Equivalence Between K-functionals Based on Continuous Linear Transforms and Applications

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K-functionals – an error measure

X — a Banach space, norm $\|\cdot\|_X$
Y — a space with a semi-norm $|\cdot|_Y$

K-functionals

$$K(f,t;X,Y) = \inf\{\|f-g\|_X + t|g|_Y : g \in Y \cap X\}$$

$Y \cap X$ is dense in X $\Longrightarrow$

$$\lim_{t\to 0+} K(f,t;X,Y) = 0$$

$$K(f,t;X_1,Y_1) \sim K(\mathcal{A}f,t;X_2,Y_2) \qquad (1)$$

Relations b/n K-functionals

Theorem

If X_1, X_2 — Banach spaces
$\mathcal{A} : X_1 \to X_2,\ \mathcal{B} : X_2 \to X_1$ *— linear operators s.t.*

1. $\|\mathcal{A}f\|_{X_2} \le c\,\|f\|_{X_1} \quad \forall f \in X_1;$
2. $|\mathcal{A}g|_{Y_2} \le c\,|g|_{Y_1} \quad \forall g \in Y_1 \cap X_1;$
3. $\|\mathcal{B}F\|_{X_1} \le c\,\|F\|_{X_2} \quad \forall F \in X_2;$
4. $|\mathcal{B}G|_{Y_1} \le c\,|G|_{Y_2} \quad \forall G \in Y_2 \cap X_2;$
5. $|f - \mathcal{B}\mathcal{A}f|_{Y_1} = 0 \quad \forall f \in X_1;$
6. $|F - \mathcal{A}\mathcal{B}F|_{Y_2} = 0 \quad \forall F \in X_2.$

Then

$$K(f, t; X_1, Y_1) \sim K(\mathcal{A}f, t; X_2, Y_2). \tag{2}$$

Moduli of smoothness

$$\omega_r(f,t)_p = \sup_{0<h\le t} \|\Delta_h^r f(\cdot)\|_{p(I)}, \tag{3}$$

finite difference with a fixed step h

$$\Delta_h^r f(x) = \begin{cases} \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(x+kh), & \text{if } x, x+rh \in I, \\ 0, & \text{otherwise.} \end{cases}$$

$$K(F,t^r; L_p(I), W_p^r(I)) \sim \omega_r(F,t)_{p(I)} \tag{4}$$

$$K(f,t^r; X_1, Y_1) \sim K(\mathcal{A}f, t^r; L_p(I), W_p^r(I)) \sim \omega_r(\mathcal{A}f, t)_{p(I)} \tag{5}$$

K-functionals with power-type weights

$$K(f,t;L_p(w)(I),W_p^r(w\varphi^r)(I))$$
$$=\inf\left\{\|w(f-g)\|_{p(I)}+t\|w\varphi^r g^{(r)}\|_{p(I)}\ :\ g\in AC_{loc}^{r-1}(I)\right\} \quad (6)$$

I	$w(x)$	$\varphi(x)$
(a,b)	$(x-a)^{\gamma_a}(b-x)^{\gamma_b}$	$(x-a)^{\lambda_a}(b-x)^{\lambda_b}$
(a,∞)	$(x-a)^{\gamma_a}(x-a+1)^{\gamma_\infty-\gamma_a}$	$(x-a)^{\lambda_a}(x-a+1)^{\lambda_\infty-\lambda_a}$

- the Ditzian-Totik modulus
- the Ivanov modulus
- the Potapov modulus, etc.

The Ditzian-Totik modulus

$$\omega_\varphi^r(f,t)_p = \sup_{0<h\le t} \|\Delta^r_{h\varphi(\cdot)} f(\cdot)\|_p, \tag{7}$$

$$\Delta^r_{h\varphi(x)} f(x) = \begin{cases} \displaystyle\sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(x+kh\varphi(x)), & \text{if } x, x+rh\varphi(x) \in I, \\ 0, & \text{otherwise.} \end{cases}$$

$\mathcal{A}$-operators

$$\rho \in \mathbb{R}, \quad i \in \mathbb{N}_0, \quad i \le r, \quad x \in (0,1), \quad f \in L_{1,loc}[0,1]$$

$$\begin{aligned}(A_i(\rho)f)(x) = x^\rho f(x) &+ \sum_{k=1}^{i} \alpha_{r,k}(\rho) x^{k-1} \int_0^x y^{-k+\rho} f(y)\,dy \\ &+ \sum_{k=i+1}^{r} \alpha_{r,k}(\rho) x^{k-1} \int_\xi^x y^{-k+\rho} f(y)\,dy,\end{aligned}$$

$$\alpha_{r,k}(\rho) = \frac{(-1)^k}{(r-1)!} \binom{r-1}{k-1} \prod_{\nu=0}^{r-1} (\rho + r - k - \nu), \quad k = 1, 2, \dots, r.$$

Best algebraic approximation

$$E_n(f)_\infty \le c\,\omega_r(\mathcal{B}f, n^{-1})_{\infty[-1,1]}, \qquad \mathcal{B} = B_1 B_2$$

$$B_1 f(x) = f\left(x + \frac{1-x^2}{2}\right) - \sum_{k=1}^{[r/2]} \beta_{r,2k}(1-x)^{2k-1} \int_0^x \frac{f\left(y + \frac{1-y^2}{2}\right)}{(1-y)^{2k}}\,dy,$$

$$B_2 f(x) = f\left(x - \frac{1-x^2}{2}\right) + \sum_{k=1}^{[r/2]} \beta_{r,2k}(1+x)^{2k-1} \int_0^x \frac{f\left(y - \frac{1-y^2}{2}\right)}{(1+y)^{2k}}\,dy,$$

Related results

- Best weighted algebraic approximation, $L_p[-1,1]$
- Bernstein polynomials, weights with $\gamma_0, \gamma_1 \in [-1,0]$

$$\|w(f-B_nf)\|_{\infty(0,1)} \sim K(f, n^{-1/2}; C(w)(0,1), W^2_\infty(w\varphi^2)(0,1))$$

$$\varphi(x) = \sqrt{x(1-x)}$$

- Kantorovich and Durrmeyer polynomials, etc.

Post-Widder and Gamma operators

Theorem

$\gamma \in \mathbb{R}$, $1 \le p \le \infty$, $f \in L_p(\chi^\gamma)(\mathbb{R}_+)$, $\mathcal{A}f(x) = e^{(\gamma+1/p)x} f(e^x)$.

1. If $\gamma \ne -1 - 1/p, -1/p$, then

$$\|\chi^\gamma(f - P_s f)\|_{p(\mathbb{R}_+)} \sim \|\chi^\gamma(f - G_s f)\|_{p(\mathbb{R}_+)} \sim \omega_2(\mathcal{A}f, s^{-1/2})_{p(\mathbb{R})} + s^{-1}\|\mathcal{A}f\|_{p(\mathbb{R})}.$$

2. If $\gamma = -1 - 1/p, -1/p$, then

$$\|\chi^\gamma(f - P_s f)\|_{p(\mathbb{R}_+)} \sim \|\chi^\gamma(f - G_s f)\|_{p(\mathbb{R}_+)} \sim \omega_2(\mathcal{A}f, s^{-1/2})_{p(\mathbb{R})} + s^{-1/2}\omega_1(\mathcal{A}f, s^{-1/2})_{p(\mathbb{R})}.$$

Best trigonometric approximation

D. Jackson, S. N. Bernstein, A. Zygmund, S. B. Stechkin:

$$E_n^T(f)_p \le c\,\omega_r(f, n^{-1})_p, \tag{8}$$

$$\omega_r(f,t)_p \le c\,t^r \sum_{0\le k\le 1/t} (k+1)^{r-1} E_k^T(f)_p, \quad 0 < t \le t_0 \tag{9}$$

A discrepancy:

$$E_n^T(f)_p = 0 \quad \Longleftrightarrow \quad f \in T_n$$

Another characterization

B — a HBS on $\mathbb{T}$:

$$\lim_{t\to t_0} \|f_t - f_{t_0}\|_B = 0, \quad \|f_t\|_B = \|f\|_B, \quad \|f\|_L \leq c\|f\|_B$$

$$f_t(x) = f(x-t)$$

$$\omega_r^T(f,t)_B = \sup_{0<h\leq t} \|\Delta_h^{2r-1}\mathfrak{F}_{r-1}f\|_B. \tag{10}$$

$$\mathfrak{F}_{r-1}f = (\delta + \mathfrak{a}) * \cdots * (\delta + (r-1)^2\mathfrak{a}) * f \tag{11}$$

$$\mathfrak{a}(x) = \frac{1}{2}|x|(2\pi - |x|), \quad x \in [-\pi,\pi], \quad \text{periodic.} \tag{12}$$

A characterization

Theorem

Let B be a HBS on $\mathbb{T}$ and $f \in B$. Then

$$E_n^T(f)_B \leq c\,\omega_r^T(f, n^{-1})_B, \quad n \geq r-1, \tag{13}$$

and

$$\omega_r^T(f,t)_B \leq c\,t^{2r-1} \sum_{r-1 \leq k \leq 1/t} (k+1)^{2r-2} E_k^T(f)_B, \quad 0 < t \leq \frac{1}{r}. \tag{14}$$

$$\omega_r^T(f,t)_B \equiv 0 \quad \Longleftrightarrow \quad f \in T_{r-1}$$

L-splines

$$Lg(x) = g^{(r)}(x) + \sum_{k=1}^{r} \varphi_k(x) g^{(r-k)}(x), \quad x \in [a, b], \tag{15}$$

$\varphi_k \in C^{r-k}[a, b]$

$$K_L(f, t)_p = \inf_{g \in W_p^r[a,b]} \{\|f - g\|_p + t^r \|Lg\|_p\}$$

— Scherer and Schumaker, 1980

$$E_n^L(f)_p \leq c\, K_L(f, 1/n)_p \tag{16}$$

for equidistant nodes

The L-modulus of smoothness

$$f \in L_p[a,b], \quad 1 \le p \le \infty$$

$$(A_L f)(x) = f(x) + \sum_{k=1}^{r} \sum_{j=0}^{r-k} (-1)^j \binom{r-k}{j} \int_a^x \frac{(x-y)^{k+j-1}}{(k+j-1)!} \varphi_k^{(j)}(y) f(y)\, dy. \quad (17)$$

The L-modulus of smoothness

$$\omega_L(f,t)_p = \omega_r(A_L f, t)_p$$

L-spline error characterization

Relation to a K-functional

$$K_L(f,1/n)_p \sim \omega_L(f,1/n)_p$$

Theorem

$$E_n^L(f)_p \leq c\,\omega_L(f,1/n)_p \tag{18}$$

$$\omega_L(f,1/n)_p \leq \frac{c}{n+1} \sum_{m=n-1}^{2n-1} E_m^L(f)_p. \tag{19}$$

for equidistant nodes

Properties of the L-modulus I

1. $\omega_L(f+g,t)_p \le \omega_L(f,t)_p + \omega_L(g,t)_p$;
2. $\omega_L(cf,t)_p = |c|\,\omega_L(f,t)_p$
3. $\omega_L(f,t)_p \le \omega_L(f,t')_p, \quad t \le t'$;
4. $\omega_L(f,t)_p \to 0, \quad t \to 0$;
5. $\omega_L(f,\lambda t)_p \le (\lambda+1)^r \omega_L(f,t)_p$;
6. $\omega_L(f,t)_p \le 2^r \|A_L\|\,\|f\|_p$;

Properties of the L-modulus II

7. $\omega_L(f,t)_p \equiv 0 \quad \text{iff} \quad f \in \ker L;$
8. $\omega_{L_2L_1}(f,t)_p \le t^{r_1}\, \omega_{L_2}(L_1 f,t)_p, \quad f \in W_p^{r_1}[a,b];$
9. $$\omega_{L_2}(L_1 f,t)_p \le c \int_0^t \frac{\omega_{L_2L_1}(f,u)_p}{u^{r_1+1}}\, du,$$
10. $\omega_{L_2L_1}(f,t)_p \le (2^{r_2} + c\,t)\, \omega_{L_1}(f,t)_p;$
11. $$\omega_{L_1}(f,t)_p \le c\, t^{r_1} \left(\int_t^{t_0} \frac{\omega_{L_2L_1}(f,u)_p}{u^{r_1+1}}\, du + \|f\|_p \right).$$