Inequalities and Symbolic Computation

Veronika Pillwein, RISC Linz

Interdisciplinary workshop on Approximation Theory, Numerical Analysis and Symbolic Computation, Sozopol

Some Packages of the Algorithmic Combinatorics Group

- HolonomicFunctions [Koutschan] discovering and proving difference and/or differential relations, closure properties
- Guess [Kauers] guessing multivariate recurrence equations, efficiently guessing minimal order univariate recurrence, differential, or algebraic equations
- SumCracker [Kauers] symbolic treatment of “admissible” sequences, *inequality proving (univariate)*

All packages availabe for download at

http://www.risc.uni-linz.ac.at/research/combinat/software/

Some Packages of the Algorithmic Combinatorics Group

- Gosper, Zeilberger [Paule, Schorn] (hypergeometric (creative) telescoping)
- MultiSum [Wegschaider, Riese] (discovering and proving hypergeometric multi-sum identities)
- Sigma [Schneider] (discovering and proving multi-sum identities including multi-sums in terms of indefinite nested sums and products)

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- *Cylindrical Algebraic Decomposition*
- *Symbolic Local Fourier Analysis*
  (VP+Stefan Takacs)
- *CAD for Special Functions Inequalities*
  (Stefan Gerhold and Manuel Kauers)
- *Schöberl's Inequality*

Cylindrical Algebraic Decomposition

Polynomial Formulas

- Everything that can be formed from variables, rational (or real algebraic) numbers, $+, -, \cdot, /, =, \neq, \leq, <, \geq, >, \neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$, $\mathrm{True}, \mathrm{False}$ according to the usual syntactic rules.

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- Examples:
  - $x^2 - (x-1)(y-1) > 0 \ \wedge \ x \geq y^2 \geq 1 \ \Rightarrow \ y^2 - 3x < 0$
  - $xz^2 - \sqrt{2}y + z^3 > 1 \ \wedge \ x > y > z > 1 \ \Rightarrow \ xyz > \frac{1}{2}$

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  - $xz^2 - \sqrt{2}y + z^3 > 1 \ \wedge \ x > y > z > 1 \ \Rightarrow \ xyz > \frac{1}{2}$
- Counterexamples:
  - $x\cos(x) - y\sin(y) > \frac{\pi}{4} \ \Rightarrow \cos(x)\sin(x) < \sin^2(x)$
  - $x\exp(y) - y^2 < xy \ \vee \ \log(y) < x \ \Rightarrow \ x^2 + y^2 < 4$

Semialgebraic Sets

- A formula Φ in n variables gives rise to a geometric object $S \subseteq \mathbb{R}^n$, its *truth region*:

$$S = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \Phi(x_1, \ldots, x_n) \equiv \mathrm{True}\}$$

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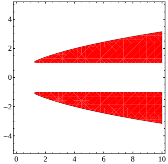

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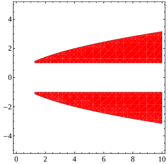

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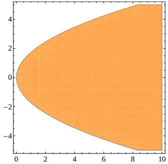

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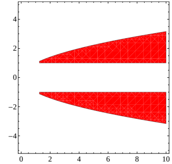

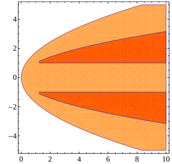

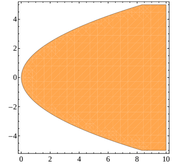

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- determine the semialgebraic set of all points $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ such that *for all* numbers $x_n \in \mathbb{R}$ a given formula is true at $(x_1 \ldots, x_{n-1}, x_n)$

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In particular, given any polynomial formula Φ *involving quantifiers* $(\forall, \exists)$ CAD can compute a polynomial formula Ψ *without quantifiers* that is equivalent (over $\mathbb{R}$) to Φ.

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Note: The execution of CAD may be *computationally expensive!*

Symbolic Local Fourier Analysis

Stefan Takacs

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- *symbolic* local Fourier analysis: *exact bounds*
- Given: Iterative procedure of the form

$$x^{(k+1)} = Ax^{(k)}$$

  the *convergence rate* is related to the *matrix norm* of A which can be estimated by the spectral radius, i.e., the *largest eigenvalue in absolute value*

The Problem

Given: matrix $A(q, c_1, c_2, \eta) \in \mathbb{R}^{8\times 8}$ with $0 < q < 1$ and

$$(c_1, c_2, \eta) \in \Omega = \{(c_1, c_2, \eta) \mid 0 \le c_1 \le c_2 < 1 \ \wedge \ \eta > 0\},$$

where $c_i = \cos(\theta_i)$ for some frequencies θ_i and $\eta = h^4/\alpha$ with mesh-size h and regularization parameter α.

Find: bound $B(q)$ for the maximal eigenvalue $\lambda_{max}(q, c_1, c_2, \eta)$ of A over $\Omega(c_1, c_2, \eta)$.

The Matrix A

$$
\begin{pmatrix}
A_{1,1} & 0 & A_{1,3} & A_{1,4} & A_{1,5} & A_{1,6} & A_{1,7} & A_{1,8} \\
0 & A_{2,2} & A_{2,3} & A_{2,4} & A_{2,5} & A_{2,6} & A_{2,7} & A_{2,8} \\
A_{3,1} & A_{3,2} & A_{3,3} & 0 & A_{3,5} & A_{3,6} & A_{3,7} & A_{3,8} \\
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A_{7,1} & A_{7,2} & A_{7,3} & A_{7,4} & A_{7,5} & A_{7,6} & A_{7,7} & 0 \\
A_{8,1} & A_{8,2} & A_{8,3} & A_{8,4} & A_{8,5} & A_{8,6} & 0 & A_{8,8}
\end{pmatrix}
$$

Common denominator of the matrix entries:

$$
\begin{aligned}
D =& 256 \big(16c_2^4c_1^4\eta + 16c_2^2c_1^4\eta + 4c_1^4\eta + 16c_2^4c_1^2\eta + 16c_2^2c_1^2\eta + 4c_1^2\eta \\
&+ 4c_2^4\eta + 4c_2^2\eta + 144c_2^4c_1^4 - 72c_2^2c_1^4 + 9c_1^4 - 72c_2^4c_1^2 - 126c_2^2c_1^2 \\
&+36c_1^2 + 9c_2^4 + 36c_2^2 + \eta + 36\big)
\end{aligned}
$$

The Matrix A

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The Numerator of $A_{1,1}$

$$
\begin{aligned}
&432q^2c_2^6c_1^6 + 3q^2\eta c_2^6c_1^6 + \eta c_2^6c_1^6 + 144c_2^6c_1^6 + 1008q^2c_2^5c_1^6 + 16q^2\eta c_2^5c_1^6 + 8\eta c_2^5c_1^6 + 720c_2^5c_1^6 + \\
&972q^2c_2^4c_1^6 + 30q^2\eta c_2^4c_1^6 + 26\eta c_2^4c_1^6 + 1476c_2^4c_1^6 + 1008q^2c_2^3c_1^6 + 28q^2\eta c_2^3c_1^6 + 44\eta c_2^3c_1^6 + 1584c_2^3c_1^6 + \\
&108q^2c_1^6 + 1080q^2c_2^2c_1^6 + 27q^2\eta c_2^2c_1^6 + 41\eta c_2^2c_1^6 + 936c_2^2c_1^6 + 12q^2\eta c_1^6 + 4\eta c_1^6 + 576q^2c_2c_1^6 + \\
&28q^2\eta c_2c_1^6 + 20\eta c_2c_1^6 + 288c_2c_1^6 + 36c_1^6 + 1008q^2c_2^6c_1^5 + 16q^2\eta c_2^6c_1^5 + 8\eta c_2^6c_1^5 + 720c_2^6c_1^5 - \\
&360q^2c_2^5c_1^5 + 80q^2\eta c_2^5c_1^5 - 64\eta c_2^5c_1^5 - 1656c_2^5c_1^5 - 3600q^2c_2^4c_1^5 + 128q^2\eta c_2^4c_1^5 - 304\eta c_2^4c_1^5 - \\
&7056c_2^4c_1^5 - 2736q^2c_2^3c_1^5 + 80q^2\eta c_2^3c_1^5 - 352\eta c_2^3c_1^5 - 5328c_2^3c_1^5 - 720q^2c_1^5 - 1872q^2c_2^2c_1^5 + \\
&80q^2\eta c_2^2c_1^5 - 184\eta c_2^2c_1^5 + 720c_2^2c_1^5 + 64q^2\eta c_1^5 - 96\eta c_1^5 - 2088q^2c_2c_1^5 + 128q^2\eta c_2c_1^5 - 160\eta c_2c_1^5 + \\
&1800c_2c_1^5 + 432c_1^5 + 972q^2c_2^6c_1^4 + 30q^2\eta c_2^6c_1^4 + 26\eta c_2^6c_1^4 + 1476c_2^6c_1^4 - 3600q^2c_2^5c_1^4 + 128q^2\eta c_2^5c_1^4 - \\
&304\eta c_2^5c_1^4 - 7056c_2^5c_1^4 - 5616q^2c_2^4c_1^4 + 108q^2\eta c_2^4c_1^4 + 2724\eta c_2^4c_1^4 + 21168c_2^4c_1^4 + 1584q^2c_2^3c_1^4 - \\
&136q^2\eta c_2^3c_1^4 - 1672\eta c_2^3c_1^4 - 3600c_2^3c_1^4 + 648q^2c_1^4 + 1404q^2c_2^2c_1^4 - 114q^2\eta c_2^2c_1^4 + 3114\eta c_2^2c_1^4 - \\
&15660c_2^2c_1^4 + 120q^2\eta c_1^4 + 616\eta c_1^4 - 576q^2c_2c_1^4 + 152q^2\eta c_2c_1^4 - 760\eta c_2c_1^4 - 2304c_2c_1^4 + 792c_1^4 + \\
&1008q^2c_2^6c_1^3 + 28q^2\eta c_2^6c_1^3 + 44\eta c_2^6c_1^3 + 1584c_2^6c_1^3 - 2736q^2c_2^5c_1^3 + 80q^2\eta c_2^5c_1^3 - 352\eta c_2^5c_1^3 - 5328c_2^5c_1^3 + \\
&1584q^2c_2^4c_1^3 - 136q^2\eta c_2^4c_1^3 - 1672\eta c_2^4c_1^3 - 3600c_2^4c_1^3 + 12384q^2c_2^3c_1^3 - 640q^2\eta c_2^3c_1^3 - 1936\eta c_2^3c_1^3 + \\
&17568c_2^3c_1^3 + 1872q^2c_1^3 + 5904q^2c_2^2c_1^3 - 580q^2\eta c_2^2c_1^3 - 1012\eta c_2^2c_1^3 + 14544c_2^2c_1^3 + 112q^2\eta c_1^3 - \\
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&5184q^2c_2^2c_1^2 - 525q^2\eta c_2^2c_1^2 + 3729\eta c_2^2c_1^2 - 15552c_2^2c_1^2 + 108q^2\eta c_1^2 + 676\eta c_1^2 - 6624q^2c_2c_1^2 - \\
&4q^2\eta c_2c_1^2 - 460\eta c_2c_1^2 + 2880c_2c_1^2 + 6948c_1^2 + 576q^2c_2^6c_1 + 28q^2\eta c_2^6c_1 + 20\eta c_2^6c_1 + 288c_2^6c_1 - \\
&2088q^2c_2^5c_1 + 128q^2\eta c_2^5c_1 - 160\eta c_2^5c_1 + 1800c_2^5c_1 - 576q^2c_2^4c_1 + 152q^2\eta c_2^4c_1 - 760\eta c_2^4c_1 - \\
&2304c_2^4c_1 + 720q^2c_2^3c_1 - 16q^2\eta c_2^3c_1 - 880\eta c_2^3c_1 - 1872c_2^3c_1 + 1440q^2c_1 - 6624q^2c_2^2c_1 - 4q^2\eta c_2^2c_1 - \\
&460\eta c_2^2c_1 + 2880c_2^2c_1 + 112q^2\eta c_1 - 240\eta c_1 - 3816q^2c_2c_1 + 176q^2\eta c_2c_1 - 400\eta c_2c_1 - 5112c_2c_1 - \\
&6048c_1 + 108q^2c_2^6 + 12q^2\eta c_2^6 + 4\eta c_2^6 + 36c_2^6 - 720q^2c_2^5 + 64q^2\eta c_2^5 - 96\eta c_2^5 + 432c_2^5 + 648q^2c_2^4 + \\
&120q^2\eta c_2^4 + 616\eta c_2^4 + 792c_2^4 + 1872q^2c_2^3 + 112q^2\eta c_2^3 - 528\eta c_2^3 - 2160c_2^3 + 1728q^2 + 108q^2c_2^2 + \\
&108q^2\eta c_2^2 + 676\eta c_2^2 + 6948c_2^2 + 48q^2\eta + 144\eta + 1440q^2c_2 + 112q^2\eta c_2 - 240\eta c_2 - 6048c_2 + 5184
\end{aligned}
$$

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2. How to find the bound?
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The Eigenvalues

Using interpolation on the characteristic polynomial of the matrix we find that the eigenvalues are

$$\lambda_1 = 0, \quad \lambda_2 = q^4, \quad {}_3\lambda_4 = \frac{1}{D}\left(e(q_2) \pm \sqrt{d(q_2)}\right),$$

each of multiplicity 2 with $q_2 = q^2$ and e, d polynomials in c_1, c_2, η and q_2.

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The Polynomial $e(q_2)$

$$\eta c_2^6 c_1^6 + 144 c_2^6 c_1^6 + 8\eta c_2^5 c_1^6 + 720 c_2^5 c_1^6 + 26\eta c_2^4 c_1^6 + 1476 c_2^4 c_1^6 + 44\eta c_2^3 c_1^6 + 1584 c_2^3 c_1^6 + 41\eta c_2^2 c_1^6 +$$
$$936 c_2^2 c_1^6 + 9\eta c_2^6 q_2^2 c_1^6 + 1296 c_2^6 q_2^2 c_1^6 - 24\eta c_2^5 q_2^2 c_1^6 - 2160 c_2^5 q_2^2 c_1^6 + 138\eta c_2^4 q_2^2 c_1^6 + 6372 c_2^4 q_2^2 c_1^6 -$$
$$132\eta c_2^3 q_2^2 c_1^6 - 4752 c_2^3 q_2^2 c_1^6 + 177\eta c_2^2 q_2^2 c_1^6 + 4968 c_2^2 q_2^2 c_1^6 + 36\eta q_2^2 c_1^6 - 60\eta c_2 q_2^2 c_1^6 - 864 c_2 q_2^2 c_1^6 +$$
$$324 q_2^2 c_1^6 + 4\eta c_1^6 + 20\eta c_2 c_1^6 + 288 c_2 c_1^6 + 6\eta c_2^6 q_2 c_1^6 + 864 c_2^6 q_2 c_1^6 + 16\eta c_2^5 q_2 c_1^6 + 1440 c_2^5 q_2 c_1^6 +$$
$$60\eta c_2^4 q_2 c_1^6 + 1944 c_2^4 q_2 c_1^6 + 88\eta c_2^3 q_2 c_1^6 + 3168 c_2^3 q_2 c_1^6 + 54\eta c_2^2 q_2 c_1^6 + 2160 c_2^2 q_2 c_1^6 + 24\eta q_2 c_1^6 +$$
$$40\eta c_2 q_2 c_1^6 + 576 c_2 q_2 c_1^6 + 216 q_2 c_1^6 + 36 c_1^6 + 8\eta c_2^6 c_1^5 + 720 c_2^6 c_1^5 - 64\eta c_2^5 c_1^5 - 1656 c_2^5 c_1^5 -$$
$$304\eta c_2^4 c_1^5 - 7056 c_2^4 c_1^5 - 352\eta c_2^3 c_1^5 - 5328 c_2^3 c_1^5 - 184\eta c_2^2 c_1^5 + 720 c_2^2 c_1^5 - 24\eta c_2^6 q_2^2 c_1^5 - 2160 c_2^6 q_2^2 c_1^5 +$$
$$32\eta c_2^5 q_2^2 c_1^5 + 2664 c_2^5 q_2^2 c_1^5 + 272\eta c_2^4 q_2^2 c_1^5 + 10224 c_2^4 q_2^2 c_1^5 + 416\eta c_2^3 q_2^2 c_1^5 + 3312 c_2^3 q_2^2 c_1^5 +$$
$$296\eta c_2^2 q_2^2 c_1^5 - 2736 c_2^2 q_2^2 c_1^5 + 32\eta q_2^2 c_1^5 + 128\eta c_2 q_2^2 c_1^5 - 792 c_2 q_2^2 c_1^5 - 144 q_2^2 c_1^5 - 96\eta c_1^5 - 160\eta c_2 c_1^5 +$$
$$1800 c_2 c_1^5 + 16\eta c_2^6 q_2 c_1^5 + 1440 c_2^6 q_2 c_1^5 + 32\eta c_2^5 q_2 c_1^5 - 1008 c_2^5 q_2 c_1^5 + 32\eta c_2^4 q_2 c_1^5 - 3168 c_2^4 q_2 c_1^5 -$$
$$64\eta c_2^3 q_2 c_1^5 + 2016 c_2^3 q_2 c_1^5 - 112\eta c_2^2 q_2 c_1^5 + 2016 c_2^2 q_2 c_1^5 + 64\eta q_2 c_1^5 + 32\eta c_2 q_2 c_1^5 - 1008 c_2 q_2 c_1^5 -$$
$$288 q_2 c_1^5 + 432 c_1^5 + 26\eta c_2^6 c_1^4 + 1476 c_2^6 c_1^4 - 304\eta c_2^5 c_1^4 - 7056 c_2^5 c_1^4 + 2724\eta c_2^4 c_1^4 + 21168 c_2^4 c_1^4 -$$
$$1672\eta c_2^3 c_1^4 - 3600 c_2^3 c_1^4 + 3114\eta c_2^2 c_1^4 - 15660 c_2^2 c_1^4 + 138\eta c_2^6 q_2^2 c_1^4 + 6372 c_2^6 q_2^2 c_1^4 + 272\eta c_2^5 q_2^2 c_1^4 +$$
$$10224 c_2^5 q_2^2 c_1^4 + 4292\eta c_2^4 q_2^2 c_1^4 + 15408 c_2^4 q_2^2 c_1^4 + 1496\eta c_2^3 q_2^2 c_1^4 + 8496 c_2^3 q_2^2 c_1^4 + 5018\eta c_2^2 q_2^2 c_1^4 + \ldots$$

The Polynomial $e(q_2)$

$$\cdots - 25740c_2^2q_2^2c_1^4 + 1064\eta q_2^2c_1^4 + 680\eta c_2q_2^2c_1^4 - 576c_2q_2^2c_1^4 + 1368q_2^2c_1^4 + 616\eta c_1^4 - 760\eta c_2c_1^4 -$$
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$$176\eta c_2^3q_2c_1^4 - 4896c_2^3q_2c_1^4 - 228\eta c_2^2q_2c_1^4 + 2808c_2^2q_2c_1^4 + 240\eta q_2c_1^4 + 80\eta c_2q_2c_1^4 + 2880c_2q_2c_1^4 +$$
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$$1936\eta c_2^3c_1^3 + 17568c_2^3c_1^3 - 1012\eta c_2^2c_1^3 + 14544c_2^2c_1^3 - 132\eta c_2^6q_2^2c_1^3 - 4752c_2^6q_2^2c_1^3 + 416\eta c_2^5q_2^2c_1^3 +$$
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$$14832c_2^2q_2^2c_1^3 + 176\eta q_2^2c_1^3 + 944\eta c_2q_2^2c_1^3 - 144c_2q_2^2c_1^3 + 720q_2^2c_1^3 - 528\eta c_1^3 - 880\eta c_2c_1^3 - 1872c_2c_1^3 +$$
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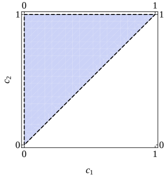
The Polynomial $e(q_2)$

$$\cdots + 740\eta c_2 q_2^2 c_1^2 + 4608 c_2 q_2^2 c_1^2 + 11844 q_2^2 c_1^2 + 676\eta c_1^2 - 460\eta c_2 c_1^2 + 2880 c_2 c_1^2 + 54\eta c_2^6 q_2 c_1^2 + 2160 c_2^6 q_2 c_1^2 - 112\eta c_2^5 q_2 c_1^2 + 2016 c_2^5 q_2 c_1^2 - 228\eta c_2^4 q_2 c_1^2 + 2808 c_2^4 q_2 c_1^2 - 616\eta c_2^3 q_2 c_1^2 + 288 c_2^3 q_2 c_1^2 - 1050\eta c_2^2 q_2 c_1^2 - 10368 c_2^2 q_2 c_1^2 + 216\eta q_2 c_1^2 - 280\eta c_2 q_2 c_1^2 - 7488 c_2 q_2 c_1^2 + 216 q_2 c_1^2 + 6948 c_1^2 + 20\eta c_2^6 c_1 + 288 c_2^6 c_1 - 160\eta c_2^5 c_1 + 1800 c_2^5 c_1 - 760\eta c_2^4 c_1 - 2304 c_2^4 c_1 - 880\eta c_2^3 c_1 - 1872 c_2^3 c_1 - 460\eta c_2^2 c_1 + 2880 c_2^2 c_1 - 60\eta c_2^6 q_2^2 c_1 - 864 c_2^6 q_2^2 c_1 + 128\eta c_2^5 q_2^2 c_1 - 792 c_2^5 q_2^2 c_1 + 680\eta c_2^4 q_2^2 c_1 - 576 c_2^4 q_2^2 c_1 + 944\eta c_2^3 q_2^2 c_1 - 144 c_2^3 q_2^2 c_1 + 740\eta c_2^2 q_2^2 c_1 + 4608 c_2^2 q_2^2 c_1 + 80\eta q_2^2 c_1 + 368\eta c_2 q_2^2 c_1 + 6120 c_2 q_2^2 c_1 + 2016 q_2^2 c_1 - 240\eta c_1 - 400\eta c_2 c_1 - 5112 c_2 c_1 + 40\eta c_2^6 q_2 c_1 + 576 c_2^6 q_2 c_1 + 32\eta c_2^5 q_2 c_1 - 1008 c_2^5 q_2 c_1 + 80\eta c_2^4 q_2 c_1 + 2880 c_2^4 q_2 c_1 - 64\eta c_2^3 q_2 c_1 + 2016 c_2^3 q_2 c_1 - 280\eta c_2^2 q_2 c_1 - 7488 c_2^2 q_2 c_1 + 160\eta q_2 c_1 + 32\eta c_2 q_2 c_1 - 1008 c_2 q_2 c_1 + 4032 q_2 c_1 - 6048 c_1 + 4\eta c_2^6 + 36 c_2^6 - 96\eta c_2^5 + 432 c_2^5 + 616\eta c_2^4 + 792 c_2^4 - 528\eta c_2^3 - 2160 c_2^3 + 676\eta c_2^2 + 6948 c_2^2 + 36\eta c_2^6 q_2^2 + 324 c_2^6 q_2^2 + 32\eta c_2^5 q_2^2 - 144 c_2^5 q_2^2 + 1064\eta c_2^4 q_2^2 + 1368 c_2^4 q_2^2 + 176\eta c_2^3 q_2^2 + 720 c_2^3 q_2^2 + 1220\eta c_2^2 q_2^2 + 11844 c_2^2 q_2^2 + 272\eta q_2^2 + 80\eta c_2 q_2^2 + 2016 c_2 q_2^2 + 9792 q_2^2 + 144\eta - 240\eta c_2 - 6048 c_2 + 24\eta c_2^6 q_2 + 216 c_2^6 q_2 + 64\eta c_2^5 q_2 - 288 c_2^5 q_2 + 240\eta c_2^4 q_2 + 1296 c_2^4 q_2 + 352\eta c_2^3 q_2 + 1440 c_2^3 q_2 + 216\eta c_2^2 q_2 + 216 c_2^2 q_2 + 96\eta q_2 + 160\eta c_2 q_2 + 4032 c_2 q_2 + 3456 q_2 + 5184$$

The Bound

- Consider extreme cases for (c_1, c_2)
- Consider the limits $\eta \to 0$ and $\eta \to \infty$



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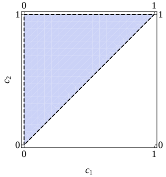

This yields the following *guess*:

$$B(q_2) = \begin{cases} lb(q_2) = \left(\frac{q_2+3}{4}\right)^2, & 0 < q_2 < Q_2, \\ rb(q_2) = q_2(q_2+1), & Q_2 \leq q_2 < 1. \end{cases},$$

with $Q_2 = \frac{1}{15}(4\sqrt{10} - 5)$.

The Proof

Need to show that

$$\lambda_{max} = \frac{1}{D}\left(e(q_2) + \sqrt{d(q_2)}\right) \leq lb(q_2), \qquad 0 < q_2 < Q_2.$$

Or equivalently

$$\sqrt{d(q_2)} \leq D\ lb(q_2) - e(q_2)$$

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Hence we need to prove that

- $R(c_1, c_2, q_2, \eta) = D\ lb(q_2) - e(q_2) \geq 0$
- $S(c_1, c_2, q_2, \eta) = (D\ lb(q_2) - e(q_2))^2 - d(q_2) \geq 0$

Sketching the Proof Steps

- Observation 1: $R(c_1, c_2, q_2, \eta)$ is *linear in η* $(0 < \eta < \infty)$:

$$R(c_1, c_2, q_2, \eta) = r_0(c_1, c_2, q_2) + r_1(c_1, c_2, q_2)\eta$$

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- A sufficient condition for $R(c_1, c_2, q_2, \eta) \geq 0$ is

$$r_0(c_1, c_2, q_2) \geq 0 \quad \text{and} \quad r_1(c_1, c_2, q_2) \geq 0$$

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$$A \Rightarrow (B \wedge C) \quad \Leftrightarrow \quad (A \Rightarrow B) \wedge (A \Rightarrow C)$$

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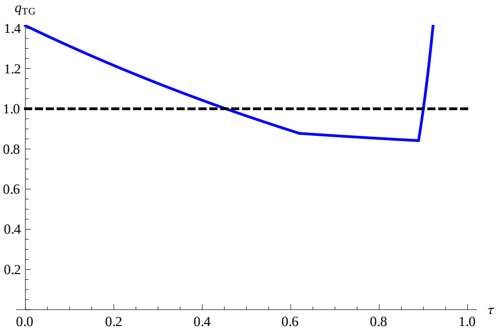
- Proceed analogously for $S(c_1, c_2, q_2, \eta)$ (here *cubic polynomials* appear)

The Bound for the Convergence Rate



Two-grid convergence factor depending on τ for $\nu = \nu_{pre} + \nu_{post} = 2 + 2$ smoothing steps

CAD for Special Functions Inequalities

Manuel Kauers and Stefan Gerhold

Input structure

Given a sequence $f(n)$ satisfying a linear recurrence with polynomial coefficients, i.e.,

given $p_i \in K[x]$ such that

$$p_r(n)f(n+r) + \cdots + p_1(n)f(n+1) + p_0(n)f(n) = 0, \quad n \geq 0,$$

and initial values $f(0), \ldots, f(r-1)$

Question: When can we prove that $f(n) \geq 0$ for all $n \geq 0$?

Gerhold-Kauers method

Given: $p_0, p_1, p_2 \in K[x]$ and initial values $f(0), f(1)$, s.t.,

$$p_2(n)f(n+2) + p_1(n)f(n+1) + p_0(n)f(n) = 0, \quad n \geq 0$$

Proof by induction: Show that

$$f(n) \geq 0 \wedge f(n+1) \geq 0 \Rightarrow -\frac{p_0(n)}{p_2(n)}f(n) - \frac{p_1(n)}{p_2(n)}f(n+1) \geq 0$$

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Generalize: To prove positivity of $f(n)$ it is sufficient to show that

$$\forall y_0, y_1 \in \mathbb{R} \forall x \in \mathbb{R} : y_0 \geq 0 \wedge y_1 \geq 0 \Rightarrow -\frac{p_0(x)}{p_2(x)}y_0 - \frac{p_1(x)}{p_2(x)}y_1 \geq 0$$

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This can be decided by a quantifier elimination algorithm!

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But it might be false even if $f(n) \geq 0$!

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Refined induction step formulas: Extend the induction hypothesis and try to show that

$$\forall\ y_0, y_1, x \in \mathbb{R} : y_0 \geq 0 \wedge y_1 \geq 0 \wedge -\frac{p_0(x)}{p_2(x)}y_0 - \frac{p_1(x)}{p_2(x)}y_1 \geq 0$$
$$\Rightarrow\ \frac{p_0(x)p_1(x+1)}{p_2(x)p_2(x+1)}y_0 + \frac{p_1(x)p_1(x+1) - p_0(x+1)p_2(x)}{p_2(x)p_2(x+1)}y_1 \geq 0$$

Turán's inequality for Legendre polynomials

For all $n \geq 0$ and all $x \in [-1, 1]$:

$$\Delta_n(x) = P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0.$$

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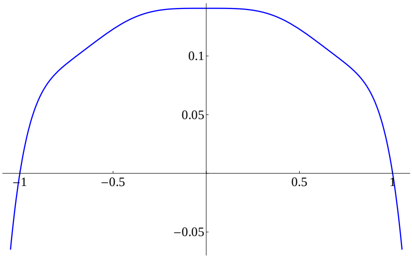

$$\Delta_4(x) = \frac{1 - x^2}{64}(35x^6 - 21x^4 + 9x^2 + 9)$$

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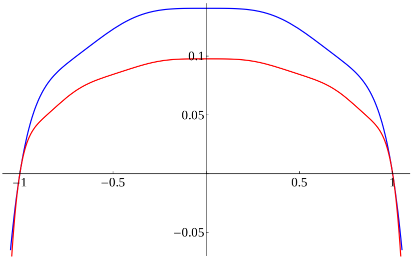

$$\Delta_6(x) = \frac{1 - x^2}{256}(693x^{10} - 1155x^8 + 690x^6 - 150x^4 + 25x^2 + 25)$$

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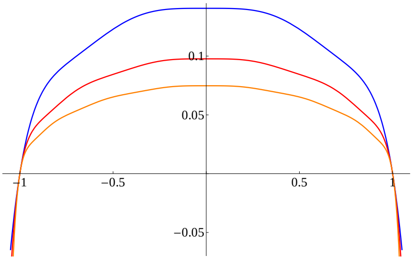

$$\Delta_8(x) = \frac{1 - x^2}{16384}(306735x^{14} - 825825x^{12} + 867867x^{10} - \ldots)$$

Proof using the Gerhold-Kauers method

- *Proof by induction:*

$$-1 \le x \le 1 \wedge n \ge 0 \wedge \Delta_n(x) \ge 0 \Longrightarrow \Delta_{n+1}(x) \ge 0$$

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$$P_{n+2}(x) = \frac{2n+3}{n+2} x P_{n+1}(x) - \frac{(n+1)}{n+2} P_n(x)$$

  and introduce *real* variables $p_{-1} \equiv P_{n-1}(x)$ and $p_0 \equiv P_n(x)$.

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- With this notation (using the recurrence) we have

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- *Consider n as a real variable*

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- The more *general* statement may be false, even if the *discrete* statement is correct
- The method may fail to *terminate*
- If it terminates, then the statement is either *proven*, or a *counterexample* is found

A Theorem of Alexander Alexandrov and Geno Nikolov

Theorem 2

Let f be a polynomial of degree at most n having only real zeros and define

$$L_k(f;x) = \sum_{j=0}^{2k} (-1)^{k-j} \frac{f^{(j)}(x)}{j!} \, \frac{f^{(2k-j)}(x)}{(2k-j)!}, \quad 1 \le k \le n.$$

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For the choice $f(x) = H_n(x)$ (the nth Hermite polynomial) the function $L_k(H_n;\cdot)$ is

- *monotonically decreasing* in $(-\infty, 0]$, and
- *monotonically increasing* in $[0, \infty)$

for all $1 \leq k \leq n$.

The explicit representation

Derivatives of Hermite polynomials satisfy the relations

$$H_n'(x) = 2nH_{n-1}(x) \quad \text{and} \quad D_x^m H_n(x) = \frac{2^m n!}{(n-m)!} H_{n-m}(x).$$

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Plugging this into

$$L_k(f;x) = \sum_{j=0}^{2k} (-1)^{k-j} \frac{f^{(j)}(x)}{j!} \, \frac{f^{(2k-j)}(x)}{(2k-j)!},$$

for $f(x) = H_n(x)$ yields

$$L_k(H_n;x) = \sum_{j=0}^{2k} (-1)^{k-j} \binom{n}{j} \binom{n}{2k-j} 4^k H_{n-j}(x) H_{n-2k+j}(x).$$

The holonomic representation

Using HolonomicFunctions a representation in terms of defining mixed difference-differential relations can be obtained directly from the sum representation:

In[2]:= $$\mathbf{Annihilator}[\sum_{j=0}^{2k}(-1)^{k-j}\binom{n}{j}\binom{n}{2k-j}4^k H_{n-j}(x)H_{n-2k+j}(x), \{\mathbf{S}[k],\mathbf{S}[n],\mathbf{Der}[x]\}]$$

Out[2]= $$\{-(k+1)(2k+1)(n+1)S_k+(-k+n+1)S_n+(n+1)xD_x - 2(n+1)\left(2k-n+2x^2\right), (n+1)D_x^2+2(k-n-1)S_n-6(n+1)xD_x + 4(n+1)\left(n+2x^2\right), S_nD_x+2(n+1)D_x-8(n+1)x, (k-n-2)S_n^2+2(n+2)\left(k+2x^2\right)S_n+4(n+1)(n+2)xD_x + 4(n+1)(n+2)\left(n-4x^2+1\right)\}$$

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- S_n denotes the forward shift in n
- the output is a list of operators annihilating the given function

A relation for the derivative

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In[3]:= $\mathbf{xfree = FindRelation[ann, Eliminate \to \{\boldsymbol{x}\}]}$

Out[3]= $\{(k+1)S_k S_n - 2(k+1)(n+1)S_k - 4(n+1)\}$

In[4]:= $\mathbf{ApplyOreOperator[xfree, \boldsymbol{y}[\boldsymbol{k}, \boldsymbol{n}]]}$

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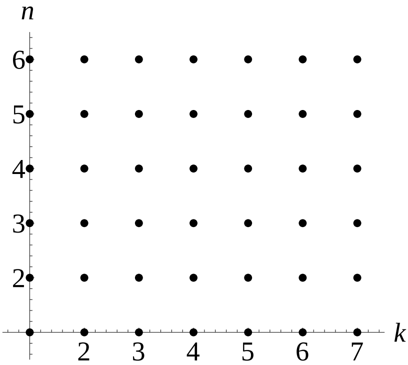
The recurrence



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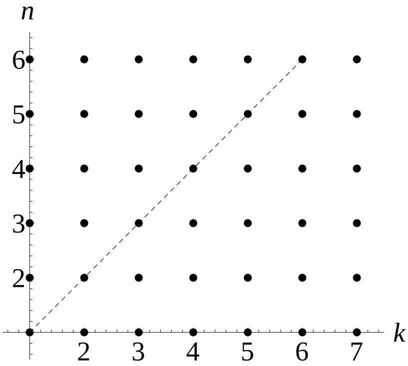

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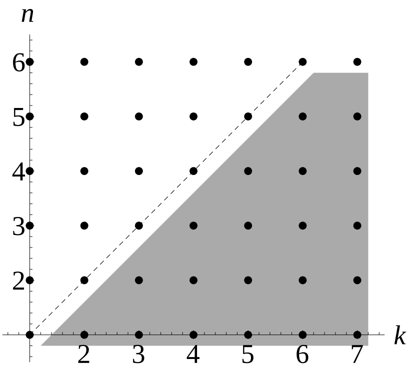

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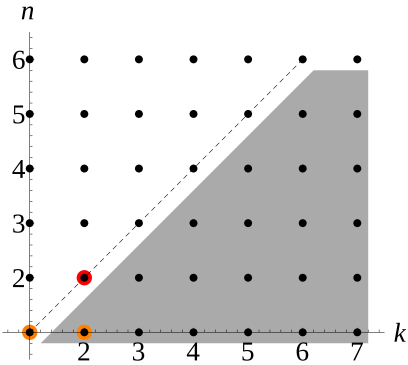

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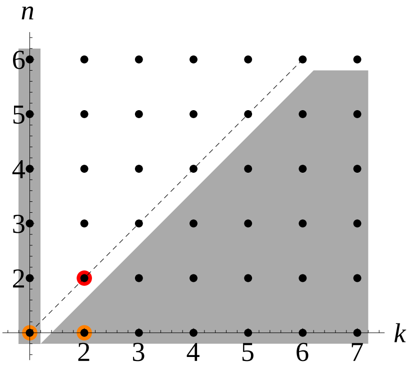

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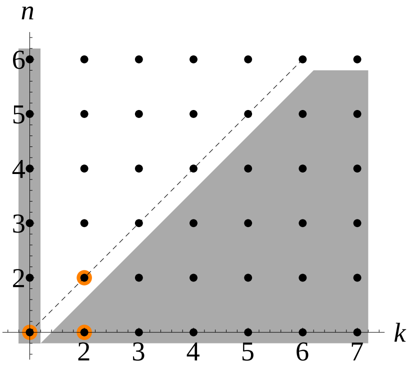

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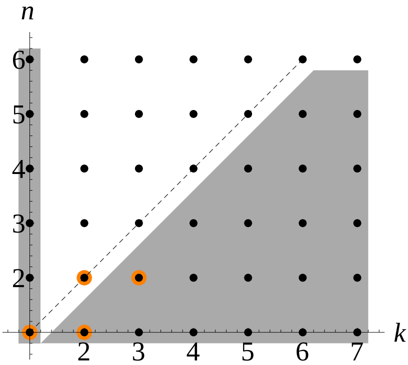

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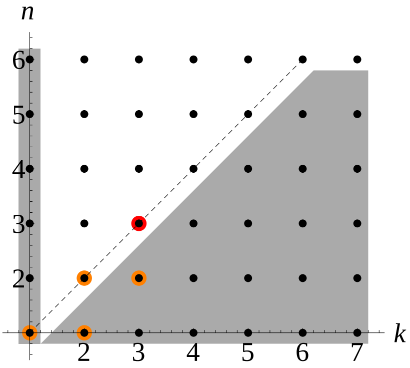

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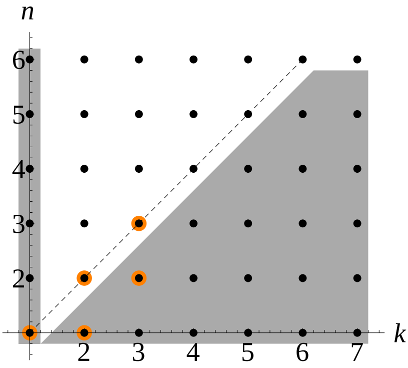

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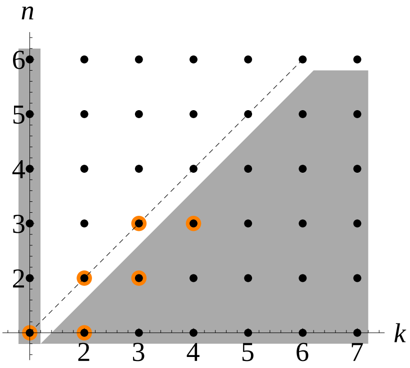

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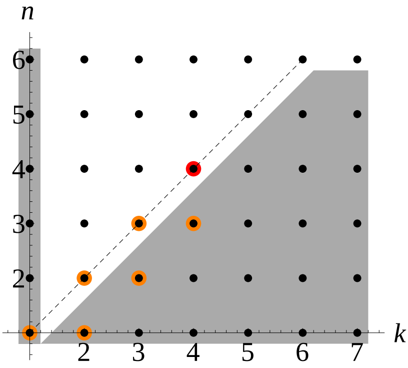

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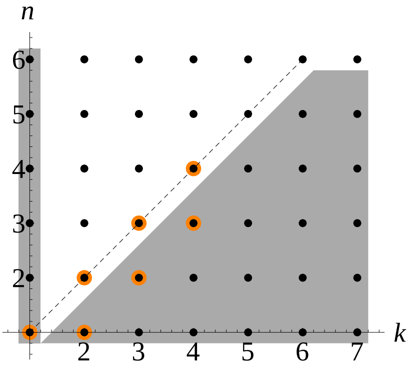

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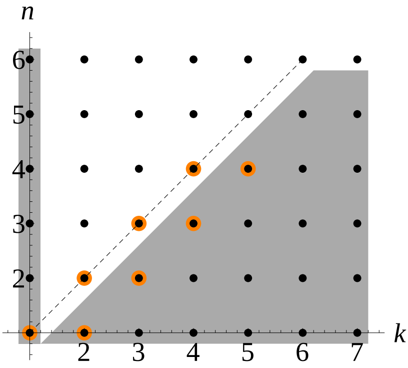

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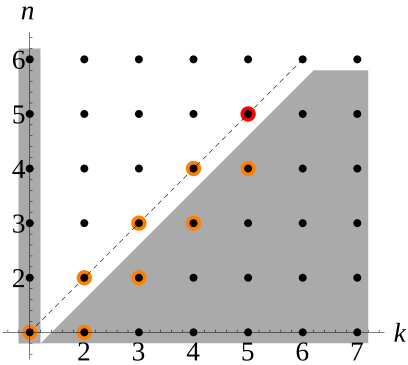

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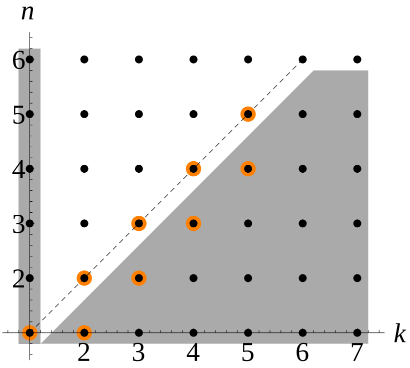

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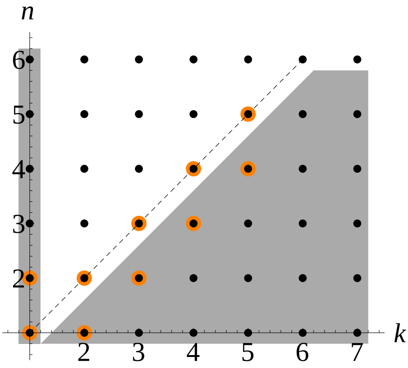

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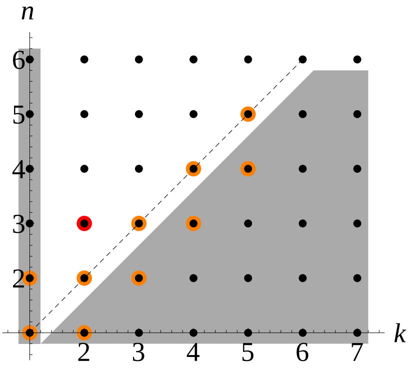

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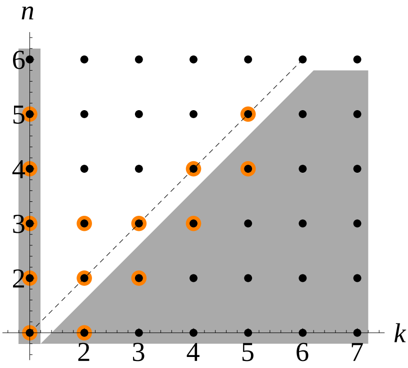

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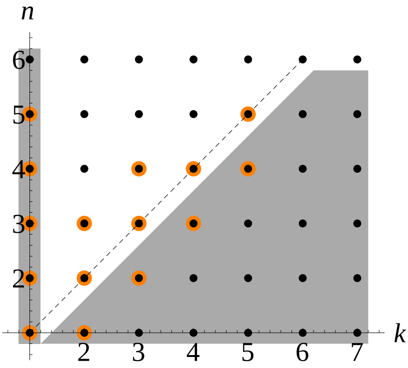

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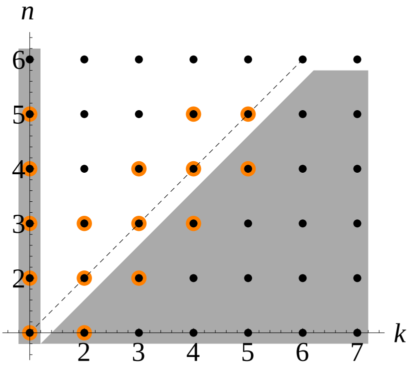

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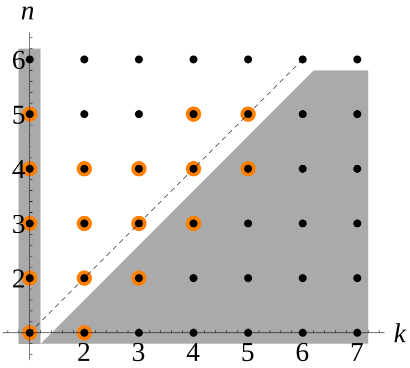

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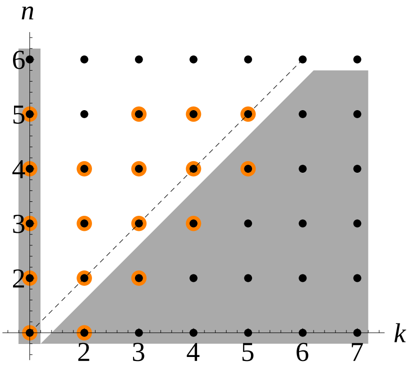

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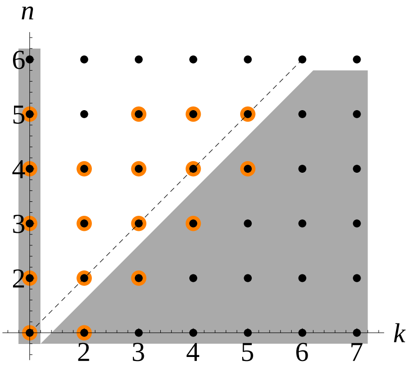

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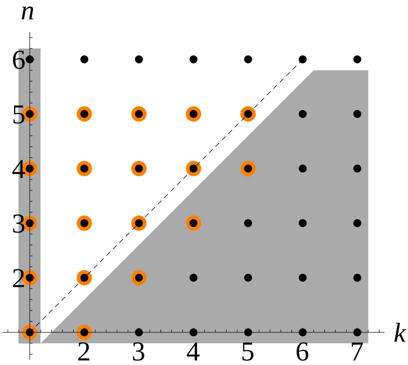

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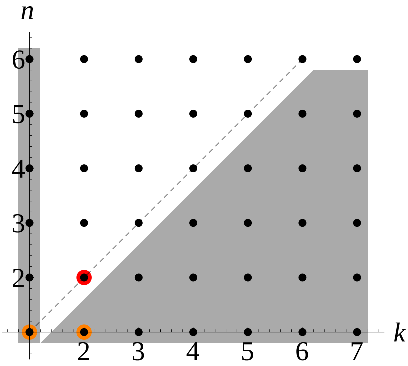

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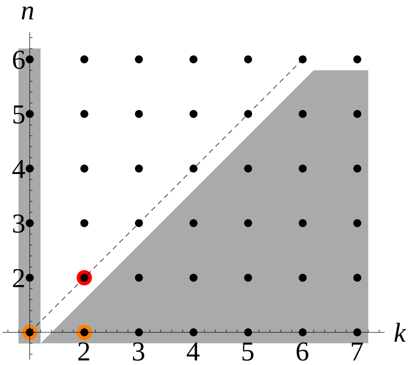

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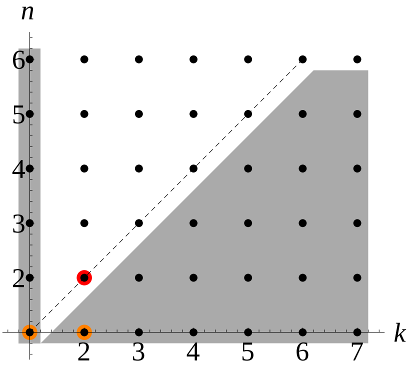

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- *Show* that $y'_{1,n}(x) \geq 0$

In[5]:= $\mathbf{ProveInequality}[\boldsymbol{8(n^2-n)}$
$\boldsymbol{(nH_{n-2}(x)H_{n-1}(x) - (n-2)H_{n-3}(x)H_n(x)) \geq 0},$
$\mathbf{Using} \rightarrow \{\boldsymbol{x \geq 0}\}, \mathbf{Variable} \rightarrow \boldsymbol{n}]$

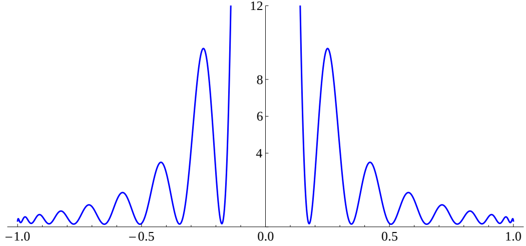
Out[5]= True

Schöberl's Inequality

Schöberl's inequality

If $-1 \le x \le 1,\ n \ge 0$, then

$$S(n,x) = \sum_{j=0}^{n} \tfrac{1}{2}(4j+1)(2n-2j+1)P_{2j}(0)P_{2j}(x) \ge 0.$$



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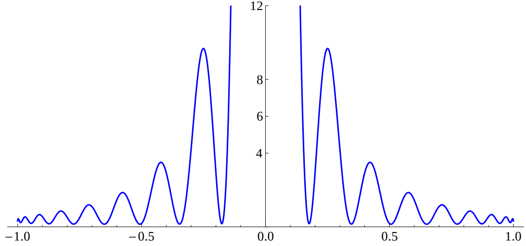
If $-1 \le x \le 1,\ n \ge 0,\ -\frac{1}{2} \le \alpha \le \frac{1}{2}$, then

$$S^{\alpha}(n,x) = \sum_{j=0}^{2n} \frac{c_j^{\alpha}}{x} \left(P_{j+1}^{(\alpha,\alpha)}(x) P_j^{(\alpha,\alpha)}(0) - P_j^{(\alpha,\alpha)}(x) P_{j+1}^{(\alpha,\alpha)}(0) \right) \ge 0.$$



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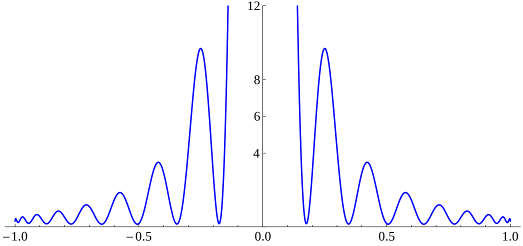

Motivation

Problem: Find a family of polynomials $(\phi_n(x))_{n\geq 0}$ such that for all polynomials $v(x)$ with $\deg(v) \leq n$:

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- kernel polynomials are not uniformly bounded in L^1

Modification of the ansatz

Schöberl modified the ansatz using gliding averages, i.e.,

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- Schöberl conjectured that $S(2n,x) \geq 0$

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If $-1 \le x \le 1,\ n \ge 0$, then

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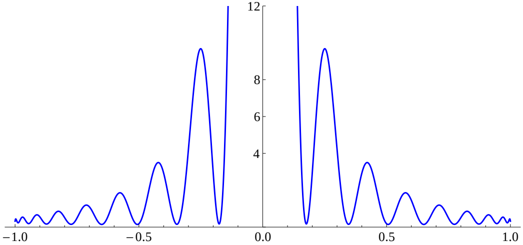


Recurrence relation for $S(n,x)$

The sums $S(n,x)$ satisfy a five term recurrence

$$\begin{aligned} 4(4n+7)(n+4)^2 S(n+4,x) &= (2n+3)^2(4n+15)S(n,x) \\ +(4n+15)(16n^2x^2 - 8n^2 + 48nx^2 - 12n + 35x^2 + 3)S(n+1,x) \\ -(-192n^2x^2 + 144n^2 - 1056nx^2 + 792n - 1260x^2 + 943)S(n+2,x) \\ -(4n+7)(16n^2x^2 - 8n^2 + 128nx^2 - 76n + 255x^2 - 173)S(n+3,x) \end{aligned}$$

- this representation makes it possible to invoke SumCracker's proving procedure [Gerhold+Kauers]
- the procedure, however, does not terminate
- a reformulation is needed!

Step 1: Decomposing $S(n, x)$

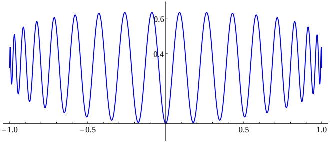
Using human insight we decompose

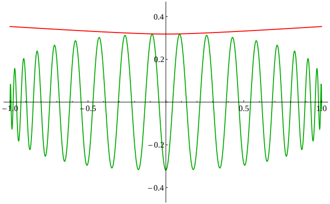
$$x^2 S(n, x) = g(n, x) + f(2n, x, 0),$$

where

$$g(n, x) = \frac{2n+1}{2}\left(x P_{2n+1}(x) - \frac{4n+2}{4n+3} P_{2n}(x)\right) P_{2n}(0),$$

$$f(n, x, y) = -\sum_{j=0}^{n} \frac{1}{(2j-1)(2j+3)} P_j(x) P_j(y)$$





Step 2: Estimating $f(2n, x)$ from below

It is a basic exercise for a student to obtain the bound

$$f(2n, x, 0) \geq \tfrac{1}{2}\left(f(2n, x, x) + f(2n, 0, 0)\right) =: e(n, x).$$

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It is a basic exercise for SumCracker to obtain the closed form

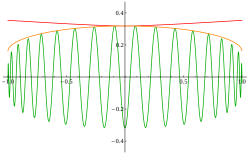
In[6]:= $\mathbf{Crack[SUM[\frac{1}{(2j-1)(2j+3)} LegendreP[\boldsymbol{j}, \boldsymbol{x}]^2, \{\boldsymbol{j}, 0, \boldsymbol{n}\}]]}$

Out[6]= $-\frac{(n+1)^2}{2n+3} P_n(x)^2 + (n+1) x P_{n+1}(x) P_n(x) - \frac{(n+1)^2}{2n+1} P_{n+1}(x)^2$

Step 3: Proving positivity of lower bound

Collecting the considerations above, the proof is completed if we can show positivity of $g(n,x)+e(n,x)$:

$$\begin{aligned} x^2 S(n,x) &= g(n,x) + f(2n,x,0) \\ &\geq g(n,x) + e(n,x) \\ &\geq 0 \end{aligned}$$



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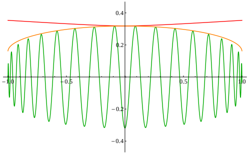

In[7]:= $\mathbf{ProveInequality}[\boldsymbol{g}[\boldsymbol{n},\boldsymbol{x}] + \boldsymbol{e}[\boldsymbol{n},\boldsymbol{x}] \geq \mathbf{0},$
$\mathbf{Using} \rightarrow \{-\mathbf{1} \leq \boldsymbol{x} \leq \mathbf{1}\}, \mathbf{Variable} \rightarrow \boldsymbol{n}]$

Out[7]= True

CAD-input for Schöberl's inequality (general case)

$$\forall n, \alpha, x, y, z, w \big((n \geq 0 \land -1 \leq x \leq 1 \land -1 \leq 2\alpha \leq 1 \land (2\alpha + 4n + 1)(y^2 + z^2)(\alpha + 2n + 1)^2 - (2\alpha + 4n + 1)(2\alpha + 4n + 3)wxz(\alpha + 2n + 1) + (2n + 1)(2\alpha + 2n + 1)(2\alpha + 4n + 3)w^2 \geq 0) \Rightarrow (2n + 3)(\alpha + 2n + 1)^2(\alpha + 2n + 3)^2(2\alpha + 2n + 3)(2\alpha + 4n + 5)y^2(\alpha + 2n + 2)^2 + (\alpha + 2n + 1)^2(64n^5 - 256x^2n^4 + 160\alpha n^4 + 464n^4 + 144\alpha^2n^3 - 512\alpha x^2n^3 - 1184x^2n^3 + 928\alpha n^3 + 1344n^3 + 56\alpha^3n^2 + 628\alpha^2n^2 - 384\alpha^2x^2n^2 - 1776\alpha x^2n^2 - 1984x^2n^2 + 2016\alpha n^2 + 1944n^2 + 8\alpha^4n + 164\alpha^3n + 912\alpha^2n - 128\alpha^3x^2n - 888\alpha^2x^2n - 1984\alpha x^2n - 1434x^2n + 1944\alpha n + 1404n + 12\alpha^4 + 120\alpha^3 + 441\alpha^2 - 16\alpha^4x^2 - 148\alpha^3x^2 - 496\alpha^2x^2 - 717\alpha x^2 - 378x^2 + 702\alpha + 405)z^2(\alpha + 2n + 2)^2 - w^2(-256n^7 + 4096x^4n^6 - 3072x^2n^6 - 896\alpha n^6 - 1728n^6 + 12288\alpha x^4n^5 + 25088x^4n^5 - 1216\alpha^2n^5 - 9216\alpha x^2n^5 - 19968x^2n^5 - 5184\alpha n^5 - 4864n^5 + 15360\alpha^2x^4n^4 + 62720\alpha x^4n^4 + 62464x^4n^4 - 800\alpha^3n^4 - 5872\alpha^2n^4 - 11008\alpha^2x^2n^4 - 49920\alpha x^2n^4 - 53120x^2n^4 - 12160\alpha n^4 - 7408n^4 - 256\alpha^4n^3 + 10240\alpha^3x^4n^3 + 62720\alpha^2x^4n^3 + 124928\alpha x^4n^3 + 81216x^4n^3 - 3104\alpha^3n^3 - 11072\alpha^2n^3 - 6656\alpha^3x^2n^3 - 47744\alpha^2x^2n^3 - 106240\alpha x^2n^3 - 74176x^2n^3 - 14816\alpha n^3 - 6592n^3 - 32\alpha^5n^2 - 752\alpha^4n^2 + 3840\alpha^4x^4n^2 + 31360\alpha^3x^4n^2 + 93696\alpha^2x^4n^2 + 121824\alpha x^4n^2 + 58320x^4n^2 - 4448\alpha^3n^2 - 10192\alpha^2n^2 - 2112\alpha^4x^2n^2 - 21696\alpha^3x^2n^2 - 76416\alpha^2x^2n^2 - 111264\alpha x^2n^2 - 57396x^2n^2 - 9888\alpha n^2 - 3424n^2 - 64\alpha^5n - 736\alpha^4n + 768\alpha^5x^4n + 7840\alpha^4x^4n + 31232\alpha^3x^4n + 60912\alpha^2x^4n + 58320\alpha x^4n + 21978x^4n - 2784\alpha^3n - 4576\alpha^2n - 320\alpha^5x^2n - 4608\alpha^4x^2n - 23296\alpha^3x^2n - 53568\alpha^2x^2n - 57396\alpha x^2n - 23340x^2n - 3424\alpha n - 960n - 32\alpha^5 - 240\alpha^4 + 64\alpha^6x^4 + 784\alpha^5x^4 + 3904\alpha^4x^4 + 10152\alpha^3x^4 + 14580\alpha^2x^4 + 10989\alpha x^4 + 3402x^4 - 640\alpha^3 - 800\alpha^2 - 16\alpha^6x^2 - 352\alpha^5x^2 - 2504\alpha^4x^2 - 8240\alpha^3x^2 - 13881\alpha^2x^2 - 11670\alpha x^2 - 3897x^2 - 480\alpha - 112)(\alpha + 2n + 2)^2 - 2(\alpha + 2n + 1)wx(128n^6 - 1024x^2n^5 + 384\alpha n^5 + 1408n^5 + 448\alpha^2n^4 - 2560\alpha x^2n^4 - 5504x^2n^4 + 3520\alpha n^4 + 5592n^4 + 256\alpha^3n^3 + 3296\alpha^2n^3 - 2560\alpha^2x^2n^3 - 11008\alpha x^2n^3 - 11488x^2n^3 + 11184\alpha n^3 + 10888n^3 + 72\alpha^4n^2 + 1424\alpha^3n^2 + 7870\alpha^2n^2 - 1280\alpha^3x^2n^2 - 8256\alpha^2x^2n^2 - 17232\alpha x^2n^2 - 11688x^2n^2 + 16332\alpha n^2 + 11258n^2 + 8\alpha^5n + 272\alpha^4n + 2278\alpha^3n + 7692\alpha^2n - 320\alpha^4x^2n - 2752\alpha^3x^2n - 8616\alpha^2x^2n - 11688\alpha x^2n - 5814x^2n + 11258\alpha n + 5940n + 16\alpha^5 + 220\alpha^4 + 1124\alpha^3 + 2669\alpha^2 - 32\alpha^5x^2 - 344\alpha^4x^2 - 1436\alpha^3x^2 - 2922\alpha^2x^2 - 2907\alpha x^2 - 1134x^2 + 2970\alpha + 1257)z(\alpha + 2n + 2)^2 \geq 0 \big)$$