Interpolation by Harmonic Polynomials Based on Radon Projections

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joint work with

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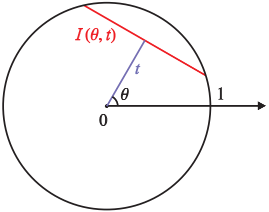
Preliminaries

Consider real bivariate functions $f(x, y)$ in the unit disk in $\mathbb{R}^2$.

Let $I(\theta, t)$ denote a chord of the unit circle at angle $\theta \in [0, 2\pi)$ and distance $t \in (-1, 1)$ from the origin.

The Radon projection $\mathcal{R}_\theta(f; t)$ of f in direction θ is defined by the line integral



$$\mathcal{R}_\theta(f; t) := \int_{I(\theta,t)} f(\mathbf{x})\, d\mathbf{x}$$

$$= \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\, ds.$$

Historical remarks

Theorem (Radon, 1917)

The Radon transform

$$f \mapsto \left\{ \mathcal{R}_\theta(f;t) : -1 \le t \le 1,\ 0 \le \theta < \pi \right\}$$

determines a differentiable function f uniquely.

Interpolation using Radon projections

Find $p \in \Pi_n^2$ (bivar. polynomials, deg. $\leq n$) such that

$$\int_I p \, dx = \gamma_I \qquad \forall I \in \mathcal{I},$$

where

- $\mathcal{I} = \left\{ I_1, I_2, \ldots, I_{\binom{n+2}{2}} \right\}$ are chords of the unit circle,
- $\gamma_I \in \mathbb{R}$ are given values.

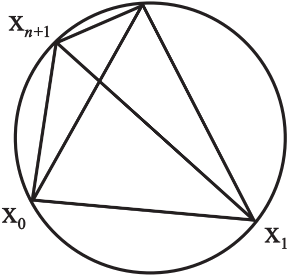
Question

For which configurations of chords $\mathcal{I}$ does this problem have a unique solution?

Previous work

Hakopian regular schemes, 1982

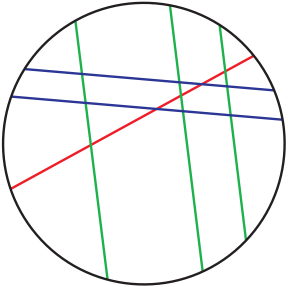
The scheme consisting of all $\binom{n+2}{2}$ chords connecting $n+2$ given points on the unit circle provides a unique interpolating polynomial in Π_n^2.



Previous work

Bojanov-G. regular schemes, 2004

A scheme consisting of $\binom{n+2}{2}$ chords partitioned into $n+1$ subsets such that the k-th subset consists of k parallel chords provides a unique interpolating polynomial in Π_n^2 (under conditions on the distances t).



Previous work

Bojanov-Xu regular schemes, 2005

A scheme consisting of $\binom{n+2}{2}$ chords partitioned into $2\lfloor(n+1)/2\rfloor + 1$ equally spaced directions such that in every direction there are $\lfloor n/2\rfloor + 1$ parallel chords provides a unique interpolating polynomial in Π_n^2 (under conditions on the distances t).

Previous work

Other regular schemes, based on the previous ones, were studied by

- G., Ismail 2006,
- G., Uluchev 2008,
- G., Hofreither, Uluchev 2010.

[Interpolation of mixed type data by bivariate polynomials, Constructive Theory of Functions, Sozopol 2010: In memory of Borislav Bojanov, I. Georgieva, C. Hofreither, R. Uluchev, `https://www.dk-compmath.jku.at/publications/dk-reports/2010-12-10/view`.]

An overview of the field may be found in

- A.S. Cavaretta, C.A. Micchelli, and A. Sharma, *Multivariate interpolation and the Radon transform*, 1980.
- F. Natterer, *The Mathematics of Computerized Tomography*, 2001.

Interpolation of harmonic functions

Assume that we know that the function to be interpolated is harmonic.

Then it's natural to work with the space $\mathcal{H}_n$ of bivariate harmonic polynomials of degree $\leq n$.

Note that $\dim \mathcal{H}_n = 2n + 1 \ll \binom{n+2}{2} = \dim \Pi_n^2$.

This means less required data and faster and cheaper computations for the same accuracy.

Interpolation with harmonic polynomials

Find $p \in \mathcal{H}_n$ (harmonic polynomials, deg. $\leq n$) such that

$$\int_I p \, dx = \gamma_I \qquad \forall I \in \mathcal{I},$$

where

- $\mathcal{I} = \{I_1, I_2, \dots, I_{2n+1}\}$ are chords of the unit circle,
- $\gamma_I \in \mathbb{R}$ are given values.

Question

For which configurations of chords $\mathcal{I}$ does this problem have a unique solution?

Previous approaches

- Adapt a proof technique used for the non-harmonic case [Bojanov, G. 2004]: construct a basis of ridge Chebyshev polynomials of Π_n.
  Ridge polynomials cannot be harmonic for degree > 1.
- Pass to a variational formulation for the Laplace equation and use functional analytic tools.
  Not clear how to show required inf-sup condition.

$\longrightarrow$ Ask for help from Symbolic Computation!

Basis of harmonic polynomials

The polynomials

$$\phi_0(x,y) = 1, \quad \phi_{k,1}(x,y) = \mathrm{Re}(x+\mathbf{i}y)^k, \quad \phi_{k,2}(x,y) = \mathrm{Im}(x+\mathbf{i}y)^k$$

for $k = 1, \ldots, n$ form a basis of $\mathcal{H}_n$. In polar coordinates:

$$\phi_{k,1}(r,\theta) = r^k \cos(k\theta), \quad \phi_{k,2}(r,\theta) = r^k \sin(k\theta),$$

where $r \in (0,1)$, $\theta \in (-\pi,\pi)$.

Using binomial theorem, we get

$$\phi_{k,1}(x,y) = \sum_{\ell=0}^{\lfloor k/2 \rfloor} \binom{k}{2\ell} (-1)^\ell x^{k-2\ell} y^{2\ell},$$

$$\phi_{k,2}(x,y) = \sum_{\ell=0}^{\lfloor k/2 \rfloor} \binom{k}{2\ell+1} (-1)^\ell x^{k-(2\ell+1)} y^{2\ell+1}.$$

Basis of harmonic polynomials

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Problem as a system of linear equations

With this basis, we have the expansion

$$p(x,y) = p_0 + \sum_{k=1}^{n} p_{k,1}\,\phi_{k,1}(x,y) + \sum_{k=1}^{n} p_{k,2}\,\phi_{k,2}(x,y),$$

and the interpolation problem can be written as a linear system

$$\begin{pmatrix} \int_{I_1} 1 & \int_{I_1} \phi_{1,1} & \cdots & \int_{I_1} \phi_{n,1} & \int_{I_1} \phi_{n,2} \\ \int_{I_2} 1 & \int_{I_2} \phi_{1,1} & \cdots & \int_{I_2} \phi_{n,1} & \int_{I_1} \phi_{n,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \int_{I_{2n+1}} 1 & \int_{I_{2n+1}} \phi_{1,1} & \cdots & \int_{I_{2n+1}} \phi_{n,1} & \int_{I_{2n+1}} \phi_{n,2} \end{pmatrix} \begin{pmatrix} p_0 \\ p_{1,1} \\ p_{1,2} \\ \vdots \\ p_{n,1} \\ p_{n,2} \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{2n+1} \end{pmatrix}$$

Radon projections of monomials

For $i, j \in \mathbb{N}_0$, using the binomial theorem,

$$\int_{I(\theta,t)} x^i y^j \, d\mathbf{x} = \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} (t\cos\theta - s\sin\theta)^i (t\sin\theta + s\cos\theta)^j \, ds$$

$$= \sum_{p=0}^{i} \sum_{q=0}^{j} \binom{i}{p} \binom{j}{q} t^{p+q} (\cos\theta)^{j+p-q} (\sin\theta)^{i-(p-q)} \times$$

$$\times \frac{(-1)^{i-p}}{i+j-p-q+1} (1-t^2)^{\frac{1}{2}(i+j-p-q+1)} \left(1 - (-1)^{i+j-p-q+1}\right).$$

Radon projections of the harmonic basis

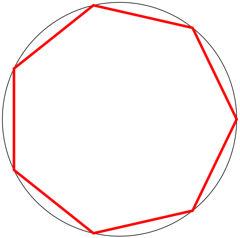
Combining the formulas for the harmonic basis and the monomial projection yields

$$\int_{I(\theta,t)} \phi_{k,1} = \sum_{\ell=0}^{\lfloor k/2 \rfloor} \binom{k}{2\ell} (-1)^\ell \sum_{p=0}^{k-2\ell} \sum_{q=0}^{2\ell} \binom{k-2\ell}{p} \binom{2\ell}{q} t^{p+q} \times$$
$$\times (\cos\theta)^{2\ell+p-q} (\sin\theta)^{k-2\ell-(p-q)} \frac{(-1)^{k-2\ell-p}}{k-p-q+1} \times$$
$$\times (1-t^2)^{\frac{1}{2}(k-p-q+1)} \left(1-(-1)^{k-p-q+1}\right)$$

and an analogous expression for $\phi_{k,2}$.

A regular polygonal scheme

For simplicity, we start with the case where the chords form a regular $(2n+1)$-sided polygon inscribed in the unit circle.



This means, for $m = 1, \ldots, 2n+1$,

$$I_m = I(\theta_m, t_m), \quad \theta_m = \frac{2\pi m}{2n+1}, \quad t_m = \cos \frac{\pi}{2n+1}.$$

HolonomicFunctions (Koutschan)

- Holonomic functions are defined by linear difference-differential equations with polynomial coefficients, e.g., Legendre polynomials $P_n(x)$:

$$(x^2-1)P_n'(x)+(n+1)xP_n(x)-(n+1)P_{n+1}(x)=0$$
$$(n+1)P_n(x)-(2n+3)xP_{n+1}(x)+(n+2)P_{n+2}(x)=0$$

- Holonomic functions are closed under operations such as addition, multiplication, summation, integration.
- The defining systems of linear difference-differential equations for these composite objects can be found algorithmically.
- An implementation is available in the Mathematica package `HolonomicFunctions` by C. Koutschan. (`http://www.risc.jku.at/research/combinat/software/`)

Symbolic simplification

The method of "creative telescoping" (developed by D. Zeilberger, implemented in `HolonomicFunctions`) delivers a recurrence relation for the integrals:

$$
\begin{aligned}
(k+5)F(k+4) - 4t\cos(\theta)(k+4)F(k+3) \\
+ 2(k+3)(2\cos(\theta)^2 + 2t^2 - 1)F(k+2) \\
- 4t\cos(\theta)(k+2)F(k+1) + (k+1)F(k) = 0
\end{aligned}
$$

where

$$F(k) = \int_{I(\theta,t)} \phi_{k,1}.$$

Similarly, $\int_{I(\theta,t)} \phi_{k,2}$ satisfies the same recurrence relation (with different initial values).

Symbolic simplification

Next, M. Petkovšek's algorithm `Hyper` is used to compute a basis of hypergeometric solutions to this recurrence. One of these solutions is

$$\frac{k+1}{k+2}\left(t\cos(\theta)+\sqrt{\sin^2(\theta)(t^2-1)}+\right.$$
$$\left.\sqrt{\cos^2(\theta)(2t^2-1)-t^2+2t\cos(\theta)\sqrt{\sin^2(\theta)(t^2-1)}}\right),$$

and the others differ by signs only.

Symbolic simplification

Using the initial values gives the linear combination of these four solutions that is a (quite complicated) closed-form representation of the integrals $\int_{I(\theta,t)} \phi_{k,1}$ and $\int_{I(\theta,t)} \phi_{k,2}$.

Further algebraic simplifications yield the simple form of the matrix entries

$$\int_{I_m} \phi_{k,1} = \alpha_k \cos(k\theta_m),$$
$$\int_{I_m} \phi_{k,2} = \alpha_k \sin(k\theta_m)$$

where

$$\alpha_k = \frac{2}{k+1} \sin \frac{(k+1)\pi}{2n+1} > 0.$$

Determinant computation

With

$$C = \begin{pmatrix} 1 & \cos(\theta_1) & \sin(\theta_1) & \dots & \cos(n\theta_1) & \sin(n\theta_1) \\ 1 & \cos(\theta_2) & \sin(\theta_2) & \dots & \cos(n\theta_2) & \sin(n\theta_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(\theta_{2n}) & \sin(\theta_{2n}) & \dots & \cos(n\theta_{2n}) & \sin(n\theta_{2n}) \\ 1 & \cos(\theta_{2n+1}) & \sin(\theta_{2n+1}) & \dots & \cos(n\theta_{2n+1}) & \sin(n\theta_{2n+1}) \end{pmatrix},$$

$$D = \operatorname{diag}(\alpha_0, \alpha_1, \alpha_1, \dots, \alpha_n, \alpha_n),$$

our system matrix has the form

$$A = CD.$$

Determinant computation

The determinant of our system matrix is then $\alpha_0 \prod_{k=1}^{n} \alpha_k^2 \times$

$$\begin{vmatrix} 1 & \cos(\theta_1) & \sin(\theta_1) & \cos(2\theta_1) & \sin(2\theta_1) & \dots & \cos(n\theta_1) & \sin(n\theta_1) \\ 1 & \cos(\theta_2) & \sin(\theta_2) & \cos(2\theta_2) & \sin(2\theta_2) & \dots & \cos(n\theta_2) & \sin(n\theta_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(\theta_{2n}) & \sin(\theta_{2n}) & \cos(2\theta_{2n}) & \sin(2\theta_{2n}) & \dots & \cos(n\theta_{2n}) & \sin(n\theta_{2n}) \\ 1 & \cos(\theta_{2n+1}) & \sin(\theta_{2n+1}) & \cos(2\theta_{2n+1}) & \sin(2\theta_{2n+1}) & \dots & \cos(n\theta_{2n+1}) & \sin(n\theta_{2n+1}) \end{vmatrix}$$

The functions $\{1, \cos(x), \sin(x), \dots, \cos(nx), \sin(nx)\}$ form a basis of the trigonometric polynomials of degree $\leq n$.

It's well known that the above determinant is nonzero provided

$$0 \leq \theta_1 < \theta_2 < \dots < \theta_{2n+1} < 2\pi.$$

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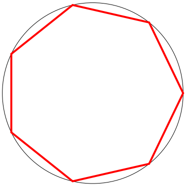
First result using symbolic tools

Theorem

The interpolation problem

$$\text{Find } p \in \mathcal{H}_n : \quad \int_I p \, dx = \gamma_I \qquad \forall I \in \mathcal{I}$$

has a unique solution if the chords $\mathcal{I}$ form a regular $(2n+1)$-sided polygon inscribed in the unit circle.



[Harmonic Interpolation Based on Radon Projections Along the Sides of Regular Polygons, *I. Georgieva, C. Hofreither, C. Koutschan, V. Pillwein, T. Thanatipanonda*], accepted for publication in CEJM

Radon projections of the harmonic basis

Symbolic result $\Longrightarrow$ Analytic result

We managed to show *analytically* the following analogue of Marr's formula for the harmonic case for general θ and t:

Theorem

$$\int_{I(\theta,t)} \phi_{k,1} = \frac{2}{k+1}\sqrt{1-t^2}\,U_k(t)\cos(k\theta),$$
$$\int_{I(\theta,t)} \phi_{k,2} = \frac{2}{k+1}\sqrt{1-t^2}\,U_k(t)\sin(k\theta),$$

where $U_k(t)$ is the Chebyshev polynomial of second kind of degree k.

The bivariate ridge polynomial in direction ϕ:

$$U_k(\phi; \mathbf{x}) := U_k(x \cos\phi + y \sin\phi).$$

The ridge polynomials $U_k(\frac{j\pi}{k+1}; \mathbf{x})$, $k = 0, \dots, n$, $j = 0, \dots, k$, form an orthonormal basis of Π_n^2 on the unit disk D.

Lemma (Marr's formula, 1974)

For each $t \in (-1, 1)$, θ and ϕ, we have

$$\mathcal{R}_\theta(U_k(\phi; \cdot); t) = \frac{2}{k+1} \sqrt{1 - t^2}\, U_k(t) \frac{\sin(k+1)(\theta - \phi)}{\sin(\theta - \phi)}.$$

Determinant computation

For $t_m = t$ constant, the determinant of our matrix is $\alpha_0 \prod_{k=1}^{n} \alpha_k^2 \times$

$$\begin{vmatrix} 1 & \cos(\theta_1) & \sin(\theta_1) & \cos(2\theta_1) & \sin(2\theta_1) & \dots & \cos(n\theta_1) & \sin(n\theta_1) \\ 1 & \cos(\theta_2) & \sin(\theta_2) & \cos(2\theta_2) & \sin(2\theta_2) & \dots & \cos(n\theta_2) & \sin(n\theta_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(\theta_{2n}) & \sin(\theta_{2n}) & \cos(2\theta_{2n}) & \sin(2\theta_{2n}) & \dots & \cos(n\theta_{2n}) & \sin(n\theta_{2n}) \\ 1 & \cos(\theta_{2n+1}) & \sin(\theta_{2n+1}) & \cos(2\theta_{2n+1}) & \sin(2\theta_{2n+1}) & \dots & \cos(n\theta_{2n+1}) & \sin(n\theta_{2n+1}) \end{vmatrix}$$

with $\alpha_k = \frac{2}{k+1}\sqrt{1-t^2}\,U_k(t) > 0$.

The functions $\{1, \cos(x), \sin(x), \dots, \cos(nx), \sin(nx)\}$ form a basis of the trigonometric polynomials of degree $\leq n$.

It's well known that the above determinant is nonzero provided

$$0 \leq \theta_1 < \theta_2 < \dots < \theta_{2n+1} < 2\pi.$$

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The functions $\{1, \cos(x), \sin(x), \dots, \cos(nx), \sin(nx)\}$ form a basis of the trigonometric polynomials of degree $\leq n$.

It's well known that the above determinant is nonzero provided

$$0 \leq \theta_1 < \theta_2 < \dots < \theta_{2n+1} < 2\pi.$$

Theorem (Existence and uniqueness)

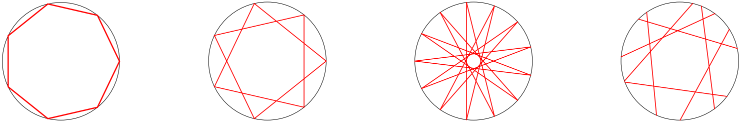
The interpolation problem

$$\text{Find } p \in \mathcal{H}_n : \quad \int_I p\,dx = \gamma_I \qquad \forall I \in \mathcal{I}$$

has a unique solution if for the chords $\mathcal{I}$ *we have*

$$0 \le \theta_1 < \theta_2 < \ldots < \theta_{2n+1} < 2\pi,$$

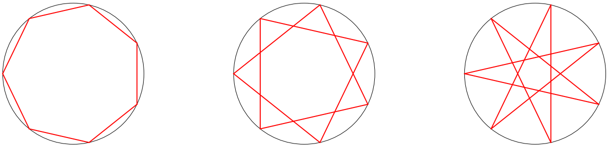
while the distances $t_m = t$ *are constant and* t *is not a zero of any* $U_1, \ldots, U_n$.



Admissible stars

Polygons which satisfy the previous conditions can be constructed by choosing $2n+1$ equally spaced points on the unit circle and joining every i-th and $(i+\ell)$-th points, as long as $2n+1$ and ℓ are relatively prime.

In particular, if $2n+1$ is prime, all $\ell = 1, \ldots, n$ are admissible.



Analytic Inverse

For the case $t_m = t$ and equally spaced angles $\theta_m = \frac{2\pi m}{2n+1}$, we even have an analytic formula for the inverse of the system matrix A in the form

$$A^{-1} = \operatorname{diag}(\beta_0, \beta_1, \beta_1, \ldots, \beta_n, \beta_n) C^\top,$$

where the β_k are row factors which depend only on t,

$$\beta_k = \begin{cases} \frac{1}{2n+1}\alpha_k^{-1}, & k = 0, \\ \frac{2}{2n+1}\alpha_k^{-1}, & k \geq 1. \end{cases}$$

Quasi-optimal solution algorithm

Recall that

$$C = \begin{pmatrix} 1 & \cos(\theta_1) & \sin(\theta_1) & \dots & \cos(n\theta_1) & \sin(n\theta_1) \\ 1 & \cos(\theta_2) & \sin(\theta_2) & \dots & \cos(n\theta_2) & \sin(n\theta_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(\theta_{2n}) & \sin(\theta_{2n}) & \dots & \cos(n\theta_{2n}) & \sin(n\theta_{2n}) \\ 1 & \cos(\theta_{2n+1}) & \sin(\theta_{2n+1}) & \dots & \cos(n\theta_{2n+1}) & \sin(n\theta_{2n+1}) \end{pmatrix}.$$

The action of the $C^\top$ is a discrete Fourier transform. With a Fast Fourier Transform (FFT), we can solve the system in slightly worse than linear ($\mathcal{O}(n)$) time.

Condition number and noisy data

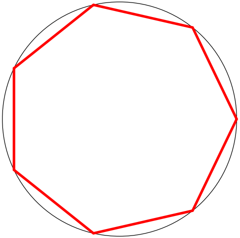
We can uniformly bound the condition number of A by

$$\kappa_2(A) \leq 2\sqrt{2}.$$

Hence errors in the input data are not amplified.
This is confirmed by numerical experiments.

A regular convex polygonal scheme

In the following, we consider the case where the chords form a regular convex $(2n+1)$-sided polygon inscribed in the unit circle.



This means, for $m = 1, \dots, 2n+1$,

$$I_m = I(\theta_m, t_m), \quad \theta_m = \frac{2\pi m}{2n+1}, \quad t_m = t = \cos\frac{\pi}{2n+1}.$$

Error estimate

Let u be harmonic in the unit disk D. Consider the interpolation problem:
Find $p^{(n)} \in \mathcal{H}_n$ (harmonic polynomials, deg. $\leq n$) such that

$$\int_I p^{(n)}\,dx = \int_I u\,dx \qquad \forall I \in \mathcal{I}.$$

Let the Fourier series of its boundary data $f = u|_{\partial D}$ be given by

$$f(\theta) = f_0 + \sum_{k=1}^{\infty}(f_k \cos(k\theta) + f_{-k}\sin(k\theta)).$$

Regularity assumption

We assume that the Fourier series converges uniformly to f. (e.g.: f is Hölder continuous.)

In this case, the series of functions of (r, θ)

$$f_0 + \sum_{k=1}^{\infty} (f_k r^k \cos(k\theta) + f_{-k} r^k \sin(k\theta))$$

converges uniformly on the unit disk, and its limit is a harmonic function, namely, u.

Error estimate in the coefficients

Theorem

Assume that $f = u|_{\partial D}$ has a uniformly convergent Fourier series and its Fourier coefficients $(f_k)_{k\in\mathbb{Z}}$ decay like $|f_k| \leq M|k|^{-s}$ with $M > 0$, $s > 1$. Let $p^{(n)} \in \mathcal{H}_n$ be obtained by our method. Then we have

$$|f_k - p_k^{(n)}| \leq MC_s n^{-s} \qquad \forall |k| \leq n,$$

where C_s is a constant which depends only on s.

Error estimate on the unit circle

Theorem

Assume that $f = u|_{\partial D}$ has a uniformly convergent Fourier series and its Fourier coefficients $(f_k)_{k\in\mathbb{Z}}$ decay like $|f_k| \le M|k|^{-s}$ with $M > 0$, $s > 1$. Let $p^{(n)} \in \mathcal{H}_n$ be obtained by our method. Then

$$\|f - p^{(n)}\|_{L^2(\partial D)} \le MCn^{-(s-1/2)}$$

with a constant C which depends only on s.

The same order of convergence as Fourier series.

Error within the unit disk

We can also derive

$$\|u - p^{(n)}\|_{L^2(D)} = \mathcal{O}(n^{-(s-1/2)}).$$

In practice, it seems that the rate is $\mathcal{O}(n^{-s})$. How to prove this is an open question.

From the maximum principle, we get that inside the disk

Theorem

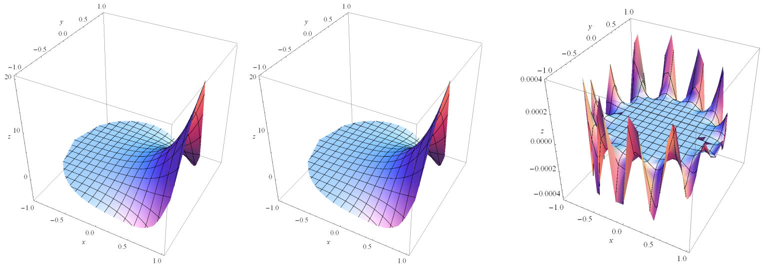
$$\|u - p^{(n)}\|_\infty = \mathcal{O}(n^{-(s-1)}).$$

Example 1

We approximate the harmonic function

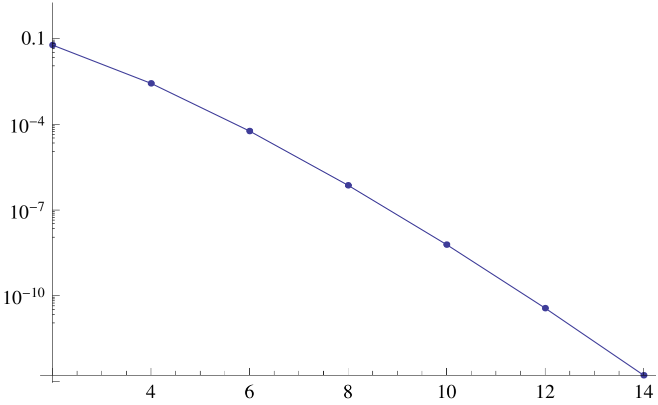
$$u(x, y) = \exp(x)\cos(y)$$

by a harmonic polynomial $p \in \mathcal{H}_n$ given the Radon projections along the edges of a regular convex $(2n+1)$-sided polygon.



$n = 12$: function u, interpolant p, error $u - p$

Example 1: Convergence



x-axis: degree of interpolating polynomial. y-axis: rel. L_2 error

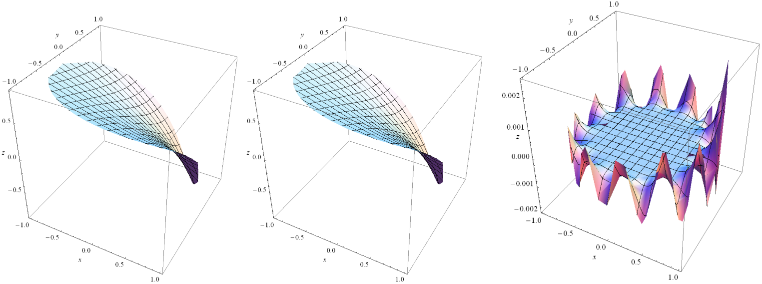
$\longrightarrow$ exponential convergence!

Example 2

We approximate the harmonic function

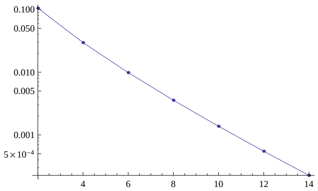
$$u(x, y) = \log \sqrt{(x-1)^2 + (y-1)^2}$$

by a harmonic polynomial $p \in \mathcal{H}_n$ given the Radon projections along the edges of a regular $(2n+1)$-sided polygon.

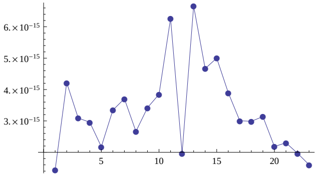


$n = 12$: function u, interpolant p, error $u - p$

Example 2: Convergence



x-axis: degree of interpolating polynomial. y-axis: rel. L_2 error

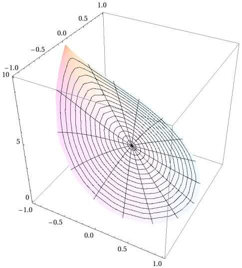


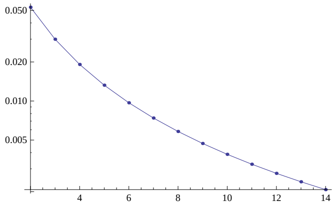
relative L_2-error for 23 regular stars for $n = 23$ (47 chords).

$\longrightarrow$ exponential convergence!

Example 3

We approximate the harmonic function with C^0 boundary data $g(\theta) = \theta^2$, $-\pi \leq \theta < \pi$, by $p \in \mathcal{H}_n$ given the Radon projections along the edges of a regular convex polygon.





function u; errors (x-axis: degree of interpolating polynomial. y-axis: rel. L_2 error)

Example 4

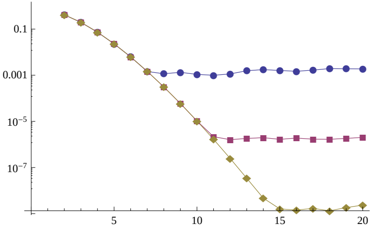
We approximate the harmonic function

$$u(x, y) = \exp(2y)\cos(2x)$$

by a harmonic polynomial given the Radon by a harmonic polynomial $p \in \mathcal{H}_n$ given the Radon projections but with artificially added measurement noise, along the edges of a regular $(2n+1)$-sided polygon.
For this, we add to the given values of the Radon projections random numbers from a normal distribution with zero mean and standard deviation ϵ.

Example 4

We see that the input function is reconstructed to the accuracy limit given by the noise level. No amplification of the noise or instabilities are observed.



Errors with noisy data. Displayed are three experiments with noise levels of 10^{-3}, 10^{-6}, 10^{-9}. x-axis: degree of interpolating polynomial. y-axis: relative L_2-error

Further work

- generalization to varying angles θ and distances t
- inhomogeneous right-hand side, $\Delta u = f$
- error estimates for more general cases
- derivation of cubature formulae for harmonic function given Radon projection type of data

Papers

- Harmonic Interpolation Based on Radon Projections Along the Sides of Regular Polygons, *I. Georgieva, C. Hofreither, C. Koutschan, V. Pillwein, T. Thanatipanonda*
  `https://www.dk-compmath.jku.at/publications/dk-reports/2011-10-20/view`
- Tomographic Reconstruction of Harmonic Functions, *I. Georgieva, C. Hofreither*
  `https://www.dk-compmath.jku.at/publications/dk-reports/2012-04-19/view`
- Interpolation of Harmonic Functions Based on Radon Projections, *I. Georgieva, C. Hofreither* - under preparation...

Grant for Young Researchers DMU 03/17

"Algorithms for Approximate Reconstruction of Functions Based on Radon Projections and PDEs"
funded by Bulgarian National Science Fund.

- Kick-off meeting: Institute of Mathematics and Informatics, Sofia, June, 22nd-24th.
  Participants: Clemens Hofreither, Christoph Koutschan, Veronika Pillwein, Alexander Alexandrov, Irina Georgieva, Rumen Uluchev.
- Clemens Hofreither- 1 month research visit at the Institute of Mathematics and Informatics, Sofia.
- Workshop "Numeric - Symbolic come together", Sozopol, August, 31st - September, 5th.

Thank you for your attention!