# Hermite Interpolation Using ERBS with Trigonometric Polynomial Local Functions

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# Outline

1. Preliminaries

2. ERBS - definition and properties

3. Hermite interpolation problem for ERBS functions

4. Algebraic polynomial local functions

5. Trigonometric polynomial local functions

6. Algorithm

7. Numerical experiments

8. Further investigations

## 1. Preliminaries

- Expo-rational B-splines / Generalized Expo-rational B-splines

  Dechevsky, Lakså, Bang (2005) / (2009)

- Hermite interpolation by trigonometric polynomials

  Salzer (1960), Kress (1972), Delvos (1993), Dryanov (1994)

  Du, Han, Jin (2004)

## 2. ERBS - definition and properties

**Definition.** Given a sequence of knots $t_0 < t_1 < \cdots < t_{n+1}$ an *expo-rational B-spline (ERBS)* associated with the knots $t_{k-1}, t_k$ and $t_{k+1}$ is defined as follows:

$$B_k(t) = \begin{cases} \int_{t_{k-1}}^{t} \psi_{k-1}(s)\,ds / \int_{t_{k-1}}^{t_k} \psi_{k-1}(s)\,ds, & \text{if } t \in (t_{k-1}, t_k], \\ \int_{t}^{t_{k+1}} \psi_k(s)\,ds / \int_{t_k}^{t_{k+1}} \psi_k(s)\,ds, & \text{if } t \in (t_k, t_{k+1}), \\ 0, & \text{if } t \notin (t_{k-1}, t_{k+1}), \end{cases}$$

where

$$\psi_k(t) = e^{-\frac{\left(t - \frac{t_k + t_{k+1}}{2}\right)^2}{(t - t_k)(t_{k+1} - t)}}.$$

## Properties:

**P1.** $B_k(t) > 0$, $t \in (t_{k-1}, t_{k+1})$, and $B_k(t) = 0$, $t \notin (t_{k-1}, t_{k+1})$;

**P2.** $\sum_{k=1}^{n} B_k(t) = 1$, i.e. $B_k(t) + B_{k+1}(t) = 1$, $t \in (t_k, t_{k+1}]$, $k = 1, \ldots, n$;

**P3.** $B_k(t_k) = 1$ and $B_k(t_i) = 1$ for $i \neq k$, $k = 1, \ldots, n$;

**P4.** $\frac{d^j}{dt^j} B_k(t_i) = 0$ if $t_{k-1} < t_k < t_{k+1}$, $k = 1, \ldots, n$, $j = 1, 2, \ldots$;

**P5.** If $t_{k-1} < t_k < t_{k+1}$, then $B_k \in C^\infty(\mathbb{R})$ and $B_k$ is analytic on $\mathbb{R} \setminus \{t_{k-1}, t_k, t_{k+1}\}$, $k = 1, \ldots, n$.

## 3. Hermite interpolation problem for ERBS functions

Given a sequence of knots $t_0 < t_1 < \cdots < t_{n+1}$ an *ERBS function* $f(t)$ is defined by

$$f(t) = \sum_{k=1}^{n} \ell_k(t) B_k(t), \quad t \in [t_1, t_n],$$

$B_k(t)$ are the ERBS and $\ell_k(t)$ are *local functions* defined on $(t_{k-1}, t_{k+1})$.

**Hermite interpolation problem (HIP).** Given:

- knots $t_0 < t_1 < \cdots < t_{n+1}$,
- multiplicities $\lambda_i$, $i = 1, \ldots, n$,
- function $g(t)$ (or the derivative values $g^{(j)}(t_i)$ only), $j = 0, \ldots, \lambda_i$, $i = 1, \ldots, n$,

find an ERBS function $f(t)$ such that

$$f^{(j)}(t_i) = g^{(j)}(t_i), \qquad j = 0, \ldots, \lambda_i, \quad i = 1, \ldots, n. \qquad (*)$$

## 4. Algebraic polynomial local functions

HIP (*) for ERBS functions

$$f(t) = \sum_{k=1}^{n} \ell_k(t) B_k(t), \quad t \in [t_1, t_n],$$

$\ell_k(t)$ is an algebraic polynomial of degree $\lambda_k - 1$, $k = 1, \ldots, n$.

Explicit form:

$$f(t) = \sum_{k=1}^{n} \left[ \sum_{j=0}^{\lambda_j - 1} \frac{(t - t_k)^j}{j!} g^{(j)}(t_k) \right] B_k(t), \quad t \in [t_1, t_n],$$

i.e.

$$\ell_k(t) = \sum_{j=0}^{\lambda_j - 1} \frac{(t - t_k)^j}{j!} g^{(j)}(t_k).$$

Dechevsky, Lakså, Bang (2005)

## 5. Trigonometric polynomial local functions

- Trigonometric polynomial:

$$\tau(x) = a_0 + \sum_{k=1}^{N} (a_k \cos kx + b_k \sin kx),$$

- half-degree trigonometric polynomials:

$$\tau(x) = \sum_{k=0}^{N} \left[ a_k \cos(k + \frac{1}{2})x + b_k \sin(k + \frac{1}{2})x \right],$$

where $a_N^2 + b_N^2 \neq 0$.

Fundamental trigonometric polynomials for HIP (*):

$$T_{r,k}(t) = \frac{2^{\lambda_r - 1}\Delta_n(t)}{\sin\frac{t-t_r}{2}} \Bigg\{ \frac{1}{(\lambda_r - 1)!}\Bigg(\phi_{r,k}^{\lambda_r - 1} - \sum_{\ell=k}^{\lambda_r - 2} \phi_{r,k}^{\ell}\, D^{\lambda_r - 1} h_{\ell,\lambda_r - 2}(0)\Bigg)$$

$$+ \sum_{s=k}^{\lambda_r - 2} \frac{(-1)^{\lambda_r - s}\phi_{r,k}^{s}}{2s!} \sum_{\ell=0}^{[(\lambda_r - s)/2]-1} \frac{a_{2\ell}(\lambda_r - 2)}{(\lambda_r - 2 - s - 2\ell)!}\, D^{\lambda_r - 2 - s - 2\ell} \cot\frac{t - t_r}{2} \Bigg\},$$

where

$$\Delta_n(t) = \prod_{j=1}^{n} \sin^{\lambda_j}\left(\frac{t - t_j}{2}\right), \qquad \Delta_{n,r}(t) = \prod_{j=1,\ j\neq r}^{n} \sin^{\lambda_j}\left(\frac{t - t_j}{2}\right),$$

$$\phi_{r,k}(t) = \frac{(t - t_r)^k}{k!\,\Delta_{n,r}(t)}, \qquad \phi_{r,k}^{s} = D^s \phi_{r,k}(t_r),$$

$$h_{s,m}(t) = \sin^{m+1}\frac{t}{2} \cdot \frac{(-1)^{m-s} 2^m}{s!} \sum_{\ell=0}^{[\frac{m-s}{2}]} \frac{a_{2\ell}(m)}{(m - s - 2\ell)!} \cdot D^{m-s-2\ell} \cot\frac{t}{2}$$

for integer $m$ and $s$, $0 \leq s \leq m$;

$a_{2\ell}(m)$ are the coefficients in the series representation of the analytic function

$$\left(\frac{x}{2\sin\frac{x}{2}}\right)^{m+1} = \sum_{j=0}^{+\infty} a_{2j}(m)x^{2j}, \qquad |x| < 2\pi.$$

We have

$$T_{r,k}^{(j)}(t_i) = \delta_{i,r}\delta_{j,k}, \qquad i,r = 1,\dots,n, \quad j,k = 0,\dots,\lambda_i - 1.$$

$\delta_{\ell,s}$ being the Kronecker's symbol.

Du, Han, Jin (2004)

**Theorem.** *Let* $0 \le t_0 < t_1 < \cdots < t_n < t_{n+1} \le 2\pi$ *be given points,* $\lambda_1, \lambda_2 \ldots, \lambda_n$ *be positive integers,* $\{d_{ij}\}_{i=1,j=0}^{n,\quad \lambda_i-1}$ *be arbitrary real numbers and*

$$f(t) = \sum_{i=1}^{n} \Big[ \sum_{j=0}^{\lambda_i-1} d_{ij} T_{i,j}(t) \Big] B_i(t), \qquad t \in [t_1, t_n],$$

*where* $T_{i,j}(t)$ *are the fundamental polynomials for the HIP (*).*

*Then ERBS function* $f(t)$ *satisfies the Hermite interpolation conditions*

$$f^{(k)}(t_r) = d_{rk}, \qquad r = 1, \ldots, n, \quad k = 0, \ldots, \lambda_r - 1.$$

## 6. Algorithm

1. Compute the coefficients $a_{2\ell}(m)$ and store in a matrix. They does not depend on the HIP data (knots and derivative values).

2. Compute the derivative values $\phi^s_{r,k} = D^s \phi_{r,k}(t_r)$.

3. Compute the derivative values $D^{\lambda_r - 1} h_{\ell,\lambda_r-2}(0)$.

4. Compute the fundamental trigonometric polynomials $T_{r,k}(t)$.

5. Compute the trigonometric polynomial local functions

$$\ell_i(t) = \sum_{j=0}^{\lambda_i - 1} g^{(j)}(t_i) T_{i,j}(t), \qquad i = 1, \ldots, n.$$

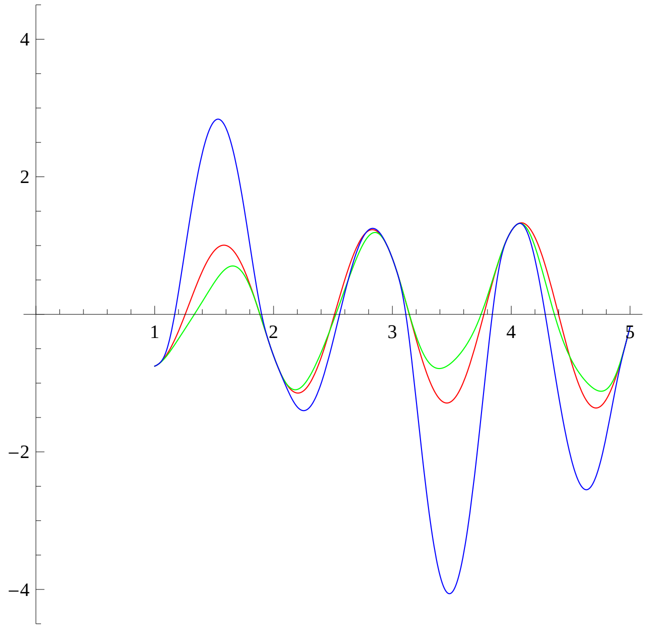
6. Compute the ERBS function using trigonometric polynomial local functions

$$f(t) = \sum_{i=1}^{n} \ell_i(t) B_i(t).$$

## 7. Numerical experiments

**Example 1.** $g(x) = \sin 5x \arctan x, \quad n = 5$

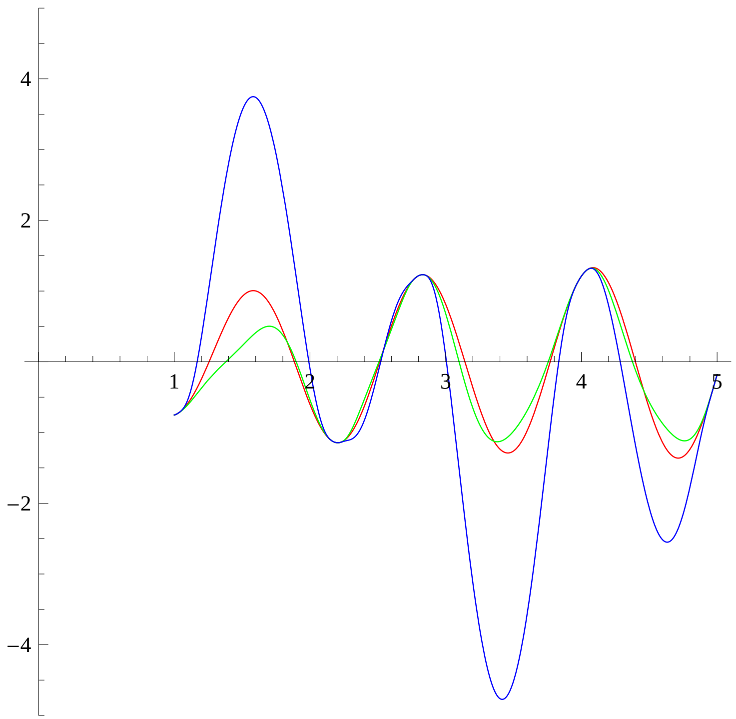
| $t_i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $\lambda_i$ | | 3 | 3 | 3 | 3 | 3 | |




Red - $g(x)$, Green - TERBS interpolant, Blue - AERBS interpolant

**Example 2.** $g(x) = \sin 5x \arctan x, \quad n = 5$

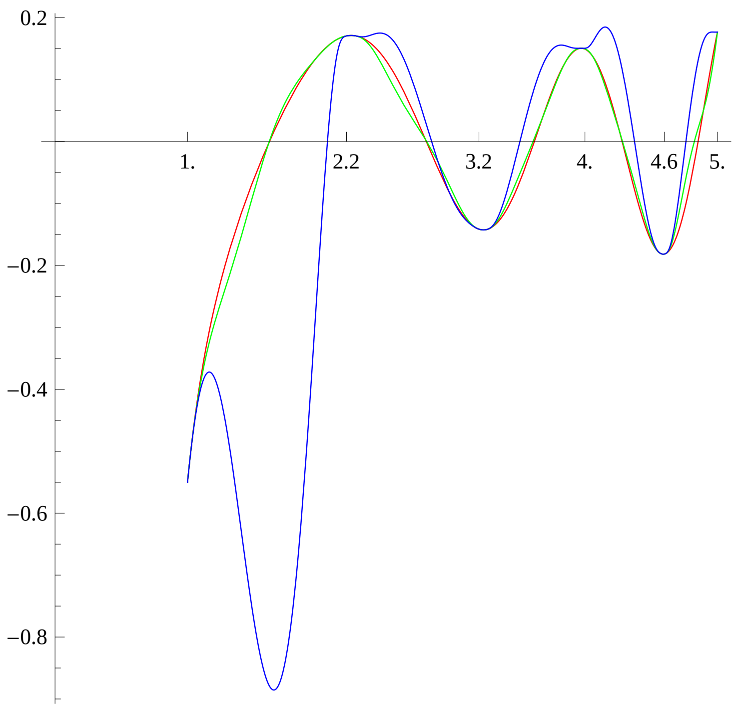
| $t_i$ | 0 | 1 | 2.2 | 2.8 | 4.1 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $\lambda_i$ | | 1 | 3 | 2 | 2 | 1 | |




Red - $g(x)$, Green - TERBS interpolant, Blue - AERBS interpolant

**Example 3.** $g(t) = \dfrac{\cos(t^2/5)}{(t-0.7)(t-6)}, \quad n = 6$

| $t_i$ | 0 | 1 | 2.236 | 3.234 | 3.972 | 4.591 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| $\lambda_i$ | | 3 | 3 | 3 | 2 | 3 | 1 | |




Red - $g(x)$, Green - TERBS interpolant, Blue - AERBS interpolant

## 8. Further investigations

- Bezier form representation
- other algorithms based on interpolation by trigonometric polynomials
- error estimates
- algorithms based on interpolation by rational functions
- algorithms with local functions of other type
- applying GERBS instead of ERBS

Thanks for your attention!