Three-Soliton Interactions for the Manakov System under Composite External Potentials

Assen Kyuldjiev

Institute for Nuclear Research and Nuclear Energy
Bulgarian Academy of Sciences, Sofia, Bulgaria

(Work done in collaboration with V. Gerdjikov from INRNE-BAS, Sofia and M. Todorov from FAMI-TU, Sofia)

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Idea of Adiabatic Approximation

The idea of the adiabatic approximation to the soliton interactions (Karpman&Solov'ev (1981)) led to effective modeling of the N-soliton trains of the perturbed scalar NLS eq.:

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u(x,t) = iR[u]. \qquad (1)$$

By N-soliton train we mean a solution of the NLSE (1) with initial condition

$$u(x, t=0) = \sum_{k=1}^{N} u_k(x, t=0), \qquad (2)$$

$$u_k(x,t) = 2\nu_k \mathrm{e}^{i\phi_k} \operatorname{sech} z_k, \quad z_k = 2\nu_k(x - \xi_k(t)), \quad \xi_k(t) = 2\mu_k t + \xi_{k,0},$$

$$\phi_k = \frac{\mu_k}{\nu_k} z_k + \delta_k(t), \quad \delta_k(t) = 2(\mu_k^2 + \nu_k^2)t + \delta_{k,0}.$$

Here μ_k are the amplitudes, ν_k – the velocities, δ_k – the phase shifts, ξ_k - the centers of solitons.

Idea of Adiabatic Approximation

The adiabatic approximation holds if the soliton parameters satisfy the restrictions

$$|\nu_k - \nu_0| \ll \nu_0, \; |\mu_k - \mu_0| \ll \mu_0, \; |\nu_k - \nu_0||\xi_{k+1,0} - \xi_{k,0}| \gg 1, \quad (3)$$

where ν_0 and μ_0 are the average amplitude and velocity respectively. In fact we have two different scales:

$$|\nu_k - \nu_0| \simeq \varepsilon_0^{1/2}, \quad |\mu_k - \mu_0| \simeq \varepsilon_0^{1/2}, \quad |\xi_{k+1,0} - \xi_{k,0}| \simeq \varepsilon_0^{-1}.$$

In this approximation the dynamics of the N-soliton train is described by a dynamical system for the $4N$ soliton parameters. In our previous works we investigate it in presence of periodic and polynomial potentials. Now, we are interested in what follow perturbation(s) by external sech-potentials:

$$iR[u] \equiv V(x)u(x,t), \qquad V(x) = \sum_s c_s \operatorname{sech}^2(2\nu_0 x - y_s). \qquad (4)$$

The latter allows us to realize the idea about localized potential wells and humps.

Potential Perturbations

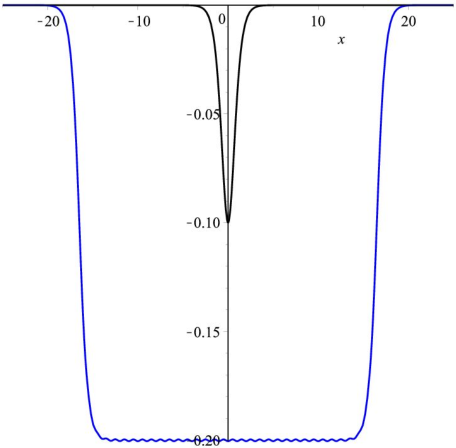

Figure: 1. Single sech-potential vs. composite potential well $V(x) = \sum_{s=0}^{32} c_s \mathrm{sech}^2(x - x_s)$, $c_s = -10^{-1}$, $x_s = -16 + sh$, $h = 1$, $s = 0, ..., 32$.

Perturbed Complex Toda Chain

In the present paper we generalize the above results to the perturbed vector NLS

$$i\vec{u}_t + \frac{1}{2}\vec{u}_{xx} + (\vec{u}^\dagger, \vec{u})\vec{u}(x,t) = iR[\vec{u}]. \qquad (5)$$

The corresponding vector N-soliton train is determined by the initial condition

$$\vec{u}(x, t=0) = \sum_{k=1}^{N} \vec{u}_k(x, t=0), \quad \vec{u}_k(x,t) = 2\nu_k e^{i\phi_k} \mathrm{sech} z_k \vec{n}_k, \qquad (6)$$

and the amplitudes, the velocities, the phase shifts, and the centers of solitons are as in Eq.(2). The phenomenology, however, is enriched by introducing a constant polarization vectors $\vec{n}_k$ that are normalized by the conditions

$$(\vec{n}_k^\dagger, \vec{n}_k) = 1, \qquad \sum_{s=1}^{n} \arg \vec{n}_{k;s} = 0.$$

Generalized Complex Toda Chain and CNSE

More precisely after involving these vectors we derive a generalized version of the CTC (GCTC) model, which allows to have in mind the polarization effects in the N-soliton train of the vector NLS.

Perturbed Complex Toda Chain Model. Initial Conditions

The corresponding model is known as the perturbed CTC model which can be written down in the form

$$\begin{aligned} \frac{d\lambda_k}{dt} &= -4\nu_0 \left(e^{Q_{k+1}-Q_k} - e^{Q_k - Q_{k-1}} \right) + M_k + iN_k, \\ \frac{dQ_k}{dt} &= -4\nu_0 \lambda_k + 2i(\mu_0 + i\nu_0)\Xi_k - iX_k, \end{aligned} \qquad (7)$$

where $\lambda_k = \mu_k + i\nu_k$ and $X_k = 2\mu_k \Xi_k + D_k$ and

$$\begin{aligned} &Q_k = -2\nu_0\xi_k + k\ln 4\nu_0^2 - i(\delta_k + \delta_0 + k\pi - 2\mu_0\xi_k), \\ &\nu_0 = \frac{1}{N}\sum_{s=1}^{N} \nu_s, \qquad \mu_0 = \frac{1}{N}\sum_{s=1}^{N} \mu_s, \qquad \delta_0 = \frac{1}{N}\sum_{s=1}^{N} \delta_s. \end{aligned} \qquad (8)$$

Variational approach and PCTC

We use the variational approach (Anderson and Lisak (1986)) and derive the GCTC model. Like the (unperturbed) CTC, GCTC is a finite dimensional completely integrable model allowing Lax representation.

The Lagrangian of the vector NLS perturbed by external potential is:

$$\begin{aligned} \mathcal{L}[\vec{u}] &= \int_{-\infty}^{\infty} dt\, \frac{i}{2} \left[(\vec{u}^\dagger, \vec{u}_t) - (\vec{u}_t^\dagger, \vec{u}) \right] - H, \\ H[\vec{u}] &= \int_{-\infty}^{\infty} dx\, \left[-\frac{1}{2}(\vec{u}_x^\dagger, \vec{u}_x) + \frac{1}{2}(\vec{u}^\dagger, \vec{u})^2 - (\vec{u}^\dagger, \vec{u}) V(x) \right]. \end{aligned} \tag{9}$$

Then the Lagrange equations of motion:

$$\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \vec{u}_t^\dagger} - \frac{\delta \mathcal{L}}{\delta \vec{u}^\dagger} = 0, \tag{10}$$

coincide with the vector NLS with external potential $V(x)$.

Variational approach and PCTC

Next we insert $\vec{u}(x,t) = \sum_{k=1}^{N} \vec{u}_k(x,t)$ (see eq. (6)) and integrate over x neglecting all terms of order ϵ and higher.
Thus after long calculations we obtain:

$$\mathcal{L} = \sum_{k=1}^{N} \mathcal{L}_k + \sum_{k=1}^{N} \sum_{n=k\pm 1} \widetilde{\mathcal{L}}_{k,n}, \qquad \mathcal{L}_{k,n} = 16\nu_0^3 e^{-\Delta_{k,n}} (R_{k,n} + R_{k,n}^*),$$
$$R_{k,n} = e^{i(\widetilde{\delta}_n - \widetilde{\delta}_k)} (\vec{n}_k^\dagger \vec{n}_n), \qquad \widetilde{\delta}_k = \delta_k - 2\mu_0 \xi_k,$$
$$\Delta_{k,n} = 2 s_{k,n} \nu_0 (\xi_k - \xi_n) \gg 1, \qquad s_{k,k+1} = -1, \qquad s_{k,k-1} = 1. \tag{11}$$

where

$$\mathcal{L}_k = -2i\nu_k \left((\vec{n}_{k,t}^\dagger, \vec{n}_k) - (\vec{n}_k^\dagger, \vec{n}_{k,t}) \right) + 8\mu_k \nu_k \frac{d\xi_k}{dt} - 4\nu_k \frac{d\delta_k}{dt} + \dots \tag{12}$$

The equations of motion are given by:

$$\frac{d}{dt}\frac{\delta \mathcal{L}}{\delta p_{k,t}} - \frac{\delta \mathcal{L}}{\delta p_k} = 0, \tag{13}$$

where $p_k = \{\delta_k, \xi_k, \mu_k, \nu_k, \vec{n}_k^\dagger\}$.

$$\begin{aligned} &\frac{\partial}{\partial t}(\mu_k + i\nu_k) = -4\nu_0\left(e^{q_{k+1}-q_k} - e^{q_k - q_{k-1}}\right) + M_k + iN_k, \\ &\frac{\partial \xi_k}{\partial t} = -\frac{1}{2\nu_k}\mathfrak{Im} h(\zeta) + \Xi[u_k], \quad \frac{\partial \delta_k}{\partial t} = 2\mu_k \frac{\partial \xi_k}{\partial t} + \mathfrak{Re} h(\zeta) + D[u_k]. \end{aligned} \tag{14}$$

Variational approach and PCTC

where $h(\zeta) = -2\zeta^2$ and

$$N_k[u] = \frac{1}{2}\mathfrak{Re}\int_{-\infty}^{\infty} R[u_k]\,\mathrm{sech}\, z_k\, \mathrm{e}^{-i\phi_k}\mathrm{d}z_k,$$

$$M_k[u] = \frac{1}{2}\mathfrak{Im}\int_{-\infty}^{\infty} R[u_k]\tanh z_k\,\mathrm{sech}\, z_k\, \mathrm{e}^{-i\phi_k}\mathrm{d}z_k,$$

$$\Xi_k[u] = \frac{1}{4}\mathfrak{Re}\int_{-\infty}^{\infty} R[u_k] z_k\,\mathrm{sech}\, z_k\, \mathrm{e}^{-i\phi_k}\mathrm{d}z_k,$$

$$D_k[u] = \frac{1}{2\nu_k}\mathfrak{Im}\int_{-\infty}^{\infty} R[u_k](1 - z_k\tanh z_k)\,\mathrm{sech}\, z_k\, \mathrm{e}^{-i\phi_k}\mathrm{d}z_k,$$

The corresponding system is a generalization of CTC:

$$\frac{d\lambda_k}{dt} = -4\nu_0 \left(e^{Q_{k+1}-Q_k} (\vec{n}_{k+1}^\dagger, \vec{n}_k) - e^{Q_k - Q_{k-1}} (\vec{n}_k^\dagger, \vec{n}_{k-1}) \right) + M_k + iN_k,$$
$$\frac{dQ_k}{dt} = -4\nu_0\lambda_k + 2i(\mu_0 + i\nu_0)\Xi_k - iX_k, \qquad \frac{d\vec{n}_k}{dt} = \mathcal{O}(\epsilon), \tag{15}$$

where again $\lambda_k = \mu_k + i\nu_k$ and the other variables are given by (8). The explicit form of M_k, N_k, Ξ_k and D_k is given by

$$M_k = \sum_s 2c_s\nu_k P(\Delta_{k,s}), \qquad N_k = 0,$$
$$\Xi_k = 0, \qquad D_k = \sum_s c_s R(\Delta_{k,s}). \tag{16}$$

where $\Delta_{k,s} = 2\nu_0\xi_k - y_s$ and the functions $P(\Delta)$ and $R(\Delta)$ are known explicitly.

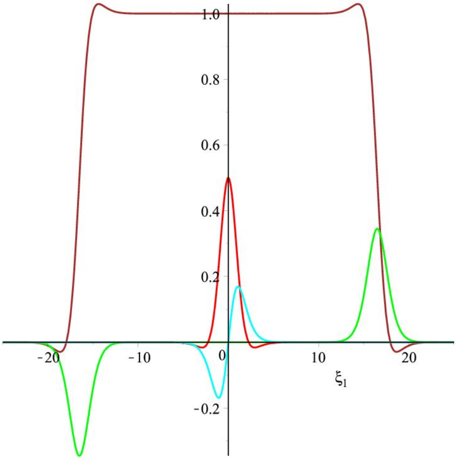

Figure: 2. *P* and *R* functions: for a single sech-potential and for the composite potential.

Now we have additional equations describing the evolution of the polarization vectors. But note, that their evolution is slow, and in addition the products $(\vec{n}^\dagger_{k+1}, \vec{n}_k)$ multiply the exponents $e^{Q_{k+1}-Q_k}$ which are also of the order of ϵ. Since we are keeping only terms of the order of ϵ we can replace $(\vec{n}^\dagger_{k+1}, \vec{n}_k)$ by their initial values

$$(\vec{n}^\dagger_{k+1}, \vec{n}_k)\Big|_{t=0} = m_{0k}^2 e^{2i\phi_{0k}}, \qquad k = 1, \dots, N-1 \qquad (17)$$

We formulate a condition on $\vec{n}_s$ that is compatible with the adiabatic approximation. We also formulate the conditions on the initial vector soliton parameters responsible for the different asymptotic regimes.
The CTC is completely integrable model; it allows Lax representation $L_t = [A.L]$, where:

$$\begin{aligned} L &= \sum_{s=1}^{N} \left(b_s E_{ss} + a_s (E_{s,s+1} + E_{s+1,s}) \right), \\ A &= \sum_{s=1}^{N} \left(a_s (E_{s,s+1} - E_{s+1,s}) \right), \end{aligned} \tag{18}$$

where $a_s = m_{0k} e^{i_{0k}} \exp((Q_{s+1} - Q_s)/2)$, $b_s = \frac{1}{2}(\mu_{s,t} + i\nu_{s,t})$ and the matrices E_{ks} are determined by $(E_{ks})_{pj} = \delta_{kp}\delta_{sj}$. The eigenvalues of L are integrals of motion and determine the asymptotic velocities.

Effects of the polarization vectors on the soliton interaction

The GCTC is also a completely integrable model because it allows Lax representation $\tilde{L}_t = [\tilde{A}.\tilde{L}]$, where:

$$\begin{aligned} \tilde{L} &= \sum_{s=1}^{N} \left(\tilde{b}_s E_{ss} + \tilde{a}_s (E_{s,s+1} + E_{s+1,s}) \right), \\ \tilde{A} &= \sum_{s=1}^{N} \left(\tilde{a}_s (E_{s,s+1} - E_{s+1,s}) \right), \end{aligned} \tag{19}$$

where $\tilde{a}_s = m_{0k} e^{i\phi_{0k}} a_s$, $b_s = \frac{1}{2}(\mu_{s,t} + i\nu_{s,t})$. Like for the scalar case, the eigenvalues of $\tilde{L}$ are integrals of motion. If we denote by $\zeta_s = \kappa_s + i\eta_s$ (resp. $\tilde{\zeta}_s = \tilde{\kappa} + i\tilde{\eta}_s$) the set of eigenvalues of L (resp. $\tilde{L}$) then their real parts κ_s (resp. $\tilde{\kappa}_s$) determine the asymptotic velocities for the soliton train described by CTC (resp. GCTC).

RTC and CTC. Asymptotic regimes

While for the RTC the set of eigenvalues ζ_s of the Lax matrix are all real, for the CTC they generically take complex values, e.g., $\zeta_s = \kappa_s + \mathrm{i}\eta_s$.
Hence, the only possible asymptotic behavior in the RTC is an asymptotically separating, free motion of the solitons. In opposite, for the CTC the real parts $\kappa_s \equiv \mathfrak{Re}\zeta_s$ of eigenvalues of the Lax matrix ζ_s determines the asymptotic velocity of the sth soliton.

Effects of the polarization vectors on the soliton interaction

Thus, starting from the set of initial soliton parameters we can calculate $L|_{t=0}$ (resp. $\tilde{L}|_{t=0}$), evaluate the real parts of their eigenvalues and thus determine the asymptotic regime of the soliton train.

Regime (i) $\kappa_k \neq \kappa_j$ $(\tilde{\kappa}_k \neq \tilde{\kappa}_j)$ for $k \neq j$ – asymptotically separating, free solitons;

Regime (ii) $\kappa_1 = \kappa_2 = \cdots = \kappa_N = 0$ $(\tilde{\kappa}_1 = \tilde{\kappa}_2 = \cdots = \tilde{\kappa}_N = 0)$ – a "bound state;"

Regime (iii) group of particles move with the same mean asymptotic velocity and the rest of the particles will have free asymptotic motion.

Varying only the polarization vectors one can change the asymptotic regime of the soliton train.

Effects of the external potentials on the GCTC. Numeric checks vs Variational approach

The predictions and validity of the CTC and GCTC are compared and verified with the numerical solutions of the corresponding CNSE using fully implicit difference scheme of Crank-Nicolson type, which conserves the energy, the mass, and the pseudomomentum. The scheme is implemented in a complex arithmetics. Such comparison is conducted for all dynamical regimes considered.

- First we study the soliton interaction of the pure Manakov model (without perturbations, $V(x) \equiv 0$) and with vanishing cross-modulation $\alpha_2 = 0$;
- 3-soliton configurations and transitions between different asymptotic regimes under the effect of well- and hump- sech-like external potential.

Three-soliton configuration in FAR. Real parts of eigenvalues of the Lax pair $\Re\zeta_1 = -0.0116$, $\Re\zeta_2 = 0$, $\Re\zeta_3 = 0.0116$. Effect of external well

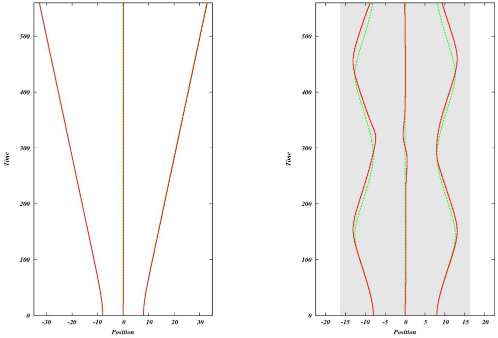

Figure: 3. Free potential behavior (left); External potential well $V(x) = \sum_{s=0}^{32} c_s \mathrm{sech}^2(x - x_s)$, $c_s = -10^{-1}$, $x_s = -16 + sh$, $h = 1$, $s = 0, ..., 32$(right).

Three-soliton configuration in BSR. Real parts of eigenvalues of the Lax pair $\Re\zeta_1 = \Re\zeta_2 = \Re\zeta_3 = 0$. Effect of external well

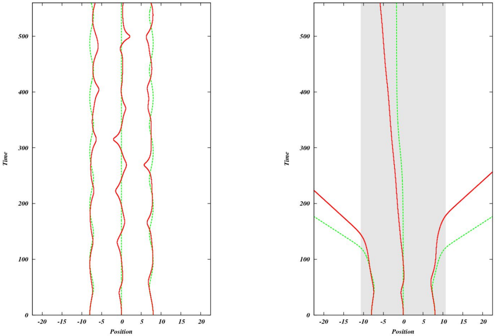

Figure: 4. Free potential behavior (left); External potential hump $V(x) = \sum_{s=0}^{12} c_s \mathrm{sech}^2(x - x_s)$, $c_s = 10^{-2}$, $x_s = -10 + sh$, $h = 5/3$, $s = 0, ..., 12$ (right).

Three-soliton configuration in MAR. Real parts of eigenvalues of the Lax pair $\mathfrak{Re}\zeta_1 = \mathfrak{Re}\zeta_2 = -0.00321$, $\mathfrak{Re}\zeta_3 = 0.00642$. Effect of external hump

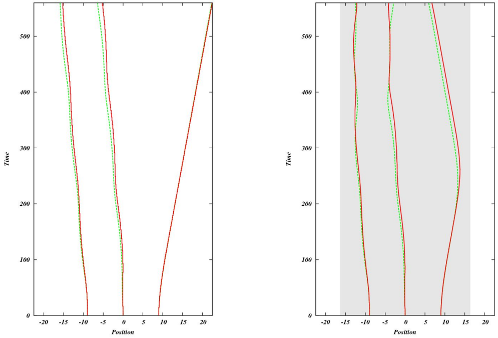

Figure: 5. Free potential behavior (left); External potential well $V(x) = \sum_{s=0}^{32} c_s \mathrm{sech}^2(x - x_s)$, $c_s = -10^{-2}$, $x_s = -16 + sh$, $h = 1$, $s = 0, \ldots, 32$.

Future work. Eight-soliton configuration in adiabatic approximation

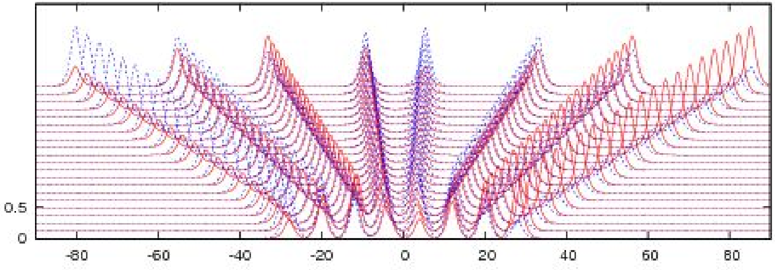

Figure: 6.

Future work. Nine-soliton configuration in adiabatic approximation

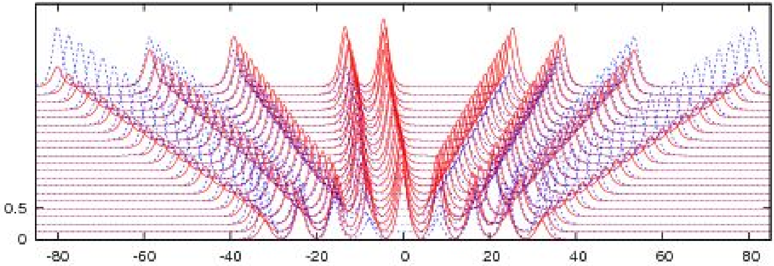

Figure: 7.

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Thanks

Thank you for your kind attention !