial\Delta$, respectively. Let

$$B_\Delta := \{f \in \pi_2 : \|f\|_\Delta \le 1\} \quad \text{and} \quad B_{\partial\Delta} := \{f \in \pi_2 : \|f\|_{\partial\Delta} \le 1\}$$

be the unit balls corresponding to these norms.

Recall that a point p of a convex set B in a linear space is *extreme* if the equality $p = \lambda p_1 + (1-\lambda)p_2$, for some $p_1, p_2 \in B$ and $\lambda \in (0,1)$, implies $p_1 = p_2 = p$.

According to the Krein-Milman theorem, every nonempty, compact and convex set B in a Banach space has an extreme point. Moreover, if E denotes the set of all extreme points of B, then B can be represented as the closure of the convex hull of E. This result motivates the study of the extreme points of the unit ball of various polynomial spaces. We refer to the papers [5, 6, 7, 2, 8, 9, 10, 4]. The description of the extreme points can be useful in finding exact constants in polynomial inequalities involving convex functionals, see [3, 14, 15, 1].

We described in [11, 12, 13] the set of all extreme points of B_Δ. In the recent paper [16] we gave a full description of the strictly definite extreme points of $B_{\partial\Delta}$ and proved certain inequalities connecting $\|f\|_{\Delta^d}$ and $\|f\|_{\partial\Delta^d}$, for $f \in \pi_2^d$ and $d \ge 2$.

Here we present the main results from [16] and prove a generalization for arbitrary simplex in $\mathbb{R}^d$. We also give a graphical illustration.

2. Results

We shall need some additional notations. Let $E_{\partial\Delta}$ be the set of all extreme points of $B_{\partial\Delta}$ and $E^-_{\partial\Delta}$ be the subset of $E_{\partial\Delta}$ which consists of the strictly concave polynomials.

We denote the vertices of Δ by $O(0,0)$, $A(1,0)$ and $B(0,1)$. The sides OA, OB and AB of Δ will be denoted by ℓ_1, ℓ_2 and ℓ_3, respectively. Let $\overrightarrow{\ell_1} = \overrightarrow{OA}$, $\overrightarrow{\ell_2} = \overrightarrow{OB}$, $\overrightarrow{\ell_3} = \overrightarrow{AB}$.

A qualitative characterization of the elements of $E^-_{\partial\Delta}$ is contained in the following theorem.

Theorem 1. *Let $p \in \pi_2$ be a strictly concave function. Then $p \in E_{\partial\Delta}$ if and only if the following conditions hold true:*

(i) *For every $i = 1,2,3$ there exists a unique point $X_i \in \operatorname{int} \ell_i$ such that $p(X_i) = 1$ and $\dfrac{\partial p}{\partial \overrightarrow{\ell_i}}(X_i) = 0$;*

(ii) $\min\{p(O), p(A), p(B)\} = -1$.

Note that all strictly convex elements of $E_{\partial\Delta}$ have the form $q = -p$, where $p \in E^{-}_{\partial\Delta}$.

The next result provides an explicit form of the elements of $E^{-}_{\partial\Delta}$.

Theorem 2. *A polynomial* $p \in \pi_2$ *is a strictly concave element of* $E_{\partial\Delta}$ *if and only if*

$$p(x,y) = a + bx + cy + dx^2 + 2exy + fy^2,$$

where

$$\begin{aligned}
&a = \gamma,\\
&b = 2\sqrt{1-\gamma}\,\big(\sqrt{1-\alpha} + \sqrt{1-\gamma}\big),\\
&c = 2\sqrt{1-\gamma}\,\big(\sqrt{1-\beta} + \sqrt{1-\gamma}\big),\\
&d = -\big(\sqrt{1-\alpha} + \sqrt{1-\gamma}\big)^2,\\
&e = \sqrt{1-\alpha}\,\sqrt{1-\beta} - \sqrt{1-\gamma}\,\big(\sqrt{1-\alpha} + \sqrt{1-\beta} + \sqrt{1-\gamma}\big),\\
&f = -\big(\sqrt{1-\beta} + \sqrt{1-\gamma}\big)^2,
\end{aligned}$$

and

$$\alpha, \beta, \gamma \in [-1, 1), \qquad \min\{\alpha, \beta, \gamma\} = -1.$$

The following theorem provides a sharp inequality between the norms of a polynomial $f \in \pi_2$ on the standard triangle and its boundary.

Theorem 3. *The inequality*

$$\|f\|_\Delta \le \frac{5}{3}\,\|f\|_{\partial\Delta} \tag{1}$$

holds true for every $f \in \pi_2$*. The equality is attained if and only if* $f = cf^*$*, where* c *is a constant and*

$$f^*(x,y) = -1 + 8(x+y) - 8(x^2 + xy + y^2). \tag{2}$$

Figure 1 presents the graph of the extremal polynomial f^*.

The next theorem contains a generalization of (1) for any triangle in $\mathbb{R}^2$.

Theorem 4. *Let* T *be a given triangle in* $\mathbb{R}^2$*. For every* $f \in \pi_2$ *we have the inequality*

$$\max_{X \in T} |f(X)| \le \frac{5}{3} \max_{X \in \partial T} |f(X)|.$$

The equality is attained if and only if $f(X) = c\,f^*(L(X))$*, where* f^* *is defined by* (2)*,* c *is a constant and* L *is an affine one-to-one transformation from* T *to* Δ*.*

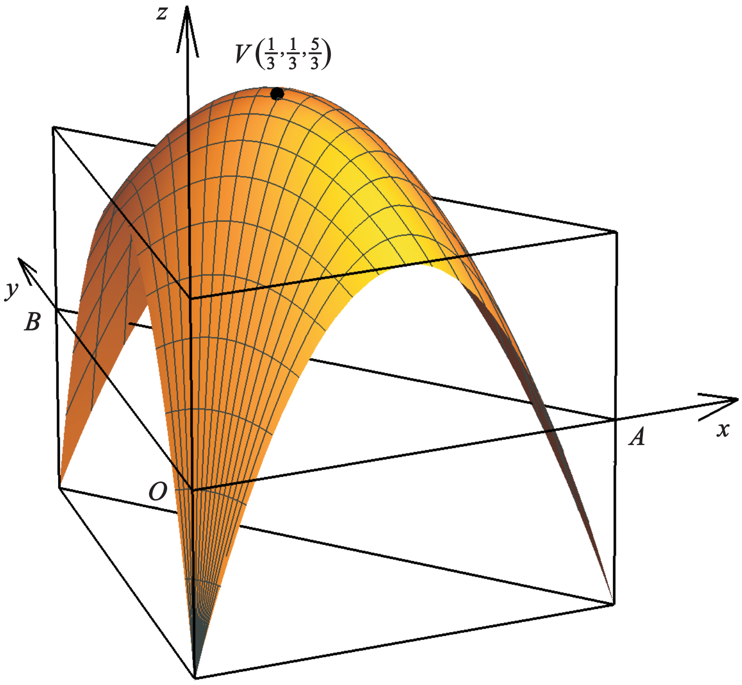

Figure 1. Graph of the extremal polynomial $f^*(x,y)$ in Theorem 3.

Another extension of the inequality (1) is for any dimension $d \geq 2$.

Let π_n^d be the set of all algebraic polynomials of d variables and of total degree not exceeding n and

$$\Delta_r^d := \Big\{(x_1,\dots,x_d) \in \mathbb{R}^d : x_i \geq 0,\, i=1,\dots,d,\, \sum_{i=1}^d x_i \leq r\Big\}, \qquad r>0.$$

Theorem 5. *The inequality*

$$\max_{X\in\Delta_r^d} |f(X)| \leq \frac{5}{3} \max_{X\in\partial\Delta_r^d} |f(X)|$$

holds true for every $f \in \pi_2^d$, $d \geq 2$.

Let us suppose that the points $v_0,\dots,v_d$ in $\mathbb{R}^d$ are affinely independent, i.e. the vectors $v_1 - v_0,\dots,v_d - v_0$ are linearly independent. The simplex determined by the points $v_0,\dots,v_d$ is the set

$$S := \Big\{x = \sum_{i=0}^d x_i v_i : x_i \geq 0,\, i=1,\dots,d,\, \sum_{i=0}^d x_i = 1\Big\}.$$

The following theorem provides an extension of Theorem 5 for arbitrary simplexes in $\mathbb{R}^d$.

Theorem 6. *Let S be a simplex in $\mathbb{R}^d$. For every $f \in \pi_2^d$, $d \geq 2$ we have the inequality*

$$\max_{X \in S} |f(X)| \leq \frac{5}{3} \max_{X \in \partial S} |f(X)|.$$

Proof. Let $v_0, \ldots, v_d$ be the vertices of S. It is easily seen that there exists a unique affine transformation $L : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$L(v_i) = e_i, \qquad i = 0, \ldots, d,$$

where $e_0 = (0, \ldots, 0)$, $e_1 = (1, 0, \ldots, 0)$, $\ldots$, $e_d = (0, \ldots, 0, 1)$. Clearly L maps S onto $\Delta^d := \Delta_1^d$ and also ∂S onto $\partial \Delta^d$.

Given $f \in \pi_2^d$ we set $g(Y) = f(L^{-1}(Y))$ for $Y \in \Delta^d$. Note that $g \in \pi_2^d$. By Theorem 5 we get

$$\begin{aligned} \max_{X \in S} |f(X)| &= \max_{Y \in \Delta^d} |g(Y)| \\ &\leq \frac{5}{3} \max_{Y \in \partial \Delta^d} |g(Y)| \\ &= \frac{5}{3} \max_{X \in \partial S} |f(X)|. \end{aligned}$$

Theorem 6 is proved. □

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