Multi-dimensional Integration and (Generalized) Harmonic Sums.

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Two Mathematica Packages

- **MultiIntegrate** allows to compute multi-dimensional integrals over hyperexponential integrands in terms of (generalized) harmonic sums. This package uses variations and extensions of the multivariate Alkmkvist-Zeilberger algorithm.
- **HarmonicSums** allows to deal with nested sums such as harmonic sums, S-sums, cyclotomic sums and cyclotomic S-sums as well as iterated integrals such as harmonic polylogarithms, multiple polylogarithms and cyclotomic polylogarithms in an algorithmic fashion.

Note that for some of the functionalities both packages rely on the **Sigma** package by Carsten Schneider.

MultiIntegrate

We use the Almkvist Zeilberger algorithm to evaluate integrals of the form

$$\int_{u_d}^{o_d} \dots \int_{u_1}^{o_1} F(n; x_1, \dots, x_d) dx_1 \dots dx_d,$$

where $F(n; x_1, \dots, x_d)$ is a hyperexponential function i. e.,

$$F(n; x_1, \dots, x_d) = q(n; x_1, \dots, x_d) \cdot e^{\frac{a(x_1, \dots, x_d)}{b(x_1, \dots, x_d)}} \cdot \left(\prod_{p=1}^{P} S_p(x_1, \dots, x_d)^{\alpha_p} \right) \cdot \left(\frac{s(x_1, \dots, x_d)}{t(x_1, \dots, x_d)} \right)^n,$$

with

$$a(x_1, \dots, x_d), b(x_1, \dots, x_d), s(x_1, \dots, x_d), t(x_1, \dots, x_d), q(n; x_1, \dots, x_d) \in \mathbb{K}[x_1, \dots, x_d]$$

and $S_p(x_1, \dots, x_d) \in \mathbb{K}[x_1, \dots, x_d], \alpha_p \in \mathbb{K}$ for $1 \leq p \leq P$. With e.g., $\mathbb{K} = \mathbb{Q}(\varepsilon)$.

Our strategy to evaluate integrals of the form

$$\int_{u_d}^{o_d} \dots \int_{u_1}^{o_1} F(n; x_1, \dots, x_d) dx_1 \dots dx_d,$$

where $F(n; x_1, \dots, x_d)$ is a hyperexponential function is:

- compute a recurrence for the integrand $F(n; x_1, \dots, x_d)$

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where $F(n; x_1, \dots, x_d)$ is a hyperexponential function is:

- compute a recurrence for the integrand $F(n; x_1, \dots, x_d)$
- use the reccurrence for the integrand to derive a recurrence for the integral
- solve the recurrence

Recurrence for the Integrand

$$F(n;x_1,\ldots,x_d)=q(n;x_1,\ldots,x_d)\cdot e^{\frac{a(x_1,\ldots,x_d)}{b(x_1,\ldots,x_d)}}\cdot\left(\prod_{p=1}^{P}S_p(x_1,\ldots,x_d)^{\alpha_p}\right)\cdot\left(\frac{s(x_1,\ldots,x_d)}{t(x_1,\ldots,x_d)}\right)^n,$$

where

$a(x_1,\ldots,x_d), b(x_1,\ldots,x_d), s(x_1,\ldots,x_d), t(x_1,\ldots,x_d), q(n;x_1,\ldots,x_d)\in\mathbb{K}[x_1,\ldots,x_d]$ and $S_p(x_1,\ldots,x_d)\in\mathbb{K}[x_1,\ldots,x_d], \alpha_p\in\mathbb{K}$ for $1\leq p\leq P$. With e.g., $\mathbb{K}=\mathbb{Q}(\varepsilon)$.

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$$F(n;x_1,\ldots,x_d)=q(n;x_1,\ldots,x_d)\cdot e^{\frac{a(x_1,\ldots,x_d)}{b(x_1,\ldots,x_d)}}\cdot\left(\prod_{p=1}^{P}S_p(x_1,\ldots,x_d)^{\alpha_p}\right)\cdot\left(\frac{s(x_1,\ldots,x_d)}{t(x_1,\ldots,x_d)}\right)^n,$$

where

$a(x_1,\ldots,x_d), b(x_1,\ldots,x_d), s(x_1,\ldots,x_d), t(x_1,\ldots,x_d), q(n;x_1,\ldots,x_d)\in\mathbb{K}[x_1,\ldots,x_d]$
and $S_p(x_1,\ldots,x_d)\in\mathbb{K}[x_1,\ldots,x_d], \alpha_p\in\mathbb{K}$ for $1\leq p\leq P$. With e.g., $\mathbb{K}=\mathbb{Q}(\varepsilon)$.

Then there exist

$L\in\mathbb{N}, e_0(n), e_1(n),\ldots,e_L(n)\in\mathbb{K}[n]$, *not all zero*, and $R_i(n;x_1,\ldots,x_d)\in\mathbb{K}(n,x_1,\ldots,x_d)$

such that

$$G_i(n;x_1,\ldots,x_d):=R_i(n;x_1,\ldots,x_d)F(n;x_1,\ldots,x_d)$$

satisfy

$$\sum_{i=0}^{L}e_i(n)F(n+i;x_1,\ldots,x_d)=\sum_{i=1}^{d}D_{x_i}G_i(n;x_1,\ldots,x_d).$$

Using the Package MultiIntegrate

In[1]:= << **Sigma.m**

In[2]:= << **HarmonicSums.m**

In[3]:= << **MultiIntegrate.m**

Sigma - A summation package by Carsten Schneider -RISC Linz- V 1.0 (7/7/11)

HarmonicSums by Jakob Ablinger -RISC Linz- Version 1.0 (1/3/12)

MultiIntegrate by Jakob Ablinger -RISC Linz- Version 1.0 (1/3/12)

In[4]:= $\mathbf{mAZ}[\frac{1}{(xy)^{n+1}(1-x-y+xy)^z}, n, \{x, y\}, f]$

Out[4]=

$$\left\{\left\{1, \frac{(-1+x)^{-z}(-1+y)^{-z}\left(\frac{1}{xy}\right)^n}{xy}\right\}, \left\{\frac{(-1+x)^{-z}(-1+y)^{-z}(xy)^{-2-n}}{xy}, \left\{\left\{nx^2y - nxy + x^2yz + x^2y - xyz - xy, x\right\}, \left\{ny^2 - ny + 2y^2 - 2y, y\right\}\right\}, \left(-1-2n-n^2-2z-2nz-z^2\right) f(1+n) + \left(4+4n+n^2\right) f(2+n)\right\}\right\}$$

Recurrence for the Integral 1

We now consider the integral

$$a(n) := \int_{u_d}^{o_d} \dots \int_{u_1}^{o_1} F(n; x_1, \dots, x_d) dx_1 \dots dx_d,$$

Where for $F(n; x_1, \dots, x_d)$ we have

$$\sum_{i=0}^{L} e_i(n) F(n+i; x_1, \dots, x_d) = \sum_{i=1}^{d} D_{x_i} G_i(n; x_1, \dots, x_d).$$

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If

$$F(n; \dots, x_{i-1}, u_i, x_{i+1}, \dots) \;=\; F(n; \dots, x_{i-1}, o_i, x_{i+1}, \dots) = 0,$$

we also have

$$G_i(n; \dots, x_{i-1}, u_i, x_{i+1}, \dots) \;=\; G_i(n; \dots, x_{i-1}, o_i, x_{i+1}, \dots) = 0$$

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$$\sum_{i=0}^{L} e_i(n) F(n+i; x_1, \dots, x_d) = \sum_{i=1}^{d} D_{x_i} G_i(n; x_1, \dots, x_d).$$

If

$$F(n; \dots, x_{i-1}, u_i, x_{i+1}, \dots) \quad = \quad F(n; \dots, x_{i-1}, o_i, x_{i+1}, \dots) = 0,$$

we also have

$$G_i(n; \dots, x_{i-1}, u_i, x_{i+1}, \dots) \quad = \quad G_i(n; \dots, x_{i-1}, o_i, x_{i+1}, \dots) = 0$$

and hence $a(n)$ satisfies the homogenous linear recurrence

$$\sum_{i=0}^{L} e_i(n) a(n+i) = 0.$$

Using the Package MultiIntegrate

In[5]:= $$\mathbf{mAZIntegrate}\left[\frac{(a-x)(-1+x)(-2+y)(y-1)}{x^m y^n}, n, \{\{x, a, 1\}, \{y, 1, 2\}\}\right]$$

Out[5]= $$\frac{-5+15a+5m-5am+n-3an-mn+amn}{(-3+m)(-2+m)(-1+m)(-3+n)(-2+n)(-1+n)}+\frac{2^{2-n}(1-3a-m+am+n-3an-mn+amn)}{(-3+m)(-2+m)(-1+m)(-3+n)(-2+n)(-1+n)}+a^{-m}\left(\frac{-15a^2+5a^3+5a^2m-5a^3m+3a^2n-a^3n-a^2mn+a^3mn}{(-3+m)(-2+m)(-1+m)(-3+n)(-2+n)(-1+n)}+\frac{2^{2-n}\left(3a^2-a^3-a^2m+a^3m+3a^2n-a^3n-a^2mn+a^3mn\right)}{(-3+m)(-2+m)(-1+m)(-3+n)(-2+n)(-1+n)}\right)$$

Recurrence for the Integral 2

We again consider the integral

$$a(n) := \int_{u_d}^{o_d} \dots \int_{u_1}^{o_1} F(n; x_1, \dots, x_d) dx_1 \dots dx_d,$$

- suppose $F(n; x_1, \dots, x_d)$ does not vanish at the bounds

Recurrence for the Integral 2

We again consider the integral

$$a(n) := \int_{u_d}^{o_d} \dots \int_{u_1}^{o_1} F(n; x_1, \dots, x_d) dx_1 \dots dx_d,$$

- suppose $F(n; x_1, \dots, x_d)$ does not vanish at the bounds
- $G_i(n; x_1, \dots, x_d)$ does not have to vanish at the bounds

Recurrence for the Integral 2

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- suppose $F(n; x_1, \dots, x_d)$ does not vanish at the bounds
- $G_i(n; x_1, \dots, x_d)$ does not have to vanish at the bounds
- then force the G_i to vanish at the integration bounds by modifying the ansatz, and look for G_i of the form

$$G_i(n; x_1, \dots, x_d) = \overline{G}_i(n; x_1, \dots, x_d)(x_i - u_i)(x_i - o_i),$$

Recurrence for the Integral 2

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$$G_i(n; x_1, \dots, x_d) = \overline{G}_i(n; x_1, \dots, x_d)(x_i - u_i)(x_i - o_i),$$

- hence $a(n)$ satisfies again a homogenous linear recurrence of the form
$$\sum_{i=0}^{L} e_i(n) a(n+i) = 0.$$

Example

$$I(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1+x_1 \cdot x_2)^n}{(1+x_1)^\varepsilon}}_{F(n):=} dx_1 dx_2.$$

Note that the integrand does not vanish at the integration bounds.

Example

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Note that the integrand does not vanish at the integration bounds.
Using our ansatz we find

$$\begin{aligned}
&2(n+1)(\varepsilon - n - 2)F(n) - (n+2)(5\varepsilon - 5n - 13)F(n+1) \\
&+(n+3)(4\varepsilon - 4n - 13)F(n+2) - (n+4)(\varepsilon - n - 4)F(n+3) \\
&= D_{x_1} F(n)(x_1 - 1)\, x_1\, (x_1 + 1)\, x_2\, ((n+3)x_1 x_2 + 2) \\
&+D_{x_2} F(n)(x_2 - 1)\, x_2\, \left(x_2^2 x_1^3(-\varepsilon + n + 4) - (\varepsilon - 3)x_2 x_1^2 + (n+2)x_2 x_1 + 1\right).
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\end{aligned}$$

Integration of this reccurence yields

$$\begin{aligned}
&2(n+1)(\varepsilon - n - 2)I(\varepsilon, n) - (n+2)(5\varepsilon - 5n - 13)I(\varepsilon, n+1) \\
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$$I(\varepsilon,n)=\int_0^1\int_0^1 \underbrace{\frac{(1+x_1\cdot x_2)^n}{(1+x_1)^\varepsilon}}_{F(n):=} dx_1 dx_2.$$

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Using our ansatz we find

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&=D_{x_1}F(n)(x_1-1)\,x_1\,(x_1+1)\,x_2\,((n+3)x_1x_2+2)\\
&+D_{x_2}F(n)(x_2-1)\,x_2\left(x_2^2x_1^3(-\varepsilon+n+4)-(\varepsilon-3)x_2x_1^2+(n+2)x_2x_1+1\right).
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&+(n+3)(4\varepsilon-4n-13)I(\varepsilon,n+2)-(n+4)(\varepsilon-n-4)I(\varepsilon,n+3)=0.
\end{aligned}$$

Solving the recurrence leads to

$$I(\varepsilon,n)=\frac{1}{n+1}\left(\sum_{i=1}^{n}\frac{1}{-i+\varepsilon-1}-2^{1-\varepsilon}\sum_{i=1}^{n}\frac{2^i}{-i+\varepsilon-1}+\frac{2^{-\varepsilon}\left(2^\varepsilon-2\right)}{\varepsilon-1}\right)$$

In[6]:= $\mathbf{mAZDirectIntegrate}\left[\frac{(1+x1*x2)^n}{(1+x1)^\epsilon}, n, \{\{x1, 0, 1\}, \{x2, 0, 1\}\}\right]$

Out[6]= $$\frac{\sum_{\iota_1=1}^{n}\frac{1}{\iota_1-\epsilon+1}}{-n-1}-\frac{2\sum_{\iota_1=1}^{n}\frac{2^{\iota_1}}{-\iota_1+\epsilon-1}}{(n+1)2^\epsilon}+\frac{2^\epsilon-2}{(n+1)(\epsilon-1)2^\epsilon}$$

In[7]:= $\mathbf{mAZDirectIntegrate}\left[\frac{(1+x1+x2+x1*x2)^n}{(1+x1)^\epsilon}, n, \{\{x1, 0, 1\}, \{x2, 0, 1\}\}\right]$

Out[7]= $$\frac{2^{-2n-\epsilon}\left(208-83\ 2^{2+n}+63\ 2^{1+2n}-2^{4+\epsilon}+7\ 2^{2+n+\epsilon}-13\ 2^{2n+\epsilon}\right)}{(1+n)(1+n-\epsilon)}$$

In[8]:= $\mathbf{mAZDirectIntegrate}\left[\frac{(1+x1+x2+x1*x2)^n}{(1+x1)^\epsilon}, n, \{\{x1, 0, \frac{1}{2}\}, \{x2, 0, 2\}\}\right]$

Out[8]= $$\frac{2^{-2n-\epsilon}\left(208-83\ 2^{2+n}+63\ 2^{1+2n}-2^{4+\epsilon}+7\ 2^{2+n+\epsilon}-13\ 2^{2n+\epsilon}\right)}{(1+n)(1+n-\epsilon)}$$

Solving recurrences using C. Schneider's Sigma package

In[9]:= **recsol=SolveRecurrence[-2(1+n)(2+n-ϵ)f[n]+(2+n)(13+5n-5ϵ)f[1+n] -(3+n)(13+4n-4ϵ)f[2+n]+(4+n)(4+n-ϵ)f[3+n]==0, f[n]]**

Out[9]= $\left\{\left\{0, \frac{1}{1+n}\right\}, \left\{0, -\frac{\sum_{i=1}^{n} \frac{1}{(-1+\epsilon-i)2^i}}{-1-n}\right\}, \left\{0, -\frac{\sum_{i=1}^{n} \frac{1}{-1+\epsilon-i}}{1+n}\right\}, \{1,0\}\right\}$

In[10]:= **init=**$\left\{\frac{2^{-\epsilon}(-2+2^{\epsilon})}{-1+\epsilon}, \frac{2^{-1-\epsilon}(8-3\ 2^{\epsilon}+2(-3+2^{\epsilon})\epsilon)}{(-2+\epsilon)(-1+\epsilon)}, \frac{2^{-\epsilon}\left(-40+11\ 2^{\epsilon}+\left(50-3\ 2^{2+\epsilon}\right)\epsilon+(-14+3\ 2^{\epsilon})\epsilon^2\right)}{3(-3+\epsilon)(-2+\epsilon)(-1+\epsilon)}\right\}$

In[11]:= **FindLinearCombination[recsol, {0,init}, n, 3]**

Out[11]= $\frac{-2+2^{\epsilon}}{(1+n)(-1+\epsilon)2^{\epsilon}} + \frac{(-12+2^{\epsilon})\sum_{i=1}^{n}\frac{1}{-1+\epsilon-i}}{(1+n)2^{\epsilon}} + \frac{16\sum_{i=1}^{n}\frac{1}{(-1+\epsilon-i)2^i}}{(1+n)2^{\epsilon}}$

Recurrence for the Integral 3

We look again at the integral

$$a(n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

Suppose that we found

$$\sum_{i=0}^{L} e_i(n) F(n+i; x_1, \ldots, x_d) = \sum_{i=1}^{d} D_{x_i} G_i(n; x_1, \ldots, x_d)$$

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By integration with respect to $x_1, \ldots, x_d$ we get

$$\begin{aligned}\sum_{i=0}^{L} e_i(n) a(n+i) &= \sum_{i=1}^{d} \int_{u_d}^{o_d} \cdots \int_{u_{i-1}}^{o_{i-1}} \int_{u_{i+1}}^{o_{i+1}} \cdots \int_{u_1}^{o_1} O_i(n) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_d \\ &\quad - \sum_{i=1}^{d} \int_{u_d}^{o_d} \cdots \int_{u_{i-1}}^{o_{i-1}} \int_{u_{i+1}}^{o_{i+1}} \cdots \int_{u_1}^{o_1} U_i(n) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_d\end{aligned}$$

with

$$\begin{aligned}O_i(n) &:= G_i(n; x_1, \ldots, x_{i-1}, o_i, x_{i+1} \ldots, x_d) \\ U_i(n) &:= G_i(n; x_1, \ldots, x_{i-1}, u_i, x_{i+1} \ldots, x_d).\end{aligned}$$

Recurrence for the Integral 3

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$$a(n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

Suppose that we found

$$\sum_{i=0}^{L} e_i(n) F(n+i; x_1, \ldots, x_d) = \sum_{i=1}^{d} D_{x_i} G_i(n; x_1, \ldots, x_d)$$

By integration with respect to $x_1, \ldots, x_d$ we get

$$\begin{aligned}\sum_{i=0}^{L} e_i(n) a(n+i) \quad = \quad & \sum_{i=1}^{d} \int_{u_d}^{o_d} \cdots \int_{u_{i-1}}^{o_{i-1}} \int_{u_{i+1}}^{o_{i+1}} \cdots \int_{u_1}^{o_1} O_i(n) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_d \\ & - \sum_{i=1}^{d} \int_{u_d}^{o_d} \cdots \int_{u_{i-1}}^{o_{i-1}} \int_{u_{i+1}}^{o_{i+1}} \cdots \int_{u_1}^{o_1} U_i(n) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_d\end{aligned}$$

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Note that there are $2 \cdot d$ integrals of dimension $d-1$, to compute.

Example

$$I(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n; x_1, x_2) :=} dx_1 dx_2,$$

Applying the algorithm leads to

$$-(n+1)F(n; x_1, x_2) + (n+2)F(n+1; x_1, x_2) = D_{x_1} 0 + D_{x_2} \frac{x_2(x_1 \cdot x_2 + 1)^{n+1}}{(1 + x_1)^\varepsilon}$$

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$$-(n+1)F(n; x_1, x_2) + (n+2)F(n+1; x_1, x_2) = D_{x_1} 0 + D_{x_2} \frac{x_2(x_1 \cdot x_2 + 1)^{n+1}}{(1 + x_1)^\varepsilon}$$

and hence it follows by integration

$$-(n+1)I(\varepsilon, n) + (n+2)I(\varepsilon, n+1) = \underbrace{\int_0^1 (x_1 + 1)^{n+1-\varepsilon} dx_1}_{I_1(n)} - \int_0^1 0 dx_1.$$

In the next step apply the algorithm to $I_1(n)$; we find

$$I_1(\varepsilon, n) = \frac{4 \cdot 2^n - 2^\varepsilon}{2^\varepsilon (n + 2 - \varepsilon)}.$$

Example

$$I(\varepsilon,n)=\int_0^1\int_0^1 \underbrace{\frac{(1+x_1\cdot x_2)^n}{(1+x_1)^\varepsilon}}_{F(n;x_1,x_2):=} dx_1 dx_2,$$

Applying the algorithm leads to

$$-(n+1)F(n;x_1,x_2)+(n+2)F(n+1;x_1,x_2)=D_{x_1}0+D_{x_2}\frac{x_2(x_1\cdot x_2+1)^{n+1}}{(1+x_1)^\varepsilon}$$

and hence it follows by integration

$$-(n+1)I(\varepsilon,n)+(n+2)I(\varepsilon,n+1)=\underbrace{\int_0^1 (x_1+1)^{n+1-\varepsilon}dx_1}_{I_1(n)}-\int_0^1 0dx_1.$$

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$$I_1(\varepsilon,n)=\frac{4\cdot 2^n-2^\varepsilon}{2^\varepsilon(n+2-\varepsilon)}.$$

Plugging in yields

$$-(n+1)I(\varepsilon,n)+(n+2)I(\varepsilon,n+1)=\frac{4\cdot 2^n-2^\varepsilon}{2^\varepsilon(n+2-\varepsilon)}.$$

Example

$$I(\varepsilon,n)=\int_0^1\int_0^1 \underbrace{\frac{(1+x_1\cdot x_2)^n}{(1+x_1)^{\varepsilon}}}_{F(n;x_1,x_2):=}dx_1dx_2,$$

Applying the algorithm leads to

$$-(n+1)F(n;x_1,x_2)+(n+2)F(n+1;x_1,x_2)=D_{x_1}0+D_{x_2}\frac{x_2(x_1\cdot x_2+1)^{n+1}}{(1+x_1)^{\varepsilon}}$$

and hence it follows by integration

$$-(n+1)I(\varepsilon,n)+(n+2)I(\varepsilon,n+1)=\underbrace{\int_0^1(x_1+1)^{n+1-\varepsilon}dx_1}_{I_1(n)}-\int_0^1 0dx_1.$$

In the next step apply the algorithm to $I_1(n)$; we find

$$I_1(\varepsilon,n)=\frac{4\cdot 2^n-2^{\varepsilon}}{2^{\varepsilon}(n+2-\varepsilon)}.$$

Plugging in yields

$$-(n+1)I(\varepsilon,n)+(n+2)I(\varepsilon,n+1)=\frac{4\cdot 2^n-2^{\varepsilon}}{2^{\varepsilon}(n+2-\varepsilon)}.$$

Solving this recurrence yields

$$I(\varepsilon,n)=\frac{1}{n+1}\left(\sum_{i=1}^{n}\frac{1}{-i+\varepsilon-1}-2^{1-\varepsilon}\sum_{i=1}^{n}\frac{2^i}{-i+\varepsilon-1}+\frac{2^{-\varepsilon}\left(2^{\varepsilon}-2\right)}{\varepsilon-1}\right).$$

In[12]:= $\mathbf{mAZIntegrate}\left[\frac{(1+x1*x2)^n}{(1+x1)^{\epsilon}}, n, \{\{x1, 0, 1\}, \{x2, 0, 1\}\}\right]$

Out[12]= $$\frac{\sum_{\iota_1=1}^{n}\frac{1}{\iota_1-\epsilon+1}}{-n-1}-\frac{2\sum_{\iota_1=1}^{n}\frac{2^{\iota_1}}{-\iota_1+\epsilon-1}}{(n+1)2^{\epsilon}}+\frac{2^{\epsilon}-2}{(n+1)(\epsilon-1)2^{\epsilon}}$$

In[13]:= $\mathbf{mAZIntegrate}\left[\frac{(1+x1+x2+x1*x2)^n}{(1+x1)^{\epsilon}}, n, \{\{x1, 0, 1\}, \{x2, 0, 1\}\}\right]$

Out[13]= $$\frac{2^{-2n-\epsilon}\left(208-83\ 2^{2+n}+63\ 2^{1+2n}-2^{4+\epsilon}+7\ 2^{2+n+\epsilon}-13\ 2^{2n+\epsilon}\right)}{(1+n)(1+n-\epsilon)}$$

In[14]:= $\mathbf{mAZIntegrate}\left[\frac{(1+x1+x2+x1*x2)^n}{(1+x1)^{\epsilon}}, n, \{\{x1, 0, \frac{1}{2}\}, \{x2, 0, 2\}\}\right]$

Out[14]= $$\frac{2^{-2n-\epsilon}\left(208-83\ 2^{2+n}+63\ 2^{1+2n}-2^{4+\epsilon}+7\ 2^{2+n+\epsilon}-13\ 2^{2n+\epsilon}\right)}{(1+n)(1+n-\epsilon)}$$

Laurent Series Expansion of the Integral

- we look again at the integral

$$\mathcal{I}(\varepsilon, n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

- assume that we can write it in the form

$$\mathcal{I}(\varepsilon, n) = \sum_{l=-L}^{\infty} \varepsilon^l I_l(n).$$

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- compute a recurrence for $\mathcal{I}(\varepsilon,n)$ in the form

$$a_0(\varepsilon,n)T(\varepsilon,n)+\cdots+a_d(\varepsilon,n)T(\varepsilon,n+d) = h_0(n)+\cdots+h_u(n)\varepsilon^u+O(\varepsilon^{u+1});$$

  use one of the methods presented above

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  $$a_0(\varepsilon, n) T(\varepsilon, n) + \cdots + a_d(\varepsilon, n) T(\varepsilon, n+d) = h_0(n) + \cdots + h_u(n)\varepsilon^u + O(\varepsilon^{u+1});$$

  use one of the methods presented above
- use algorithm FLSR implemented in Sigma.

Using the Package MultiIntegrate

In[15]:= **mAZExpandedIntegrate[$\frac{(1+x1*x2)^n}{(1+x1)^\epsilon}$, n, { ϵ, 0, 2}, {{x1, 0, 1}, {x2, 0, 1}}]**

Out[15]=

$$\left\{\left\{\frac{S[1,\{2\},n]}{n+1}-\frac{S[1,n]}{n+1}+\frac{2^{n+1}-1}{(n+1)^2},-\frac{\log(2)S[1,\{2\},n]}{n+1}+\frac{S[2,\{2\},n]}{n+1}-\frac{S[2,n]}{n+1}+\frac{\log(2)\left(-2^{n+1}\right)(n+1)+2^{n+1}-1}{(n+1)^3},\frac{\log(2)^2S[1,\{2\},n]}{2n+2}-\frac{\log(2)S[2,\{2\},n]}{n+1}+\frac{S[3,\{2\},n]}{n+1}-\frac{S[3,n]}{n+1}+\frac{\log(2)^2 2^n(n+1)^2-\log(2)2^{n+1}(n+1)+2^{n+1}-1}{(n+1)^4}\right\},0,2\right\}$$

In[16]:= **mAZExpandedIntegrate[$\frac{(1+x1+x2+x1*x2)^n}{(1+x1)^\epsilon}$, n, { ϵ, 0, 1}, {{x1, 0, 1}, {x2, 0, 1}}]**

Out[16]=

$$\left\{\left\{\frac{\left(2^{n+1}-1\right)^2}{(n+1)^2},-\frac{\left(2^{n+1}-1\right)\left(-2^{n+1}+2^{n+1}n\log(2)+2^{n+1}\log(2)+1\right)}{(n+1)^3}\right\},0,1\right\}$$

In[17]:= **mAZExpandedIntegrate[$\frac{(1+x1+x2+x1*x2)^n}{(1+x1)^\epsilon}$, n, { ϵ, 0, 1}, {{x1, 0, $\frac{1}{2}$}, {x2, 0, 2}}]**

Out[17]=

$$\left\{\left\{-\frac{2^{-n-1}\left(2^{n+1}-3^{n+1}\right)\left(3^{n+1}-1\right)}{(n+1)^2},-\frac{2^{-n-1}\left(3^{n+1}-1\right)\left(2^{n+1}-3^{n+1}+3^{n+1}n\log\left(\frac{3}{2}\right)+3^{n+1}\log\left(\frac{3}{2}\right)\right)}{(n+1)^3}\right\},0,1\right\}$$

Using the Package MultiIntegrate

The following integral occurs in the computation of special Feynman diagrams in particle physics:

$$\int_0^1\int_0^1\Bigg(\frac{(s(x-1)+t(u-1)+1)^n}{(w-1)(z-1)(sx-s+tu-t-u+1)(sx-s+tu-t-x+1)}\\
+\frac{1}{(z-1)(sx-s+tu-t-x+1)}\frac{(z(-s+tu-t+1)+x((s-1)z+1))^n}{-swx+sw+sxz-sz-tuw+tuz+tw-tz+uw-u-w-xz+x+z}\\
+\frac{1}{(w-1)(sx-s+tu-t-u+1)}\frac{(u((t-1)w+1)-w(s(-x)+s+t-1))^n}{swx-sw-sxz+sz+tuw-tuz-tw+tz-uw+u+w+xz-x-z}\Bigg)\,dudx$$

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$$+\frac{1}{(z-1)(sx-s+tu-t-x+1)}\frac{(z(-s+tu-t+1)+x((s-1)z+1))^n}{-swx+sw+sxz-sz-tuw+tuz+tw-tz+uw-u-w-xz+x+z}$$
$$+\frac{1}{(w-1)(sx-s+tu-t-u+1)}\frac{(u((t-1)w+1)-w(s(-x)+s+t-1))^n}{swx-sw-sxz+sz+tuw-tuz-tw+tz-uw+u+w+xz-x-z}\Bigg)dudx$$

$$=\frac{1}{(w-1)(z-1)(s+t-1)}\Bigg(S_{1,1}\left(-\frac{(s+t-1)w(z-1)}{sw-sz+z-1},\frac{sw-sz+z-1}{z-1};n\right)-S_{1,1}\left(-\frac{(s+t-1)w(z-1)}{sw-sz+z-1},\frac{z(sw-sz+z-1)}{w(z-1)};n\right)$$
$$-S_{1,1}\left(-\frac{(s+t-1)w(z-1)}{sw-sz+z-1},\frac{(t-1)(sw-sz+z-1)}{(s+t-1)(z-1)};n\right)+S_{1,1}\left(\frac{s+t-1}{t-1},(1-t)w;n\right)+S_{1,1}\left(\frac{s+t-1}{t-1},-\frac{(s-1)(t-1)}{s+t-1};n\right)$$
$$+S_{1,1}\left(-\frac{(s+t-1)w(z-1)}{sw-sz+z-1},\frac{(sw-sz+z-1)(tz-1)}{(s+t-1)w(z-1)};n\right)-S_{1,1}\left(\frac{s+t-1}{t-1},1-t;n\right)+S_{1,1}\left(\frac{s+t-1}{s-1},-\frac{(s-1)(t-1)}{s+t-1};n\right)$$
$$+S_{1,1}\left(\frac{(s+t-1)(w-1)z}{(t-1)w-tz+1},\frac{-tw+w+tz-1}{w-1};n\right)-S_{1,1}\left(\frac{s+t-1}{t-1},-\frac{(t-1)(sw-1)}{s+t-1};n\right)+S_{1,1}\left(\frac{s+t-1}{s-1},(1-s)z;n\right)$$
$$-S_{1,1}\left(\frac{(s+t-1)(w-1)z}{(t-1)w-tz+1},\frac{(s-1)(-tw+w+tz-1)}{(s+t-1)(w-1)};n\right)-S_{1,1}\left(\frac{s+t-1}{s-1},1-s;n\right)-S_{1,1}\left(\frac{s+t-1}{s-1},-\frac{(s-1)(tz-1)}{s+t-1};n\right)$$
$$-S_{1,1}\left(\frac{(s+t-1)(w-1)z}{(t-1)w-tz+1},-\frac{w((t-1)w-tz+1)}{(w-1)z};n\right)+S_{1,1}\left(\frac{(s+t-1)(w-1)z}{(t-1)w-tz+1},-\frac{(sw-1)((t-1)w-tz+1)}{(s+t-1)(w-1)z};n\right)$$
$$-S_{1,1}(1,(1-s)z;n)+S_{1,1}(1,z-sz;n)+S_2((1-s)z;n)-S_{1,1}(1,(1-t)w;n)+S_{1,1}(1,w-tw;n)-S_2((-s-t+1)w;n)$$
$$-S_2((-s-t+1)z;n)+2S_2(-s-t+1;n)-S_2(1-s;n)+S_2((1-t)w;n)-S_2(1-t;n)\Bigg)$$

(Generalized) Harmonic Sums

Definition (Harmonic Sums (H-Sums))

For $c_i \in \mathbb{Z}^*$ and $n \in \mathbb{N}$ we define

$$S_{c_1,\ldots,c_k}(n) = \sum_{n \geq i_1 \geq i_2 \geq \cdots \geq i_k \geq 1} \frac{\operatorname{sign}(c_1)^{i_1}}{i_1^{|c_1|}} \cdots \frac{\operatorname{sign}(c_k)^{i_k}}{i_k^{|c_k|}}$$

k is called the depth and $w = \sum_{i=1}^{k} |c_i|$ is called the weight of the harmonic sum $S_{c_1,\ldots,c_k}(n)$.

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$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i}$$

$$S_{2,-3}(n) = \sum_{i=1}^{n} \frac{\sum_{j=1}^{i} \frac{(-1)^j}{j^3}}{i^2}$$

Algebraic Relations

$$
\begin{aligned}
S_{1,1}(n) &= \frac{1}{2}\left(S_1(n)^2 + S_2(n)\right) \\
S_{1,1,1}(n) &= \frac{1}{6}\left(S_1(n)^3 + 3S_1(n)S_2(n) + 2S_3(n)\right) \\
S_{2,1,3}(n) &= S_2(n)S_4(n) + S_6(n) + S_3(n)S_{2,1}(n) - S_{4,2}(n) \\
&\quad + S_{5,1}(n) - S_{2,3,1}(n) - S_{3,2,1}(n)
\end{aligned}
$$

Harmonic Polylogarithms (H-Logs)

We define $f : \{0, 1, -1\} \times (0, 1) \mapsto \mathbb{R}$ by

$$\begin{aligned} f(0, x) &= \frac{1}{x}, \\ f(1, x) &= \frac{1}{1 - x}, \\ f(-1, x) &= \frac{1}{1 + x}. \end{aligned}$$

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\end{aligned}
$$

Definition (Harmonic Polylogarithms (H-Logs))

Let $m_i \in \{-1, 0, 1\}$ we define for $x \in (0, 1)$:

$$
\begin{aligned}
\mathrm{H}(x) &= 1, \\
\mathrm{H}_{m_1,m_2,\ldots,m_k}(x) &= \begin{cases} \frac{1}{k!}(\log x)^k, & \text{if } (m_1,\ldots,m_k) = \mathbf{0} \\ \int_0^x f_{m_1}(y)\mathrm{H}_{m_2,\ldots,m_k}(y)dy, & \text{otherwise.} \end{cases}
\end{aligned}
$$

Example

$$
\begin{aligned}
\mathrm{H}_1(x) &= \int_0^x \frac{1}{1-x_1} dx_1 = -\log(1-x) \\
\mathrm{H}_{-1}(x) &= \int_0^x \frac{1}{1+x_1} dx_1 = \log(1+x) \\
\mathrm{H}_{1,0,-1}(x) &= \int_0^x \frac{1}{1-x_1} \int_0^{x_1} \frac{1}{x_2} \int_0^{x_2} \frac{1}{1+x_3} dx_3 dx_2 dx_1
\end{aligned}
$$

Mellin Transform

Definition (Mellin Transform of Harmonic Polylogarithms)

Let $f(x)$ be a harmonic polylogarithm and let $n \in \mathbb{N}$, the Mellin transform is defined by:

$$M(f(x), n) = \int_0^1 x^n f(x) dx$$

Since harmonic polylogarithms are locally integrable functions on $(0, 1)$ their Mellin transform is finite.

Depth 1 cases

$$M\left(\mathrm{H}_0(x), n\right) = -\frac{1}{(n+1)^2}$$

$$M\left(\mathrm{H}_1(x), n\right) = \frac{1}{(n+1)^2} + \frac{\mathrm{S}_1(n)}{n+1}$$

$$\begin{aligned} M\left(\mathrm{H}_{-1}(x), n\right) &= \frac{(-1)^n}{n+1}\left(\mathrm{S}_{-1}(n) + \log(2)\right) \\ &\quad + \frac{\log(2)}{n+1} - \frac{1}{(n+1)^2} \end{aligned}$$

In[1]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger -RISC Linz- Version 1.0 (1/3/12)

In[2]:= **Mellin[H[-1,x],x,n]**

Out[2]= $$\frac{-\left(\left((-1)^n+1\right)(n+1)\mathrm{S}[-1,\infty]+(-1)^n(n+1)\mathrm{S}[-1,n]-1\right.}{(n+1)^2}$$

In[3]:= **Mellin[H[1,0,1,x],x,n]**

Out[3]=
$$\frac{(n+1)^3\mathrm{S}[1,n]\mathrm{S}[2,\infty]+(n+1)^2\mathrm{S}[2,\infty]-(n+1)^3\mathrm{S}[2,1,n]-(n+1)\mathrm{S}[1,n]-1}{(n+1)^4}$$

In[4]:= **Mellin[H[-1,1,0,x],x,n]**

Out[4]= $$\frac{1}{(n+1)^4}\Big(-2(-1)^n(n+1)^3\mathrm{S}[3,\infty]-2(n+1)^3\mathrm{S}[3,\infty]+(-1)^n(n+1)^3\mathrm{S}[-2,-1,\infty]+(n+1)^3\mathrm{S}[-2,-1,\infty]+(-1)^n(n+1)^3\mathrm{S}[-1,-2,\infty]+(n+1)^3\mathrm{S}[-1,-2,\infty]-(-1)^n(n+1)^3\mathrm{S}[-1,2,n]+(n+1)^2\mathrm{S}[2,n]+1\Big)$$

Using the Package HarmonicSums

In[5]:= **InvMellin[S[-1, n], n, x]**

Out[5]= $$\delta_{1-x}\mathrm{S}[-1,\infty]+\frac{(-1)^n}{x+1}$$

In[6]:= **InvMellin[S[-1, 2, n], n, x]**

Out[6]= $$\delta_{1-x}(-2\mathrm{S}[3,\infty]+\mathrm{S}[-2,-1,\infty]+\mathrm{S}[-1,-2,\infty])-\frac{(-1)^n\mathrm{H}[1,0,x]}{x+1}$$

In[7]:= **InvMellin[S[-1, 1, 3, n], n, x]**

Out[7]= $$\begin{aligned}&\delta_{1-x}(6\mathrm{S}[5,\infty]-\mathrm{S}[-3,-2,\infty]-2\mathrm{S}[-2,-3,\infty]-3\mathrm{S}[-1,-4,\infty]-\\&\mathrm{S}[2,3,\infty]-2\mathrm{S}[3,2,\infty]-3\mathrm{S}[4,1,\infty]+\mathrm{S}[-3,-1,1,\infty]+\\&\mathrm{S}[-2,-2,1,\infty)+\mathrm{S}[-2,-1,2,\infty]+\mathrm{S}[-1,-3,1,\infty]+\\&\mathrm{S}[-1,-2,2,\infty]+\mathrm{S}[-1,-1,3,\infty])+\frac{(-1)^n\mathrm{H}[1,1,0,0,x]}{x+1}\end{aligned}$$

Differential Relations

- harmonic sums can be represented as the Mellin transform of harmonic polylogarithms:

$$S_{1,2}(n) = \int_0^1 \frac{(x^n - 1)\mathrm{H}_{1,0}(x)}{1-x} dx$$

- we can differentiate the right hand side with respect to n:

$$\begin{aligned} \frac{d}{dn} S_{1,2}(n) &= \int_0^1 \frac{x^n \mathrm{H}_0(x)\mathrm{H}_{1,0}(x)}{1-x} dx \\ &= \int_0^1 \frac{x^n \mathrm{H}_{0,1,0}(x)}{1-x} dx + 2 \int_0^1 \frac{x^n \mathrm{H}_{1,0,0}(x)}{1-x} dx \end{aligned}$$

- evaluation of the integrals leads to differential relations:

$$\frac{d}{dn} S_{1,2}(n) = \frac{\zeta_2^2}{2} + 2\,\zeta_3\, S_1(n) - 2 S_{1,3}(n) - S_{2,2}(n)$$

Half-Integer Relations

- Let $a_i \in \mathbb{N}$, then we have the relation:

$$\sum S_{\pm a_m,\pm a_{m-1},\dots,\pm a_1}(2n) = \frac{1}{2^{\sum_{i=1}^{m} a_i - m}} S_{a_m,a_{m-1},\dots,a_1}(n)$$

  where we sum on the left hand side over the 2^m possible combinations concerning $\pm$.

- this leads to half-integer relations:

$$\frac{1}{8} S_{2,3}(n) = S_{2,3}(2n) + S_{2,-3}(2n) + S_{-2,3}(2n) + S_{-2,-3}(2n).$$

Number of Basic Harmonic Sums

Weight	Number of							
	All	N_A	N_D	N_H	N_{AD}	N_{AH}	N_{DH}	N_{ADH}
1	2	2	2	1	2	1	1	1
2	6	3	4	4	1	2	3	1
3	18	8	12	14	5	6	10	4
4	54	18	36	46	10	15	32	9
5	162	48	108	146	30	42	100	27
6	486	116	324	454	68	107	308	65
7	1458	312	972	1394	196	294	940	187
8	4374	810	2916	4246	498	780	2852	486

Number of Basic Harmonic Sums

For $w \geq 2$ we have

$$
\begin{aligned}
All(w) &= 2 \cdot 3^{w-1} \\
N_A(w) &= \frac{1}{w} \sum_{d|w} \mu\left(\frac{w}{d}\right) 3^d \\
N_D(w) &= 4 \cdot 3^{w-2} \\
N_H(w) &= 2 \cdot 3^{w-1} - 2^{w-1} \\
N_{AD}(w) &= \frac{1}{w} \sum_{d|w} \mu\left(\frac{w}{d}\right) 3^d - \frac{1}{w-1} \sum_{d|w-1} \mu\left(\frac{w-1}{d}\right) 3^d \\
N_{AH}(w) &= \frac{1}{w} \sum_{d|w} \mu\left(\frac{w}{d}\right) \left[3^d - 2^d\right] \\
N_{DH}(w) &= 4 \cdot 3^{w-2} - 2^{w-2} \\
N_{ADH}(w) &= \frac{1}{w} \sum_{d|w} \mu\left(\frac{w}{d}\right) \left[3^d - 2^d\right] - \frac{1}{w-1} \sum_{d|w-1} \mu\left(\frac{w-1}{d}\right) \left[3^d - 2^d\right]
\end{aligned}
$$

Using the Package HarmonicSums

In[8]:= **ComputeHSumBasis[2,n,UseDifferentiation−>True]**

Out[8]=
$$
\begin{aligned}
\{\ S[1,-1,n] &\rightarrow -\mathrm{Diff}[S[-1,n],1]+S[-2,\infty]+S[-1,n]S[1,n]-S[-1,1,n],\\
S[1,1,n] &\rightarrow \frac{1}{2}S[1,n]^2-\frac{1}{2}\mathrm{Diff}[S[1,n],1]+\frac{1}{2}S[2,\infty],\\
S[-1,-1,n] &\rightarrow \frac{1}{2}S[-1,n]^2-\frac{1}{2}\mathrm{Diff}[S[1,n],1]+\frac{1}{2}S[2,\infty],\\
S[-2,n] &\rightarrow S[-2,\infty]-\mathrm{Diff}[S[-1,n],1],\\
S[2,n] &\rightarrow S[2,\infty]-\mathrm{Diff}[S[1,n],1],\\
\mathrm{Half}[S[1,1,n],n] &\rightarrow \frac{1}{2}S[-1,n]^2+S[1,n]S[-1,n]+\frac{1}{2}S[1,n]^2\\
&\quad -\mathrm{Diff}[S[-1,n],1]-\mathrm{Diff}[S[1,n],1]+S[-2,\infty]+S[2,\infty]\ \}
\end{aligned}
$$

Using the Package HarmonicSums

Only algebraic relations:

In[9]:= **ComputeHSumBasis[2,n]**

Out[9]= $$\Big\{\{S[-2,n],S[2,n],S[-1,1,n]\},\Big\{S[1,-1,n]\to S[1,-1,n],S[1,1,n]\to \frac{1}{2}S[2,n]+\frac{1}{2}(2S[1,1,n]-S[2,n]),S[-1,-1,n]\to \frac{1}{2}S[2,n]+\frac{1}{2}(2S[-1,-1,n]-S[2,n])\Big\}\Big\}$$

Using the Package HarmonicSums

Only algebraic relations:

In[11]:= **ComputeHSumBasis[2,n]**

Out[11]= $\Big\{\{S[-2,n],S[2,n],S[-1,1,n]\},\Big\{S[1,-1,n]\to S[1,-1,n],S[1,1,n]\to \frac{1}{2}S[2,n]+\frac{1}{2}(2S[1,1,n]-S[2,n]),S[-1,-1,n]\to \frac{1}{2}S[2,n]+\frac{1}{2}(2S[-1,-1,n]-S[2,n])\Big\}\Big\}$

Algebraic relations using C. Schneider's `Sigma` package:

In[12]:= **SigmaReduce[{S[-2, n], S[2, n], S[-1, -1, n], S[-1, 1, n], S[1, -1, n], S[1, 1, n]}, n]**

Out[12]= $\Big\{S[-2,n],S[2,n],\frac{1}{2}\left(S[-1,n]^2+S[2,n]\right),S[-1,1,n],S[-2,n]+S[-1,n]S[1,n]-S[-1,1,n],\frac{1}{2}\left(S[1,n]^2+S[2,n]\right)\Big\}$

Definition (S-Sums)

For $c_i \in \mathbb{N}$ and $x_i \in \mathbb{R}^*$ we define

$$\mathrm{S}_{c_1,\ldots,c_k}(x_1,\ldots,x_k;n) = \sum_{n \geq i_1 \geq i_2 \geq \cdots \geq i_k \geq 1} \frac{x_1^{i_1}}{i_1^{c_1}} \cdots \frac{x_k^{i_k}}{i_k^{c_k}}.$$

k is called the depth and $w = \sum_{i=1}^{k} c_i$ is called the weight of the S-sum $\mathrm{S}_{c_1,\ldots,c_k}(x_1,\ldots,x_k;n)$.

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$$\mathrm{S}_{2,3}(4,3;n) = \sum_{i=1}^{n} \frac{4^i \sum_{j=1}^{i} \frac{3^j}{j^3}}{i^2}$$

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$$\mathrm{S}_{2,3}(4,3;n) = \sum_{i=1}^{n} \frac{4^i \sum_{j=1}^{i} \frac{3^j}{j^3}}{i^2}$$

$$\mathrm{S}_{2,3}(1,-1;n) = \sum_{i=1}^{n} \frac{1^i \sum_{j=1}^{i} \frac{(-1)^j}{j^3}}{i^2} = S_{2,-3}(n)$$

Example

Multiple Polylogarithms (M-Logs)

Let $a \in \mathbb{R}$ and

$$q = \begin{cases} a, & \text{if } a > 0 \\ \infty, & \text{otherwise} \end{cases}$$

We define f as follows:

$$f_a : (0, q) \mapsto \mathbb{R}$$

$$f_a(x) = \begin{cases} \frac{1}{x}, & \text{if } a = 0 \\ \frac{1}{|a| - \operatorname{sign}(a)\, x}, & \text{otherwise.} \end{cases}$$

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Definition (Multiple Polylogarithms (M-Logs))

Let $m_i \in \mathbb{R}$ and let $q = \min_{m_i > 0} m_i$ we define for $x \in (0, q)$:

$$\mathrm{H}(x) = 1,$$
$$\mathrm{H}_{m_1, m_2, \ldots, m_k}(x) = \begin{cases} \frac{1}{k!}(\log x)^k, & \text{if } (m_1, \ldots, m_k) = \mathbf{0} \\ \int_0^x f_{m_1}(y) \mathrm{H}_{m_2, \ldots, m_k}(y) dy, & \text{otherwise.} \end{cases}$$

Example

$$
\begin{aligned}
\mathrm{H}_1(x) &= \int_0^x \frac{1}{1-x_1}\,dx_1 = -\log(1-x) \\
\mathrm{H}_{-2}(x) &= \int_0^x \frac{1}{2+x_1}\,dx_1 = \log(2+x) - \log(2) \\
\mathrm{H}_{2,0,-\frac{1}{2}}(x) &= \int_0^x \frac{1}{2-x_1} \int_0^{x_1} \frac{1}{x_2} \int_0^{x_2} \frac{1}{\frac{1}{2}+x_3}\,dx_3\,dx_2\,dx_1
\end{aligned}
$$

Integral representation of S-sums with depth 1

Let $m \in \mathbb{N}$, $b \in \mathbb{R}^*$ and $n \in \mathbb{N}$ then

$$
\begin{aligned}
\mathrm{S}_1(b;n) &= \int_0^b \frac{x_1^n - 1}{x_1 - 1} dx_1 \\
\mathrm{S}_2(b;n) &= \int_0^b \frac{1}{x_2} \int_0^{x_2} \frac{x_1^n - 1}{x_1 - 1} dx_1 dx_2 \\
\mathrm{S}_m(b;n) &= \int_0^b \frac{1}{x_m} \int_0^{x_m} \frac{1}{x_{m-1}} \cdots \int_0^{x_3} \frac{1}{x_2} \int_0^{x_2} \frac{x_1^n - 1}{x_1 - 1} dx_1 dx_2 \cdots dx_{m-1} dx_m
\end{aligned}
$$

Integral representation of S-sums with arbitrary depth

Let $m_i \in \mathbb{N}$, $b_i \in \mathbb{R}^*$ and $n \in \mathbb{N}$ then

$$S_{m_1,m_2,\ldots,m_k}(b_1,b_2,\ldots,b_k;n) =$$

$$\int_0^{b_1\cdots b_k} \frac{dx_{k_{m_k}}}{x_{k_{m_k}}} \int_0^{x_{k_{m_k}-1}} \frac{dx_{k_{m_k-1}}}{x_{k_{m_k-1}}} \cdots \int_0^{x_{k_3}} \frac{dx_{k_2}}{x_{k_2}} \int_0^{x_{k_2}} \frac{dx_{k_1}}{x_{k_1} - b_1\cdots b_{k-1}}$$

$$\int_{b_1\cdots b_{k-1}}^{x_{k_1}} \frac{dx_{k-1_{m_{k-1}}}}{x_{k-1_{m_{k-1}}}} \int_0^{x_{k-1_{m_{k-1}-1}}} \frac{dx_{k-1_{m_{k-1}-1}}}{x_{k-1_{m_{k-1}-1}}} \cdots \int_0^{x_{k-1_3}} \frac{dx_{k-1_2}}{x_{k-1_2}}$$

$$\int_0^{x_{k-1_2}} \frac{dx_{k-1_1}}{x_{k-1_1} - b_1\cdots b_{k-2}}$$

$$\vdots$$

$$\int_{b_1 b_2}^{x_{3_1}} \frac{dx_{2_{m_2}}}{x_{2_{m_2}}} \int_0^{x_{2_{m_2-1}}} \frac{dx_{2_{m_2-1}}}{x_{2_{m_2-1}}} \cdots \int_0^{x_{2_3}} \frac{dx_{2_2}}{x_{2_2}} \int_0^{x_{2_2}} \frac{dx_{2_1}}{x_{2_1} - b_1}$$

$$\int_{b_1}^{x_{2_1}} \frac{dx_{1_{m_1}}}{x_{1_{m_1}}} \int_0^{x_{1_{m_1-1}}} \frac{dx_{1_{m_1-1}}}{x_{1_{m_1-1}}} \cdots \int_0^{x_{1_3}} \frac{dx_{1_2}}{x_{1_2}} \int_0^{x_{1_2}} \frac{{x_{1_1}}^n - 1}{x_{1_1} - 1} dx_{1_1}$$

Integral Representation of S-sums: Example

- we consider the sum

$$\mathrm{S}_{1,2,1}\left(\frac{1}{3},\frac{1}{2},1;n\right)$$

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$$\mathrm{S}_{1,2,1}\left(\frac{1}{3},\frac{1}{2},1;n\right) = \int_0^{\frac{1}{6}}\frac{1}{x_1-\frac{1}{6}}\int_{\frac{1}{6}}^{x_1}\frac{1}{x_2}\int_0^{x_2}\frac{1}{x_3-\frac{1}{3}}\int_{\frac{1}{3}}^{x_3}\frac{x_4^n-1}{x_4-1}dx_4dx_3dx_2dx_1$$

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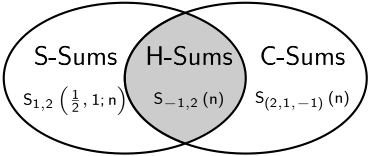
- integration by parts leads to integrals of depth one:

$$\begin{aligned}S_{1,2,1}\left(\frac{1}{3},\frac{1}{2},1;n\right) &= -\left(H_{\frac{1}{3},0,\frac{1}{6}}(\frac{1}{6})-H_{\frac{1}{3}}(\frac{1}{6})H_{0,\frac{1}{6}}(\frac{1}{6})\right)\int_{\frac{1}{6}}^{\frac{1}{3}}\frac{x^n-1}{x-1}dx\\ &+H_{0,\frac{1}{6}}(\frac{1}{6})\int_0^{\frac{1}{6}}\frac{x^n-1}{x-1}H_{\frac{1}{3}}(x)dx-\int_0^{\frac{1}{6}}\frac{x^n-1}{x-1}H_{\frac{1}{3},0,\frac{1}{6}}(x)dx\end{aligned}$$

In[13]:= **InvMellin[S[1,2,1,{1/3,1/2,1}, n], n, x]**

Out[13]=
$$-\frac{6^{-n}\mathrm{H}[2,0,1,x]}{-6+x}+\frac{6^{-n}\mathrm{H}[2,x]\mathrm{S}[2,\infty]}{-6+x}-\frac{6^{-n}\mathrm{S}[3,\infty]}{-6+x}+\frac{3^{-n}\mathrm{S}[3,\infty]}{-3+x}-\frac{3^{-n}\mathrm{S}[2,\infty]\mathrm{S}[1,\{\frac{1}{2}\},\infty]}{3-x}+\frac{6^{-n}\mathrm{S}[2,\infty]\mathrm{S}[1,\{\frac{1}{2}\},\infty]}{6-x}+\frac{6^{-n}\mathrm{S}[1,2,\{\frac{1}{2},2\},\infty]}{-6+x}-\frac{3^{-n}\mathrm{S}[1,2,\{\frac{1}{2},2\},\infty]}{-3+x}+\delta_{1-x}\Bigg(-\mathrm{S}[4,\infty]-\mathrm{S}[3,\infty]\mathrm{S}[1,\left\{\frac{1}{6}\right\},\infty]+\mathrm{S}[3,\infty]\mathrm{S}[1,\left\{\frac{1}{3}\right\},\infty]-\mathrm{S}[2,\infty]\mathrm{S}[1,\left\{\frac{1}{6}\right\},\infty]\mathrm{S}[1,\left\{\frac{1}{2}\right\},\infty]+\mathrm{S}[2,\infty]\mathrm{S}[1,\left\{\frac{1}{3}\right\},\infty]\mathrm{S}[1,\left\{\frac{1}{2}\right\},\infty]-\mathrm{S}[2,\infty]\mathrm{S}[2,\left\{\frac{1}{2}\right\},\infty]+\mathrm{S}[2,\infty]\mathrm{S}[1,1,\left\{\frac{1}{6},3\right\},\infty]+\mathrm{S}[1,\left\{\frac{1}{6}\right\},\infty]\mathrm{S}[1,2,\left\{\frac{1}{2},2\right\},\infty]-\mathrm{S}[1,\left\{\frac{1}{3}\right\},\infty]\mathrm{S}[1,2,\left\{\frac{1}{2},2\right\},\infty]+\mathrm{S}[1,3,\left\{\frac{1}{6},6\right\},\infty]+\mathrm{S}[2,2,\left\{\frac{1}{2},2\right\},\infty]-\mathrm{S}[1,1,2,\left\{\frac{1}{6},3,2\right\},\infty]\Bigg)$$

Connection between these structures



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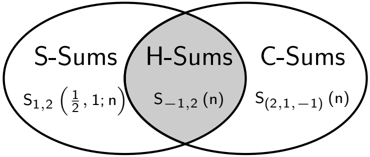

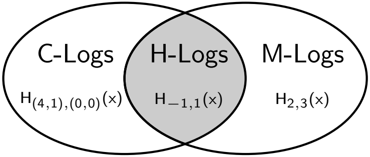

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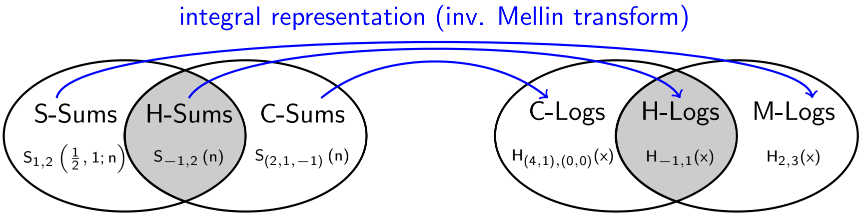

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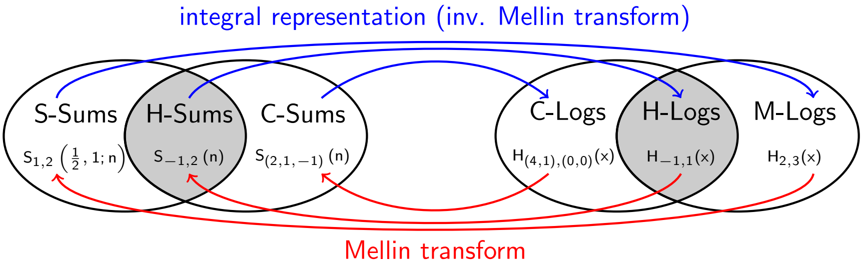

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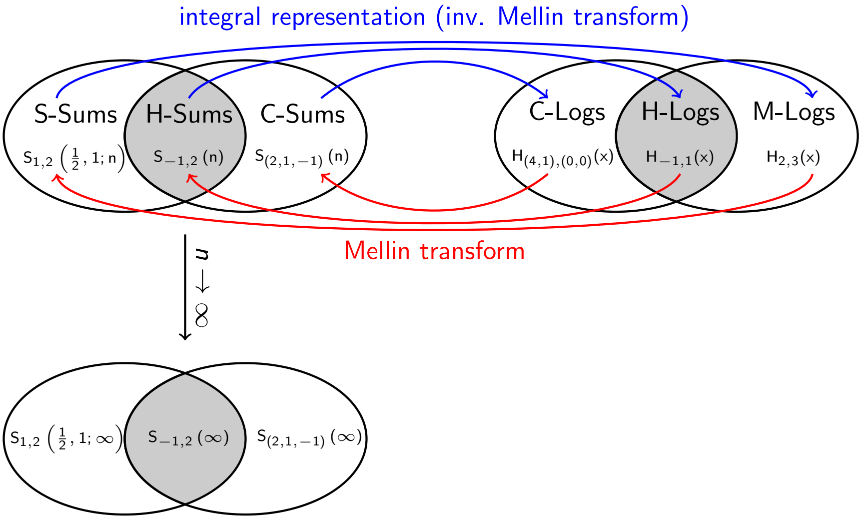

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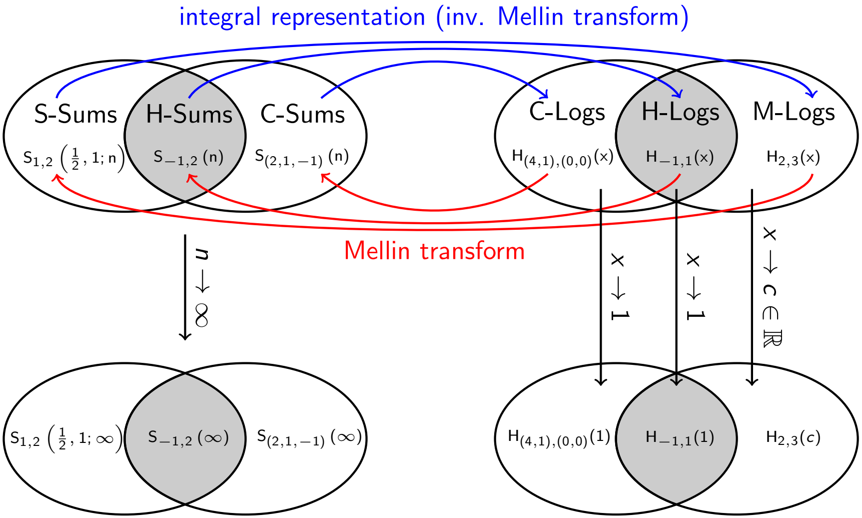

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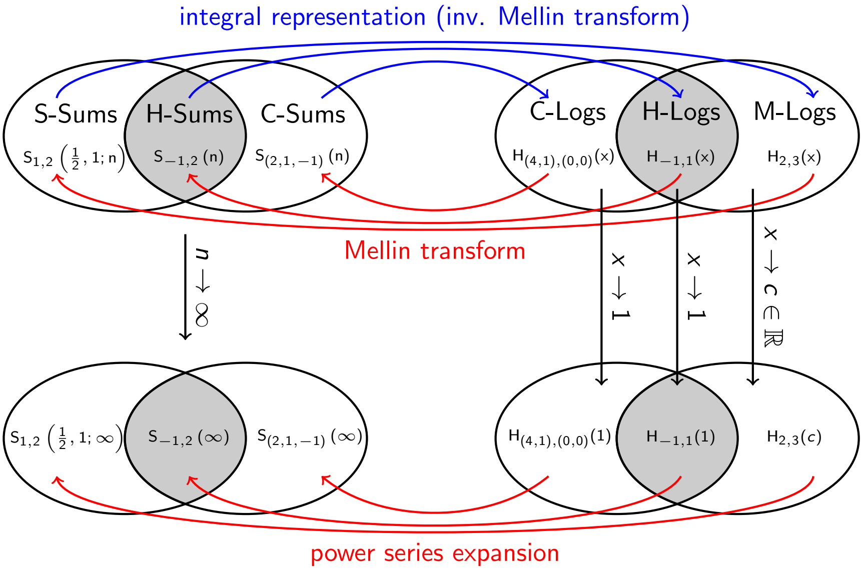

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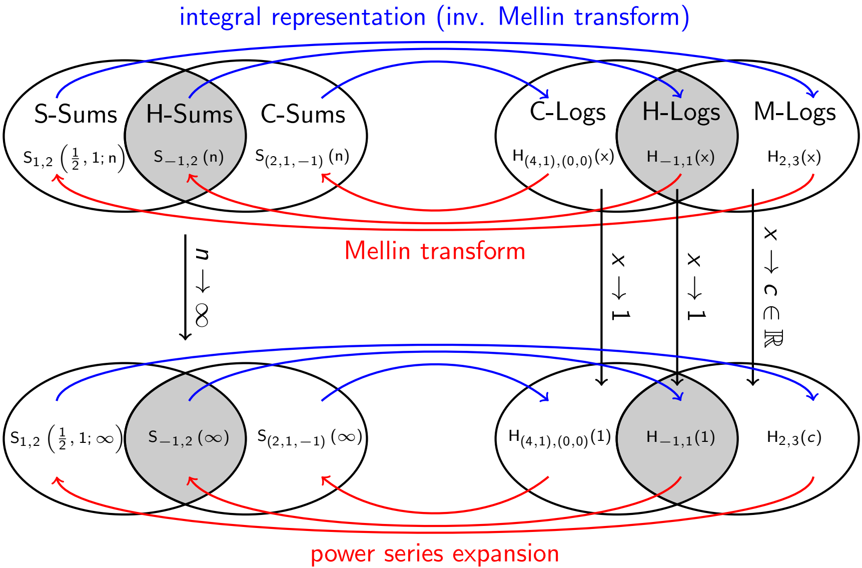

In[14]:= **HToS[H[-1, 3, x] + H[{4, 1}, x]]**

$$\text{Out[14]=}\quad \sum_{\tau_1=1}^{\infty}\frac{3^{-\tau_1}x^{\tau_1}}{\tau_1^2}-\sum_{\tau_1=1}^{\infty}\frac{x^{4\tau_1}}{4\tau_1}-\sum_{\tau_1=1}^{\infty}\frac{(-x)^{\tau_1}\mathrm{S}[1,\{-\frac{1}{3}\},\tau_1]}{\tau_1}+\sum_{\tau_1=1}^{\infty}\frac{x^{-2+4\tau_1}}{-2+4\tau_1}$$

$$\text{In[15]:=}\quad \mathbf{SToH}[\sum_{\tau_1=1}^{\infty}\frac{(-x)^{\tau_1}\mathrm{S}[1,\{-\frac{1}{3}\},\tau_1]}{\tau_1}]$$

$$\text{Out[15]=}\quad -\mathrm{H}[-1,3,x]+\mathrm{H}[0,3,x]$$

$$\text{In[16]:=}\quad \mathbf{SinfToH[S[2,\infty] - S[-1,-1,\infty] + S[1,\left\{\frac{1}{3}\right\},\infty]]}$$

$$\text{Out[16]=}\quad -\mathrm{H}[-1,1,1]+\mathrm{H}[3,1]$$

In[17]:= **HToSinf[H[-1, 1, 1] + H[3, 1]]**

$$\text{Out[17]=}\quad \mathrm{S}[2,\infty]-\mathrm{S}[-1,-1,\infty]+\mathrm{S}[1,\left\{\frac{1}{3}\right\},\infty]$$

We say that the function $f : \mathbb{R} \to \mathbb{R}$ is expanded in an asymptotic series

$$f(x) \sim \sum_{k=0}^{\infty} \frac{a_k}{x^k},\ x \to \infty,$$

where a_k are constants, if for all $K \geq 0$

$$R_K(x) = f(x) - \sum_{k=0}^{K} \frac{a_k}{x^k} = o\left(\frac{1}{x^K}\right),\ x \to \infty.$$

Why do we need these expansions of harmonic sums?

E.g.,

- for limits of the form

$$\lim_{n\to\infty} n\left(S_2(n)-\zeta_2-S_{2,2}(n)+\frac{7\,\zeta_2^2}{10}\right)=\zeta_2-1$$

- for the approximation of the values of analytic continued harmonic sums at the complex plane

$$S_{2,-3}(-20+10i)\approx -0.795096-0.105476i$$

Let $\varphi(x)$ be analytic at $x = 1$, with

$$\varphi(1-x) = \sum_{k=0}^{\infty} a_k x^k.$$

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Consider the Mellin transform

$$\Omega(n, \varphi(x)) = \int_0^1 x^n \varphi(x) dx$$

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For $\mathrm{Re}(n) > 0$, $\Omega(n, \varphi(x))$ is given by the factorial series

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$$\Omega(n, \varphi(x)) = \sum_{k=0}^{\infty} \frac{a_k k!}{(n+1)(n+2)\cdots(n+k+1)}$$

- $\Omega(n, \varphi(x))$ can be continued analytically to values of $n \in \mathbb{C}$ as a meromorphic function.
- $\Omega(n, \varphi(x))$ has an analytic asymptotic representation.
- The poles of $\Omega(n, \varphi(x))$ are situated at the negative integers.

Asymptotic Expansions

- this basic idea can be turned into an algorithm

$$S_{-1,3}(n) \sim (-1)^n\left(-\frac{1}{4n^3}+\frac{5}{8n^4}-\frac{5}{8n^5}-\frac{5}{16n^6}\right)+\frac{3\ \log(2)\,\zeta_3}{4}+(-1)^n\left(\frac{1}{2n}-\frac{1}{4n^2}+\frac{1}{8n^4}-\frac{1}{4n^6}\right)\zeta_3-\frac{19\,\zeta_2^2}{40}$$

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- the algorithm can be extended to cyclotomic harmonic sums

$$\begin{aligned} S_{(2,1,2),(1,0,1)}(n) \;\sim\; & -4\log(2)S_{(2,1,-1)}(\infty)^2-8\log(2)S_{(2,1,-1)}(\infty)-4\log(2)+\frac{1}{9n^3}-\frac{1}{4n}+\frac{7\zeta_3}{4} \\ & +\left(-\frac{11}{48n^3}+\frac{1}{4n^2}-\frac{1}{4n}\right)(\log(n)+\gamma) \end{aligned}$$

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- extension to a subset of the S-Sums:

$$\begin{aligned} S_{2,1}(1,\tfrac{1}{3})(n) \sim\ & -S_{1,2}\left(\frac{1}{3},1;\infty\right)+3^{-n}\left(\frac{3}{2n^3}-\frac{3}{4n^2}+\frac{1}{2n}\right)H_{0,3}(1)+3^{-n}\left(-\frac{3}{2n^3}+\frac{3}{4n^2}-\frac{1}{2n}\right)S_2\left(\frac{1}{3};\infty\right) \\ & +\left(3^{-n}\left(-\frac{3}{2n^3}+\frac{1}{2n^2}+3^n\left(-\frac{1}{6n^3}+\frac{1}{2n^2}-\frac{1}{n}\right)\right)+\zeta_2\right)S_1\left(\frac{1}{3};\infty\right) \\ & +S_3\left(\frac{1}{3};\infty\right)+H_3(1)3^{-n}\left(\frac{3}{2n^3}-\frac{1}{2n^2}\right)+\frac{3^{-n}}{4n^3} \end{aligned}$$

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$$S_{-1,3}(n) \sim (-1)^n\left(-\frac{1}{4n^3}+\frac{5}{8n^4}-\frac{5}{8n^5}-\frac{5}{16n^6}\right)+\frac{3\ \log(2)\ \zeta_3}{4}+(-1)^n\left(\frac{1}{2n}-\frac{1}{4n^2}+\frac{1}{8n^4}-\frac{1}{4n^6}\right)\zeta_3-\frac{19\ \zeta_2^2}{40}$$

- the algorithm can be extended to cyclotomic harmonic sums

$$\begin{aligned}S_{(2,1,2),(1,0,1)}(n) \sim\ & -4\log(2)S_{(2,1,-1)}(\infty)^2-8\log(2)S_{(2,1,-1)}(\infty)-4\log(2)+\frac{1}{9n^3}-\frac{1}{4n}+\frac{7\zeta_3}{4}\\ & +\left(-\frac{11}{48n^3}+\frac{1}{4n^2}-\frac{1}{4n}\right)(\log(n)+\gamma)\end{aligned}$$

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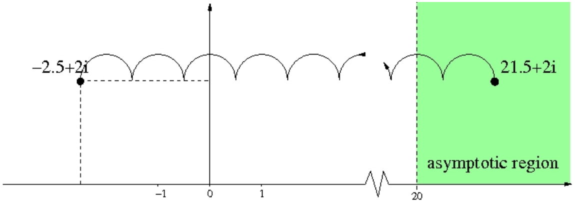
$$\begin{aligned}S_{2,1}(1,\frac{1}{3})(n) \sim\ & -S_{1,2}\left(\frac{1}{3},1;\infty\right)+3^{-n}\left(\frac{3}{2n^3}-\frac{3}{4n^2}+\frac{1}{2n}\right)H_{0,3}(1)+3^{-n}\left(-\frac{3}{2n^3}+\frac{3}{4n^2}-\frac{1}{2n}\right)S_2\left(\frac{1}{3};\infty\right)\\ & +\left(3^{-n}\left(-\frac{3}{2n^3}+\frac{1}{2n^2}+3^n\left(-\frac{1}{6n^3}+\frac{1}{2n^2}-\frac{1}{n}\right)\right)+\zeta_2\right)S_1\left(\frac{1}{3};\infty\right)\\ & +S_3\left(\frac{1}{3};\infty\right)+H_3(1)3^{-n}\left(\frac{3}{2n^3}-\frac{1}{2n^2}\right)+\frac{3^{-n}}{4n^3}\end{aligned}$$

- extension to a subset of the cyclotomic S-Sums:

$$S_{(2,1,2)}\left(\frac{1}{2};n\right) \sim -\frac{\pi^2-6\left(8H_{0,1}\left(\frac{1}{\sqrt{2}}\right)+\log^2(2)\right)}{24\sqrt{2}}-2^{-n}\left(-\frac{293}{8n^5}+\frac{99}{16n^4}-\frac{5}{4n^3}+\frac{1}{4n^2}+2^n\right)$$

Consider the harmonic sum $S_{-1,3}(2.5+2i)$, we have:

$$
\begin{aligned}
S_{-1,3}(n) &\sim (-1)^n\left(-\frac{1}{4n^3}+\frac{5}{8n^4}-\frac{5}{8n^5}-\frac{5}{16n^6}\right)+\frac{3\ \ln 2\ z3}{4} \\
&\quad +(-1)^n\left(\frac{1}{2n}-\frac{1}{4n^2}+\frac{1}{8n^4}-\frac{1}{4n^6}\right)z3-\frac{19\ z2^2}{40} \\
S_{-1,3}(n-1) &= S_{-1,3}(n)-(-1)^n\frac{S_{-1,3}(n)}{n}
\end{aligned}
$$



- $S_{-1,3}(-2.5+2i) = -0.795096 - 0.105476i$

Using the Package HarmonicSums

In[18]:= **SExpansion[S[-1, 3, n], n, 10]**

Out[18]=

$$(-1)^n\left(-\frac{1}{4n^3}+\frac{5}{8n^4}-\frac{5}{8n^5}-\frac{5}{16n^6}+\frac{31}{24n^7}+\frac{133}{96n^8}-\frac{169}{24n^9}-\frac{163}{16n^{10}}\right)+\frac{3\ \ln 2\ z3}{4}+(-1)^n\left(\frac{1}{2n}-\frac{1}{4n^2}+\frac{1}{8n^4}-\frac{1}{4n^6}+\frac{17}{16n^8}-\frac{31}{4n^{10}}\right)z3-\frac{19\ z2^2}{40}$$

In[19]:= **GetApproximation[S[-1,3,n], {-2.5, 2}]**

Out[19]= $-0.795096 - 0.105476\ i$

In[20]:= **HLimit[n*(S[2, n] - z2 - S[2, 2, n] + 7*z2^2/10), n]**

Out[20]= $-1+z2$

Definition (Cyclotomic Harmonic Sums (C-Sums))

Let $a_i, k \in \mathbb{N}^*$, $b_i, n \in \mathbb{N}$ and $c_i \in \mathbb{Z}^*$ we define

$$\mathrm{S}_{(a_1,b_1,c_1),(a_2,b_2,c_2),\dots,(a_k,b_k,c_k)}(n) =$$
$$= \sum_{n \geq i_1 \geq \cdots \geq i_k \geq 1} \frac{\mathrm{sign}(c_1)^{i_1}}{(a_1 i_1 + b_1)^{|c_1|}} \frac{\mathrm{sign}(c_2)^{i_2}}{(a_2 i_2 + b_2)^{|c_2|}} \cdots \frac{\mathrm{sign}(c_k)^{i_k}}{(a_k i_k + b_k)^{|c_k|}}$$

k is called the depth and $w = \sum_{i=1}^{k} |c_i|$ is called the weight of the cyclotomic harmonic sum $\mathrm{S}_{(a_1,b_1,c_1),\dots,(a_k,b_k,c_k)}(n)$.

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$$\mathrm{S}_{(2,1,2),(3,2,1)}(n) = \sum_{i=1}^{n} \frac{\sum_{j=1}^{i} \frac{1}{3j+2}}{(2i+1)^2}$$

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$$\mathrm{S}_{(1,0,2),(1,0,-3)}(n) = \sum_{i=1}^{n} \frac{\sum_{j=1}^{i} \frac{(-1)^j}{(1j+0)^3}}{(1i+0)^2} = S_{2,-3}(n)$$

Definition (Cyclotomic S-Sums (CS-Sums))

Let $a_i, c_i, n, k \in \mathbb{N}^*$, $b_i \in \mathbb{N}$ and $x_i \in \mathbb{R}^*$ we define

$$S_{(a_1,b_1,c_1),(a_2,b_2,c_2),\dots,(a_k,b_k,c_k)}(x_1, x_2, \dots, x_k, n) =$$

$$= \sum_{n \geq i_1 \geq \cdots \geq i_k \geq 1} \frac{x_1^{i_1}}{(a_1 i_1 + b_1)^{c_1}} \frac{x_2^{i_2}}{(a_2 i_2 + b_2)^{c_2}} \cdots \frac{x_k^{i_1}}{(a_k i_k + b_k)^{c_k}}$$

k is called the depth and $w = \sum_{i=1}^{k} c_i$ is called the weight of the Cyclotomic S-Sums $S_{(a_1,b_1,c_1),\dots,(a_k,b_k,c_k)}(x_1, \dots, x_k; n)$.

Definition (Cyclotomic S-Sums (CS-Sums))

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$$S_{(2,1,2),(3,2,1)}(4, 5, n) \quad = \quad \sum_{i=1}^{n} \frac{i^4 \sum_{j=1}^{i} \frac{j^5}{3j+2}}{(2i+1)^2}$$

Definition (Cyclotomic S-Sums (CS-Sums))

Let $a_i, c_i, n, k \in \mathbb{N}^*$, $b_i \in \mathbb{N}$ and $x_i \in \mathbb{R}^*$ we define

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$$= \sum_{n \geq i_1 \geq \cdots \geq i_k \geq 1} \frac{x_1^{i_1}}{(a_1 i_1 + b_1)^{c_1}} \frac{x_2^{i_2}}{(a_2 i_2 + b_2)^{c_2}} \cdots \frac{x_k^{i_1}}{(a_k i_k + b_k)^{c_k}}$$

k is called the depth and $w = \sum_{i=1}^{k} c_i$ is called the weight of the Cyclotomic S-Sums $\mathrm{S}_{(a_1,b_1,c_1),\dots,(a_k,b_k,c_k)}(x_1,\dots,x_k;n)$.

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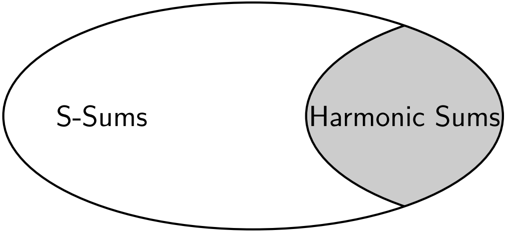
$$\mathrm{S}_{(1,0,2),(1,0,3)}(1,-1,n) = \sum_{i=1}^{n} \frac{\sum_{j=1}^{i} \frac{(-1)^j}{(1j+0)^3}}{(1i+0)^2} = S_{2,-3}(n)$$

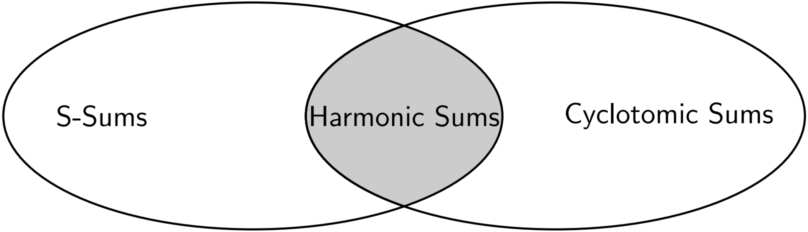
Connection between these nested sums

Harmonic Sums

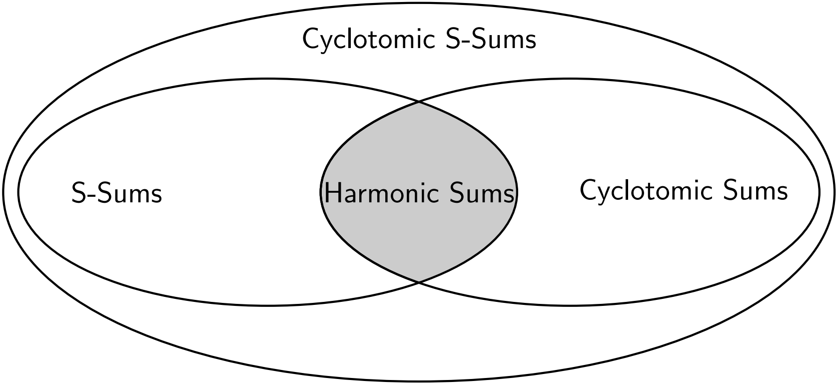
Connection between these nested sums



Connection between these nested sums



Connection between these nested sums



Cyclotomic Harmonic Polylogarithms (C-Logs)

We now define the alphabet

$$\mathfrak{A} := \left\{\frac{1}{x}\right\} \cup \left\{ \frac{x^l}{\Phi_k(x)} \middle| k \in \mathbb{N}^*, 0 \leq l < \varphi(k) \right\},$$

where $\Phi_k(x)$ denotes the kth cyclotomic polynomial and φ is Euler's totient function.

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where $\Phi_k(x)$ denotes the kth cyclotomic polynomial and φ is Euler's totient function.

Definition (Cyclotomic Harmonic Polylogarithms)

Let $m_i \in \mathfrak{A}$ we define for $x \in (0,1)$:

$$\begin{aligned} \mathrm{H}(x) &= 1, \\ \mathrm{H}_{m_1,m_2,\ldots,m_k}(x) &= \begin{cases} \frac{1}{k!}(\log x)^k, & \text{if } (m_1,\ldots,m_k) = (\frac{1}{x},\ldots,\frac{1}{x}) \\ \int_0^x m_1 \mathrm{H}_{m_2,\ldots,m_k}(y)dy, & \text{otherwise.} \end{cases} \end{aligned}$$

k is called the depth of $\mathrm{H}_{\mathbf{m}}(x)$.

Using the Package HarmonicSums

In[21]:= **Mellin[H[{4, 1}, x], x, 2 n]**

$$\text{Out[21]}= \frac{(-1)^n \mathrm{H}[\{4,0\},1]}{2n+1} + \frac{\mathrm{H}[\{4,1\},1]}{2n+1} - \frac{(-1)^n \mathrm{S}[\{\{2,1,-1\}\},n]}{2n+1} - \frac{(-1)^n}{2n+1} - \frac{1}{(2n+1)(2n+3)} + \frac{1}{(2n+1)(2(n+1)+1)}$$

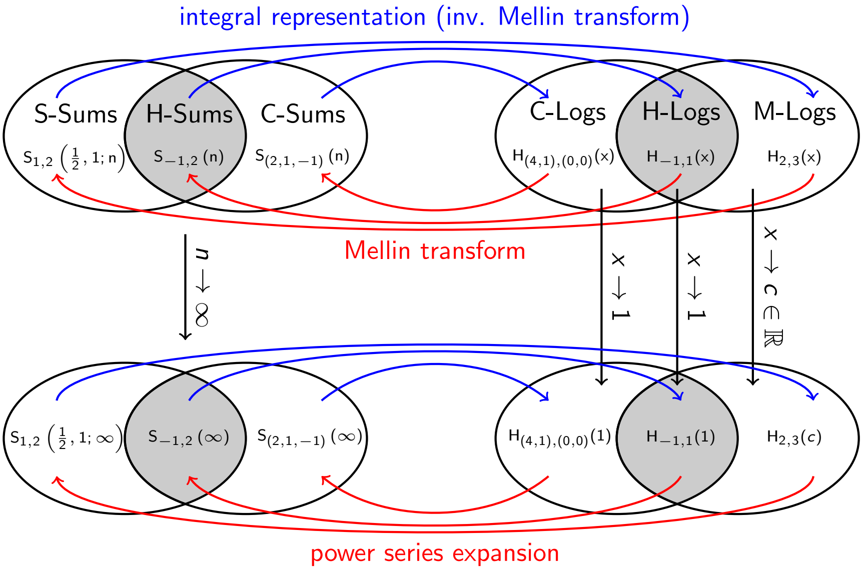
In[22]:= **Mellin[H[{3, 1}, x], x, 3 n]**

$$\text{Out[22]}= \frac{\mathrm{H}[\{3,0\},1]}{3n+1} + \frac{2\mathrm{H}[\{3,1\},1]}{3n+1} - \frac{\mathrm{S}[\{\{3,1,1\}\},n]}{3n+1} + \frac{\mathrm{S}[1,n]}{3(3n+1)} - \frac{1}{3n+1} + \frac{1}{(3n+1)(3n+4)} - \frac{1}{(3n+1)(3(n+1)+1)}$$

In[23]:= **Mellin[H[{5, 2}, x], x, 5 n]**

$$\text{Out[23]}= \frac{\mathrm{H}[\{5,2\},1]}{5n+1} - \frac{\mathrm{H}[\{5,3\},1]}{5n+1} + \frac{\mathrm{S}[\{\{5,4,1\}\},n]}{5n+1} - \frac{\mathrm{S}[1,n]}{5(5n+1)} + \frac{1}{4(5n+1)} - \frac{1}{(5n+1)(5n+4)}$$

Connection between these structures



Cyclotomic Harmonic Sums at Infinity

We are interested in the limits:

$$\lim_{N\to\infty} \mathrm{S}_{\{a_1,b_1,c_1\},\{a_2,b_2,c_2\},\ldots,\{a_k,b_k,c_k\}}(N) = \sigma_{\{a_1,b_1,c_1\},\{a_2,b_2,c_2\},\ldots,\{a_k,b_k,c_k\}},$$

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There is a huge amount of relations between theese values for

example we have:

$$\begin{aligned} S_{(2,1,1),(1,0,1)}(\infty) &= \frac{1}{6} - \frac{1}{2}S_{(1,0,-1)}(\infty)^2 + \frac{1}{2}S_{(1,0,1)}(\infty) + \frac{1}{8}S_{(1,0,1)}(\infty)^2 \\ &\quad + \frac{4}{3}S_{(2,1,-1)}(\infty) + \frac{2}{3}S_{(2,1,-1)}(\infty)^2 - S_{(2,1,1)}(\infty) \\ &\quad + \frac{1}{2}S_{(1,0,1)}(\infty)S_{(2,1,1)}(\infty) - \frac{1}{2}S_{(2,1,1)}(\infty)^2 \end{aligned}$$

Cyclotomic Harmonic Sums at Infinity

We consider for example the iteration of the summands

$$\frac{1}{k^{l_1}}, \qquad \frac{(-1)^k}{k^{l_2}}, \qquad \frac{1}{(2k+1)^{l_3}}, \qquad \frac{(-1)^k}{(2k+1)^{l_4}}$$

with $k \geq 1, \quad l_i, k \in \mathbb{N} \setminus \{0\}$.

weight	N_S	A	SH	A + SH	A + sh + H_1	A + SH + H_1 + H_2	A + SH + H_1 + H_2 + M
1	4	4	4	4	4	3	3
2	20	10	13	3	3	2	1
3	100	40	46	6	6	5	3
4	500	150	163	10	10	9	6
5	2500	624	650	21	21	19	13
6	12500	2580	2635	36	36	34	25

Using the Package HarmonicSums

In[24]:= **ComputeCSumInfBasis[2,{{1,0},{2,1}}]**

Out[24]=

$$\begin{array}{l}
\Big\{\{S[\{1,0,-1\},\{2,1,-1\},\infty]\},\Big\{S[\{2,1,1\},\{2,1,1\},\infty]\to\\
\frac{1}{2}\left(1+4S[\{2,1,-1\},\infty]+2S[\{2,1,-1\},\infty]^2+S[\{2,1,1\},\infty]^2\right),S[\{2,1,1\},\{1,0,1\},\infty]\to\\
\frac{1}{24}\left(4+32S[\{2,1,-1\},\infty]+16S[\{2,1,-1\},\infty]^2-12S[\{2,1,1\},\infty]^2-12S[-1,\infty]^2+12S[\{2,1,1\},\infty](-2+S[1,\infty])+12S[1,\infty]+3S[1,\infty]^2\right),S[\{1,0,1\},\{2,1,1\},\infty]\to\\
\frac{1}{24}\left(-4-32S[\{2,1,-1\},\infty]-16S[\{2,1,-1\},\infty]^2+12S[\{2,1,1\},\infty]^2+12S[-1,\infty]^2+12S[\{2,1,1\},\infty](-2+S[1,\infty])+12S[1,\infty]-3S[1,\infty]^2\right),S[1,1,\infty]\to\\
\frac{1}{6}\left(8+16S[\{2,1,-1\},\infty]+8S[\{2,1,-1\},\infty]^2+3S[1,\infty]^2\right),S[\{2,1,1\},\{2,1,-1\},\infty]\to\\
\frac{1}{2}(-2-4S[\{2,1,1\},\infty]-S[\{1,0,-1\},\{2,1,-1\},\infty]-2S[-1,\infty]-\\
S[\{2,1,-1\},\infty]S[-1,\infty]+2S[1,\infty]+\\
S[\{2,1,-1\},\infty]S[1,\infty]),S[\{2,1,1\},\{1,0,-1\},\infty]\to\\
1+2S[\{2,1,-1\},\infty]+S[\{2,1,-1\},\infty]^2+(1+\\
S[\{2,1,1\},\infty])S[-1,\infty],S[\{1,0,1\},\{2,1,-1\},\infty]\to\\
2+2(2+S[\{2,1,-1\},\infty])S[\{2,1,1\},\infty]+S[\{1,0,-1\},\{2,1,-1\},\infty]+\\
2S[-1,\infty]-2S[1,\infty],S[1,-1,\infty]\to-\frac{1}{2}S[-1,\infty](S[-1,\infty]-\\
2S[1,\infty]),S[\{2,1,-2\},\infty]\to-1-2S[\{2,1,1\},\infty]-\\
S[\{1,0,-1\},\{2,1,-1\},\infty]-S[-1,\infty]+S[1,\infty],S[-2,\infty]\to\\
\frac{1}{3}(-4)(1+S[\{2,1,-1\},\infty])^2,S[\{2,1,-1\},\{2,1,1\},\infty]\to\\
\frac{1}{2}(-S[\{1,0,-1\},\{2,1,-1\},\infty]+S[\{2,1,-1\},\infty](2S[\{2,1,1\},\infty]+\\
S[-1,\infty]-S[1,\infty])),S[\{2,1,-1\},\{1,0,1\},\infty]\to\\
-2-4S[\{2,1,1\},\infty]-S[\{1,0,-1\},\{2,1,-1\},\infty]-S[-1,\infty]+\\
2S[1,\infty]+S[\{2,1,-1\},\infty](-2-2S[\{2,1,1\},\infty]+\\
S[1,\infty]),S[\{1,0,-1\},\{2,1,1\},\infty]\to\\
-1-4S[\{2,1,-1\},\infty]-S[\{2,1,-1\},\infty]^2,S[-1,1,\infty]\to\\
\frac{1}{6}\left(-8-16S[\{2,1,-1\},\infty]-8S[\{2,1,-1\},\infty]^2+3S[-1,\infty]^2\right),S[\{2,1,-1\},\{2,1,-1\},\infty]\to\\
\frac{1}{2}\left(1+4S[\{2,1,-1\},\infty]+3S[\{2,1,-1\},\infty]^2\right),S[\{2,1,-1\},\{1,0,-1\},\infty]\to\\
-2S[\{2,1,1\},\infty]-S[\{1,0,-1\},\{2,1,-1\},\infty]+\\
S[\{2,1,-1\},\infty]S[-1,\infty]+S[1,\infty],S[-1,-1,\infty]\to\\
\frac{1}{6}\left(8+16S[\{2,1,-1\},\infty]+8S[\{2,1,-1\},\infty]^2+3S[-1,\infty]^2\right),S[\{2,1,2\},\infty]\to\\
1+4S[\{2,1,-1\},\infty]+2S[\{2,1,-1\},\infty]^2,S[2,\infty]\to\\
\frac{8}{3}(1+S[\{2,1,-1\},\infty])^2\Big\}\Big\}
\end{array}$$

Higher Cyclotomy: Weight $w = 1$

l	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
sums	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
basis	2	3	4	5	6	6	8	9	8	10	12	10	14	14	11
new basis sums	2	1	2	2	4	1	6	4	4	3	10	2	12	5	3

Table: The number of the weight w = 1 cyclotomic harmonic sums up to cyclotomy l = 15, the basis elements at fixed value of l, and the new basis elements in ascending sequence.

Let p, p_i, q be pairwise distinct primes > 2, and let k, k_i be positive integers. The number of basis elements at w = 1 and cyclotomy l are given by

$$\varphi(l) = \begin{cases} l+1, & l = 1 \text{ or } l = 2^k \\ (p-1)p^{k-1}+2, & l = p^k \\ 2\varphi\left(2^{k-1}\prod_{i=1}^n p_i^{k_i}\right) - n - 1, & l = 2^k \prod_{i=1}^n p_i^{k_i} \\ (q-1)\varphi\left(\prod_{i=1}^n p_i^{k_i}\right) - n(q-2) - q + 3, & l = q\prod_{i=1}^n p_i^{k_i} \\ q\,\varphi\left(q^{k-1}\prod_{i=1}^n p_i^{k_i}\right) - (n+2)(q-1), & l = q^k \prod_{i=1}^n p_i^{k_i},\ k > 1\ . \end{cases}$$

Using the Package HarmonicSums

In[25]:= **ComputeCSumInfBasis[1,{{1, 0}, {2, 1}, {3, 1}, {3, 2}, {4, 1}, {4, 3}}]**

Out[25]=

$$\Big\{\Big\{S[\{1,0,-1\},\infty], S[\{1,0,1\},\infty], S[\{2,1,-1\},\infty], S[\{3,1,-1\},\infty], S[\{3,1,1\},\infty], S[\{4,1,-1\},\infty], S[\{4,3,-1\},\infty]\Big\},$$

$$\Big\{\log(3) \to -\tfrac{1}{3}S[\{1,0,-1\},\infty] - \tfrac{2}{3}S[\{1,0,1\},\infty] - S[\{3,1,-1\},\infty] +$$
$$2S[\{3,1,1\},\infty] + 1, S[\{4,3,1\},\infty] \to -\tfrac{3}{4}S[\{1,0,-1\},\infty] +$$
$$\tfrac{1}{4}S[\{1,0,1\},\infty] - \tfrac{1}{2}S[\{2,1,-1\},\infty] - \tfrac{5}{6}, S[\{4,1,1\},\infty] \to$$
$$-\tfrac{3}{4}S[\{1,0,-1\},\infty] + \tfrac{1}{4}S[\{1,0,1\},\infty] + \tfrac{1}{2}S[\{2,1,-1\},\infty] -$$
$$\tfrac{1}{2}, S[\{3,2,1\},\infty] \to -\tfrac{1}{3}S[\{1,0,-1\},\infty] - S[\{3,1,-1\},\infty] +$$
$$S[\{3,1,1\},\infty] - \tfrac{1}{2}, S[\{2,1,1\},\infty] \to$$
$$-S[\{1,0,-1\},\infty] + \tfrac{1}{2}S[\{1,0,1\},\infty] - 1, S[\{3,2,-1\},\infty] \to$$
$$\tfrac{2}{3}S[\{1,0,-1\},\infty] + S[\{3,1,-1\},\infty] + \tfrac{1}{2}\Big\}\Big\}$$

l	N_S	SH	A	A + SH	$A + SH + H_1$	$A + SH + H_1 + H_2$	$A + SH + H_1 + H_2 + M$
1	6	4	3	1	1	1	1
2	20	13	10	3	3	2	1
3	42	27	21	7	6	6	5
4	72	46	36	12	11	10	3
5	110	70	55	19	17	17	16
6	156	99	78	27	25	24	5
7	210	133	105	37	34	34	33
8	272	172	136	48	45	44	12
9	342	216	171	61	57	57	52
10	420	265	210	75	71	70	22
11	506	319	253	91	86	86	85
12	600	378	300	108	103	102	21
13	702	442	351	127	121	121	120
14	812	551	406	147	141	140	49
15	930	585	465	169	162	162	145
16	1056	664	528	192	185	184	50
17	1190	748	595	217	209	209	208
18	1332	837	666	243	235	234	63
19	1482	931	741	271	262	262	261
20	1640	1030	820	300	291	290	74

Table: Number of basis elements of the weight w = 2 cyclotomic harmonic sums up to cyclotomy l = 20 after applying the quasi-shuffle algebra of the sums (A), the shuffle algebra of the cyclotomic harmonic polylogarithms (SH), and the three multiple argument relations (H_1, H_2, M) of the sums.