Обобщено вълново уравнение. Парадигма на Бусинеск и концепция за квази–частици за уединени вълни

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План на изложението

- Постановка на проблемите. Парадигма на Бусинеск. Вълните като квазичастици
- Интегруемост и неинтегруемост. Нелинейност и дисперсия
- Числени методи, техники и алгоритми
- Избор и построяване на начални условия
- Хамилтонова структура и закони за запазване
- Динамика на квазичастици и поляризация
- По-важни резултати и дискусия. Бъдещи цели и предстоящи изследвания

Уравнението на Бусинеск е исторически първият модел на повърхнинни вълни в плитък течен слой, който отчита и нелинейността, и дисперсията. Балансът между нелинейността (водеща до по-стръмен профил) и дисперсията (стремяща се към изглаждане) поддържа формата на вълната (wave of transition). От този баланс се пораждат уединени вълни с поведение на частици, наречени от Забуски и Крускал солитони (collision property) в специални случаи. В КС, движеща се заедно с центъра на бягащата вълна уравнението на Бусинеск се редуцира до уравнение на Кортевег-де Фриз. Преобладаващата част от получените досега аналитични и числени резултати са за едномерния случай. Не се знае почти нищо за нестационарни решения, включващи взаимодействие между солитони. Известни са отделни статии за решения с ограничена валидност, напр. уравнението на Кадомцев-Петвиашвили.

Откритието, че локализираните решения (имащи крайна енергия в безкрайна област) могат да запазят своята индивидуалност след сблъсък, доведе до понятието квазичастица. От особен интерес са обобщените вълнови уравнения, съдържащи нелинейност и дисперсия (напр. уравнението на Бусинеск и нелинейното уравнение на Шрьодингер) и нелинейните еволюционни уравненя (напр. уравнението на Кортевег-де Фриз). Квази-частиците са много важни при интерпретиране на дуализма вълна-частица във физиката.
Повечето физически системи не са напълно интегруеми дори и в случая на една пространствена променлива) и само численият подход може да доведе до пълното изясняване на физическия механизъм на взаимодействията.

Ефективни числени методи, техники и алгоритми

- Неявни диференчни схеми, използващи итерационни методи за решаване на линеаризирани системи с разредени матрици;
- Неявни диференчни схеми, основани на покоординатно разцепване на бихармоничните оператори;
- Спектрален и/или псевдоспектрален метод, използващ бързо преобразование на Фурие и апроксимация по метода на колокациите за нелинейните членове;
- Спектрален метод на Гальоркин, основан на специална пълна ортонормирана система от функции от класа $L_2[-\infty, \infty]$.

Парадигма на Бусинеск. Концепция за квази-частици

Уравнението, изведено от Бусинеск през 1871 г., наречено "Original Boussinesq Equation"

$$\frac{\partial^2 h}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left[(gH)h + \frac{3g}{2}h^2 + \frac{gH^3}{3}\frac{\partial^2 h}{\partial x^2}\right] \tag{1}$$

е напълно интегруемо, но некоректно в смисъл на Адамар заради положителния знак на четвъртата производна. Въвеждайки ефекта на повърхностното напрежение Кортевег-де Фриз получават т.н. Proper Boussinesq Equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left[u - \alpha u^2 + \beta\frac{\partial^2 u}{\partial x^2}\right] \tag{2}$$

което е коректно по Адамар (коректно поставена начална задача) само когато $\beta < 0$. С течение на времето уравнението е "подобрявано" в голям брой изследвания и публикации. Самата смяна на грешния знак пред четвъртата производна води до т.н. 'good' или 'proper' Bousinesq Equation.

Парадигма на Бусинеск. Концепция за квази-частици

Некоректността може да бъде отстранена, ако допуснем, че $\frac{\partial}{\partial t} \approx \frac{\partial}{\partial x}$ (известно като релация на линейния импеданс) и разменим съответните производни (въпросната релация е източник на голямо количество нефизични резултати):

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left[u - \alpha u^2 + \beta \frac{\partial^2 u}{\partial t^2}\right]. \tag{3}$$

Новополученото уравнение се нарича "improper" или RLWE (уравнение на Бенджамин-Бона-Махони). Нека отбележим, че разгледаните дотук версии на уравнението на Бусинеск притежават локализирани решения от тип *sech*. Напр. решение на уравнението на Бусинеск има явния вид

$$u = -\frac{3}{2}\frac{c^2-1}{\alpha}\mathrm{sech}^2\left(\frac{x-ct}{2}\sqrt{\frac{c^2-1}{\beta}}\right).$$

Обобщено вълново уравнение

Повечето аналитични и числени резултати, получени досега са за едномерни задачи, докато в 2 и повече измерения получаването на аналитични резултати е проблемно. Всички двумерни уравнения на Бусинеск са неинтегруеми. През 2001 г. Христов показва, че съгласуваната реализация на метода на Бусинеск води до следното обобщено вълново уравнение (Generalized Wave Equation) за функция $f = \phi(x, y, z = 0; t)$:

$$\begin{aligned} & f_{tt} + 2\beta \nabla f \cdot \nabla f_t + \beta f_t \Delta f \\ & + \tfrac{3\beta^2}{2} (\nabla f)^2 \Delta f - \Delta f + \frac{\beta}{6} \Delta^2 f - \frac{\beta}{2} (\Delta f)_{tt} = 0 \end{aligned}$$

с Хамилтонова плътност

$$\mathcal{H} = \frac{1}{2} \left[f_t^2 + (\nabla f)^2 - \tfrac{1}{4} \beta^2 (\nabla f)^4 + \tfrac{1}{6} \beta (\nabla f)^2 + \tfrac{1}{2} \beta (\nabla f_t)^2 \right].$$

Обобщено вълново уравнение

Това уравнение е възможно най-строгото амплитудно уравнение, което може да бъде изведено. То описва повърхнинни вълни на невискозен плитък слой, когато височината на вълната е голяма в сравнение с дълбочината на слоя. Уравнението е изведено едва през 2001 г. Наред с него множество несъгласувани все още са обект на изследване. Най-популярни са версиите с квадратична нелинейност, които са полезни от парадигмична гледна точка.

Парадигма на Бусинеск. Парадигмично уравнение на Бусинеск

Ако заменим f_t с f_x в квадратичния нелинеен член, получаваме

$$u_{tt} = (u + \frac{3\beta}{2}u^2 + \frac{\beta}{2}u_{tt} - \frac{\beta}{6}u_{xx})_{xx},$$

което се нарича Парадигмично уравнение на Бусинеск (Boussinesq Paradigm Equation). Това уравнение не е било известно на Бусинеск. Макар и опростено, то не е инвариантно относно Галилееви трансформации на системата.

Нека да разгледаме уравнението, когато потенциалът на скоростта и височината на повърхността не зависят от координатата y. Имайки предвид, че f е потенциалът на дъното на слоя, въвеждаме вертикална скорост $u = f_x$ и спомагателна функция q. По този начин получаваме т.н. Dispersive Wave System, която може да се приеме за родоначалник на различните уравнения на Бусинеск:

$$u_t + \frac{\alpha}{2}(u^2)_x = q_{xx},$$
$$q_t + \alpha u q_x = u - \beta_2 u_{xx} + \beta_1 u_{tt},$$

където β_1 и β_2 са дисперсионни коефициенти, $\beta_1 = 3\beta_2 = \beta$; $\alpha = \beta$ е амплитуден параметър.

Парадигмичното уравнение на Бусинеск в 2D има вида

$$u_{tt} = \Delta[u - \alpha u^2 + \beta_1 u_{tt} - \beta_2 \Delta u] \qquad (4)$$

където u е отклонението на повърхността, $\beta_1, \beta_1 > 0$ са два дисперсионни коефициента, α е амплитуден параметър.

Парадигма на Бусинеск. Концепция за квази-частици

Парадигма на Бусинеск с нелинейности от трета и пета степен (Cubic-Quintic Boussinesq Paradigm Equation)

$$u_{tt} = \Delta[u - \alpha(u^3 - \sigma u^5) + \beta_1 u_{tt} - \beta_2 \Delta u] \qquad (5)$$

където u отново е отклонение на повърхността, $\beta_1, \beta_1 > 0$ са дисперсионни коефициенти. Законът за енергията има вида

$$\frac{dE}{dt} = 0 \quad \text{with}$$

$$E = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[(\nabla u_t)^2 + u^2 - \frac{1}{2}u^4 + \frac{\sigma}{3}u^6 + \beta_1 u^2 + \beta_2(\nabla u)^2]dxdy.$$

За разлика от Парадигмичното уравнение с квадратична нелинейност, когато амплитудата расте, петата степен в CQBPE за достатъчно големи σ ще доминира и ще поддържа функционала на енергиятаа положителен, което от своя страна ще ограничава нарастването на амплитудата. Нашите изследвания показват, че за $\sigma > \frac{3}{16}$ не може да се очаква избухване (blow-up) на решението.

Парадигма на Бусинеск. (Енегийно)-съгласувана система на Бусинеск

Това е по-нататъшно обобщение на Парадигмичното уравнение на Бусинеск. Изведена е строго от модела на Бусинеск.

$$\frac{\partial \chi}{\partial t} = -\beta \nabla \cdot \chi \nabla f - \Delta f + \frac{\beta}{6} \Delta^2 f - \frac{\beta}{2} \frac{\partial^2 \Delta f}{\partial t^2} , \tag{6a}$$

$$\frac{\partial f}{\partial t} = -\frac{\beta}{2} (\nabla f)^2 - \chi, \tag{6b}$$

Парадигма на Бусинеск. (Енергийно)-съгласувана система на Бусинеск

След известно количество преобразования получаваме следния закон за баланса на енергията

$$E = \frac{1}{2} \int_D \left[\chi^2 + (1+\beta\chi)(\nabla f)^2 + \frac{\beta}{6}(\Delta f)^2 + \frac{\beta}{2}(\nabla f_t)^2 \right] dx$$

$$\frac{dE}{dt} = \oint_{\partial D} \left[(1+\beta\chi) f_t \frac{\partial f}{\partial n} + \frac{\beta}{2} f_t \frac{\partial f_{tt}}{\partial n} + \frac{\beta}{2} f_t \frac{\partial \Delta f}{\partial n} - \frac{\beta}{2} \Delta f \frac{\partial f_t}{\partial n} \right] ds, \tag{7}$$

което ни дава основание да наречем тези уравнения "Енергийно-съгласувана парадигма на Бусинеск".

Граничните условия имат вида

$$u|_{x=-L_1,L_2} = 0 \qquad q_x|_{x=-L_1,L_2} = 0.$$

и гарантират консервативност на пълната енергия в интервала $[-L_1, L_2]$.

Формулиране на проблема. Избор на гранични условия

Началното условие е суперпозиция от две солитонни вълни, движещи се в противоположни посоки с фазови скорости c_l и c_r

$$u(x, t = 0) = \frac{a \operatorname{sgn}(c)}{\frac{|c|-1}{2} + \cosh^2[b(x - X - ct)]}$$

където $a = \frac{c^2-1}{\alpha}$, $b = \sqrt{\frac{c^2-1}{2(\beta_1 c^2 - \beta_2)}}$. Sech-подобните решения съществуват в две области -– дозвукова: $0 < c < 1/\sqrt{3}$ и свръхзвукова: $c > 1$. Тъй като дозвуковата област не допуска съществуването на дълги вълни за малки β и не е физично значима, ние се интересуваме само от свръхзвуковите режими.

Формулиране на проблема. Закони за запазване

Дефинираме “маса”, M, (псевдо)импулс, P, и (псевдо)енергия, E:

$$M = \int_{-L_1}^{L^2} u \mathrm{d}x, \quad P = \int_{-L_1}^{L^2} (uq_x - \frac{\beta}{2} u_t u_x) dx,$$

$$E = \frac{1}{2} \int_{-L_1}^{L^2} \left(u^2 + q_x^2 + \frac{\beta}{2} u^3 + \frac{\beta}{2} u_t^2 + \frac{\beta}{6} u_x^2 \right) \mathrm{d}x,$$

Тук $-L_1$ и L_2 са краищата на интервала, в който се разглежда решението. В сила са следните закони за запазване

$$\frac{dM}{dt} = 0, \quad \frac{dP}{dt} = -\frac{\beta}{2} u_x^2 \Big|_{x=-L_1}^{x=L_2}, \quad \frac{dE}{dt} = 0.$$

Построяваме консервативна схема за системата, която е инвариантна относно Галилееви трансформации. Въбеждаме равномерна мрежа в интервала $[-L_1, L_2]$, $x_i = -L_1 + (i-1)h$, $h = (L_1 + L_2)/(N-1)$, N е броят на възлите. Използваме Нютонова квазилинеаризация в комбинация с вътрешни итерации. Това ни дава възможност да направим диференчната схема неявна и относно нелинейните членове.

Построяваме робастна икономична консервативна схема с вътрешна итерация по нелинейност:

$$\frac{u_i^{n+1,k}-u_i^n}{\tau}=\frac{q_{i+1}^{n+1/2,k}-2q_i^{n+1/2,k}+q_{i-1}^{n+1/2,k}}{h^2}$$

$$-\frac{\beta}{8h}\left[(u_{i+1}^{n+1,k-1})^2-(u_{i-1}^{n+1,k-1})^2+(u_{i+1}^n)^2-(u_{i-1}^n)^2\right]$$

$$\frac{q_i^{n+1/2,k}-q_i^{n-1/2}}{\tau}=-\frac{\beta}{8h}\left(q_{i+1}^{n+1/2,k-1}-q_{i-1}^{n+1/2,k}+q_{i+1}^{n-1/2}-q_{i-1}^{n-1/2}\right)$$

$$\times\,(u_i^{n+1,k}+u_i^{n-1})$$

$$-\frac{\beta}{12h^2}\left[(u_{i+1}^{n+1,k}-2u_i^{n+1,k}+u_{i-1}^{n+1,k})+(u_{i+1}^{n-1}-2u_i^{n-1}+u_{i-1}^{n-1})\right]$$

$$+\frac{\beta}{2}\frac{u_i^{n+1,k}-2u_i^n+u_i^{n-1}}{\tau^2}+\frac{u_i^{n+1,k}+u_i^{n-1}}{2}.$$

Може да се докаже, че горната апроксимация осигурява консервативност на енергията и масата на диференчно ниво за произволен потенциал $U(u)$, т.е. $M^{n+1} = M^n$ и $E^{n+1/2} = E^{n-1/2}$.

Функции на Ейри

Функцията на Ейри се дефинира с помощта на несобствения интеграл

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right)\mathrm{d}t.$$

Тя е функция с възвратна точка – осцилира за $x < 0$ и затихва експоненциално за $x > 0$. Асимптотичното поведение в отрицателната част се описва от функцията

$$\mathrm{Ai}(-x) \sim \frac{\sin\left(\frac{2}{3}x^{3/2} + \frac{\pi}{4}\right)}{\sqrt{\pi}x^{1/4}}.$$

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$

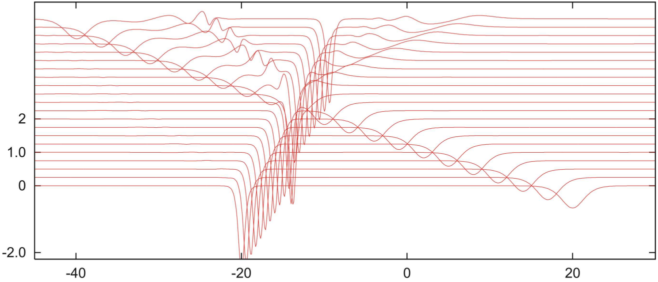


Рис.: 1. Short-times evolution. Forming of accompanying excitations

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$

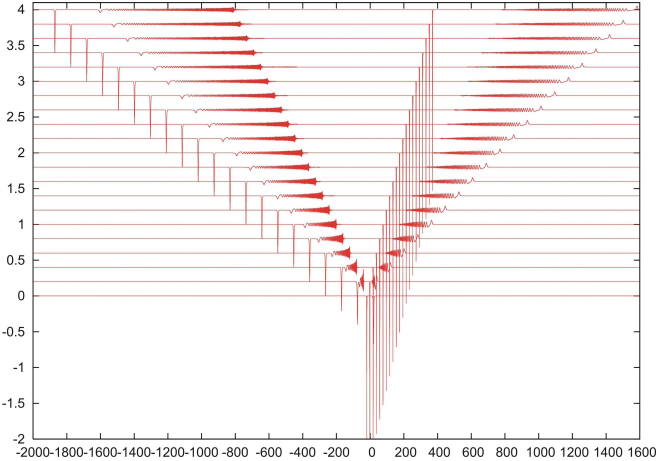


Рис.: 2. Long-times evolution.

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$

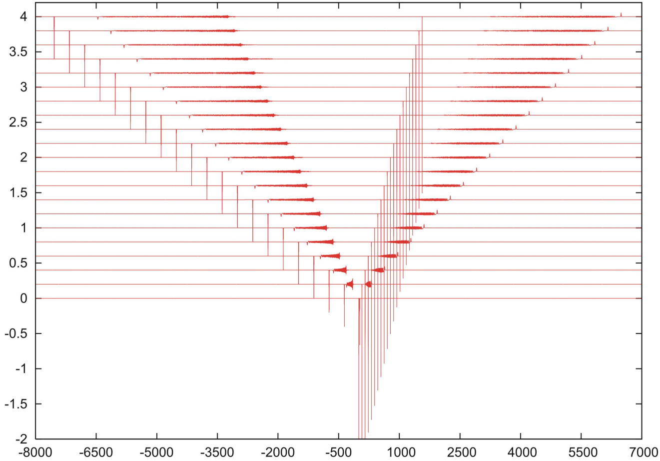


Рис.: 3. Very long-times evolution.

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$

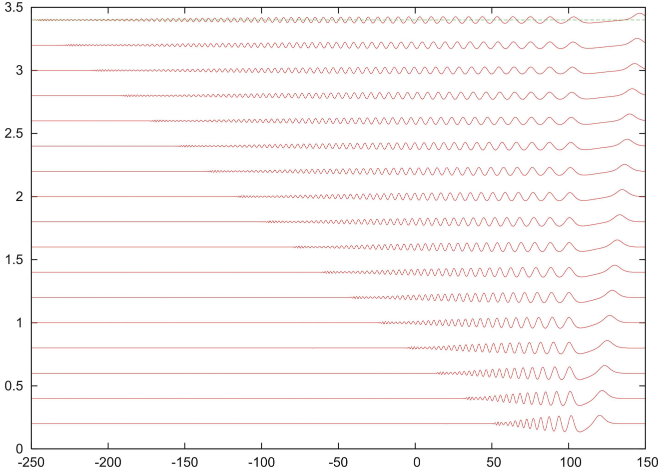


Рис.: 4. The evolution of the left excitation (shifted).

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$

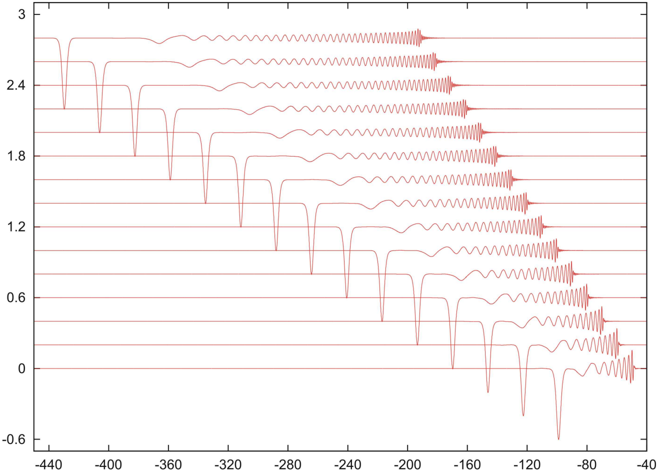


Рис.: 5. The right left-going soliton with its trail. Middle-times evolution.

Head-on collision. $c_l = 0.22$, $c_r = -1.2$, $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = -3$, $\tau = 0.05$, $h = 0.1$, $M = -5.02002$, $E = 3.584$

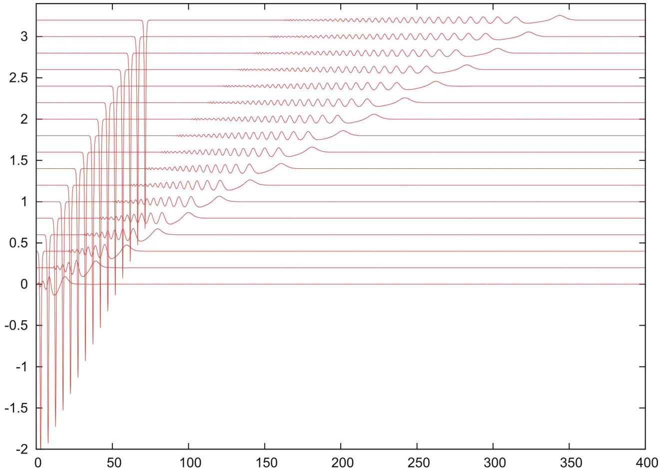


Рис.: 6. The left right-going soliton with its trail. Middle-times evolution.

Problem Formulation: Equations

CNLSE is system of nonlinearly coupled Schrödinger equations (called the Gross-Pitaevskii or Manakov-type system):

$$\begin{aligned} \mathrm{i}\psi_t &= \beta\psi_{xx} + \left[\alpha_1|\psi|^2 + (\alpha_1 + 2\alpha_2)|\phi|^2\right]\psi(+\Gamma\phi), \\ \mathrm{i}\phi_t &= \beta\phi_{xx} + \left[\alpha_1|\phi|^2 + (\alpha_1 + 2\alpha_2)|\psi|^2\right]\phi(+\Gamma\psi), \end{aligned} \tag{8}$$

where:
β is the dispersion coefficient;
α_1 describes the self-focusing of a signal for pulses in birefringent media;
$\Gamma = \Gamma_r + \mathrm{i}\Gamma_i$ is the magnitude of linear coupling. Γ_r governs the oscillations between states termed as breathing solitons, while Γ_i describes the gain behavior of soliton solutions.
α_2 (called cross-modulation parameter) governs the nonlinear coupling between the equations.

When $\alpha_2 = 0$, no nonlinear coupling is present despite the fact that "cross-terms" proportional to α_1 appear in the equations. For $\alpha_2 = 0$, the solutions of the two equations are identical, $\psi \equiv \phi$, and equal to the solution of single NLSE with nonlinearity coefficient $\alpha = 2\alpha_1$.

For $\Gamma = 0$, CNSE is alternately called the Gross-Pitaevskii equation or an equation of Manakov type. It was derived independently by Gross and Pitaevskii to describe the behavior of Bose-Einstein condensates as well as optic pulse propagation. It was solved analytically for the case $\alpha_2 = 0$, $\beta = \frac{1}{2}$ by Manakov via Inverse Scattering Transform who generalized an earlier result by Zakharov & Shabat for the scalar cubic NLSE (i.e. Eq.(2-ψ) with $\varphi(x, t) = 0$). Recently, Chow, Nakkeeran, and Malomed studied periodic waves in optic fibers using a version of CNSE with $\Gamma \neq 0$.

Integrability and Nonintegrability

The single NLSE has the form

$$i\psi_t + \beta\psi_{xx} + \alpha|\psi|^2\psi = 0 \tag{9}$$

As far as the applications in nonlinear optics are concerned, the above equation describes the single-mode wave propagation in a fiber. Depending on the sign of (9) admits single and multiple sech-solutions (*bright solitons*), as well as tanh-profile, or *dark soliton* solutions. In this paper we concentrate on the case of bright solitons.

A dynamical system with infinite number of conservation laws (integrals) is called integrable. NSE is an integrable dynamical system. CNSE is a nonintegrable generalisation of NSE.

Define "mass", M, (pseudo)momentum, P, and energy, E:

$$M \stackrel{\text{def}}{=} \frac{1}{2\beta}\int_{-L_1}^{L_2}(|\psi|^2+|\phi|^2)\mathrm{d}x, \quad P \stackrel{\text{def}}{=} -\int_{-L_1}^{L_2}\mathcal{I}(\psi\bar{\psi}_x+\phi\bar{\phi}_x)\mathrm{d}x,$$

$$E \stackrel{\text{def}}{=} \int_{-L_1}^{L_2}\mathcal{H}\mathrm{d}x, \quad \text{where} \tag{10}$$

$$\mathcal{H} \stackrel{\text{def}}{=} \beta\left(|\psi_x|^2+|\phi_x|^2\right)-\tfrac{1}{2}\alpha_1(|\psi|^4+|\phi|^4) \\ -(\alpha_1+2\alpha_2)\left(|\phi|^2|\psi|^2\right)-2\Gamma[\Re(\bar{\psi}\bar{\phi})]$$

is the Hamiltonian density of the system. Here $-L_1$ and L_2 are the left end and the right end of the interval under consideration. The following conservation/balance laws hold, namely

$$\frac{dM}{dt}=0, \qquad \frac{dP}{dt}=\mathcal{H}\Big|_{x=L_2}-\mathcal{H}\Big|_{x=-L_1}, \qquad \frac{dE}{dt}=0, \tag{11}$$

We concern ourselves with the soliton solutions whose modulation amplitude is of general form (non-sech) and which are localized envelops on a propagating carrier wave. This allows us to play various scenario of initial polarization. Unfortunately in sech-case the initial polarization can be only linear. Then we assume that for each of the functions ϕ, ψ the initial condition is of the form of a single propagating soliton, namely

$$\begin{Bmatrix} \psi(x,t) \\ \phi(x,t) \end{Bmatrix} = \begin{Bmatrix} A^{\psi} \\ A^{\phi} \end{Bmatrix} \operatorname{sech}\left[b(x-X-ct)\right] \exp\left\{ \mathrm{i}\left[\frac{c}{2\beta}(x-X)-nt\right]\right\}.$$

$$b^2 = \frac{1}{\beta}\left(n+\frac{c^2}{4\beta}\right), \quad A = b\sqrt{\frac{2\beta}{\alpha_1}}, \quad u_c = \frac{2n\beta}{c}, \tag{12}$$

where X is the spatial position (center of soliton), c is the phase speed, n is the carrier frequency, and b^{-1} – a measure of the support of the localized wave.

Problem Formulation: Choice of Initial Conditions

We assume that for each of the functions ϕ, ψ the initial condition has the general type

$$\begin{aligned} \psi &= A_\psi(x + X - c_\psi t) \exp\left\{\mathrm{i}\left[n_\psi t - \tfrac{1}{2} c_\psi (x - X - c_\psi t) + \delta_\psi\right]\right\} \\ \phi &= A_\phi(x + X - c_\phi t) \exp\left\{\mathrm{i}\left[n_\phi t - \tfrac{1}{2} c_\phi (x - X - c_\phi t) + \delta_\phi\right]\right\}, \end{aligned} \tag{13}$$

where c_ψ, c_ϕ are the phase speeds and X's are the initial positions of the centers of the solitons; n_ψ, n_ϕ are the carrier frequencies for the two components; δ_ψ and δ_ϕ are the phases of the two components. Note that the phase speed must be the same for the two components ψ and ϕ. If they propagate with different phase speeds, after some time the two components will be in two different positions in space, and will no longer form a single structure. For the envelopes (A_ψ, A_ϕ), $\theta \equiv \arctan(\max|\phi|/\max|\psi|)$ is a polarization angle.

Problem Formulation: Generation of Initial Conditions

Generally the carrier frequencies for the two components $n_\psi \neq n_\phi$ – elliptic polarization. When $n_\psi = n_\phi$ – circular polarization. If one of them vanishes – linear polarization (sech soliton, $\theta = 0; 90^\circ$).
In general case the initial condition is solution of the following system of nonlinear conjugated equations

$$
\begin{aligned}
A''_\psi + \left(n_\psi + \tfrac{1}{4}c_\psi^2\right) A_\psi + \left[\alpha_1 A_\psi^2 + (\alpha_1 + 2\alpha_2) A_\phi^2\right] A_\psi &= 0 \\
A''_\phi + \left(n_\phi + \tfrac{1}{4}c_\phi^2\right) A_\phi + \left[\alpha_1 A_\phi^2 + (\alpha_1 + 2\alpha_2) A_\psi^2\right] A_\phi &= 0.
\end{aligned}
\tag{14}
$$

The system admits bifurcation solutions since the trivial solution obviously is always present.

We solve the auxiliary conjugated system (14) with asymptotic boundary conditions using Newton method and the initial approximation of sought nontrivial solution is sech-function. The final solution, however, is not obligatory sech-function. It is a two-component polarized soliton solution.

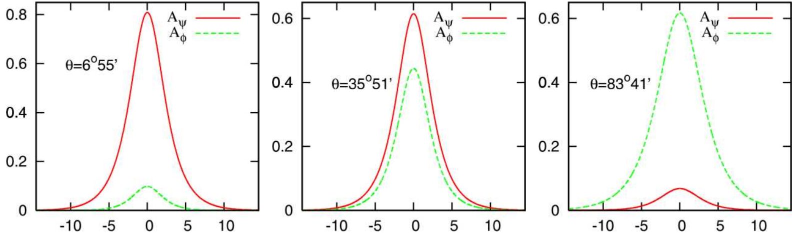


Рис.: 7. Amplitudes A_ψ and A_ϕ for $c_l = -c_r = 1$, $\alpha_1 = 0.75$, $\alpha_2 = 0.2$. Left:$n_\psi = -0.68$; middle: $n_\psi = -0.55$; right: $n_\psi = -0.395$.

Another dimension of complexity is introduced by the phases of the different components. The initial difference in phases can have a profound influence on the polarizations of the solitons after the interaction and the magnitude of the full energy. The relative shift of real and imaginary parts is what matters in this case.

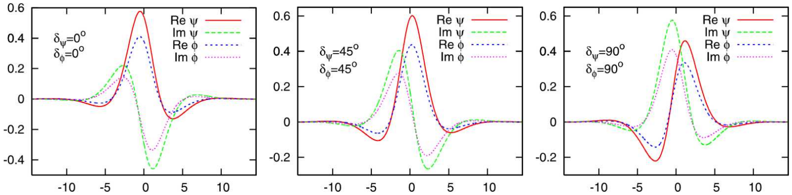


Рис.: 8. Real and imaginary parts of the amplitudes from the case shown in the middle panel of Figure 7 and the dependence on phase angle.

Problem Formulation: Initial Conditions

After completing the initial conditions our aim is to understand better the influence of the initial polarization and initial phase difference on the particle-like behavior of the localized waves. We call a localized wave a quasi-particle (QP) if it survives the collision with other QPs (or some other kind of interactions) without losing its identity.

Linearly Coupled Problem Formulation: Equations and Initial Conditions

For the linearly coupled system of NLSE the magnitude of linear coupling Γ_r generates breathing the solitons although noninteracting The initial conditions must be

$$\Psi = \psi \cos(\Gamma t) + \mathrm{i}\phi \sin(\Gamma t), \quad \Phi = \phi \cos(\Gamma t) + \mathrm{i}\psi \sin(\Gamma t), \tag{15}$$

where ϕ and ψ are assumed to be sech-solutions of (8) for $\alpha_2 = 0$. Hence (8) posses solutions, which are combinations of interacting solitons oscillating with frequency Γ_r and their motion gives rise to the so-called rotational polarization.

Problem Formulation: Conservation Laws

In all considered cases we found that a conservation of the total polarization is present. Only for the linearly CNLSE ($\alpha_2 = 0$) the total polarizations breathe with an amplitude evidently depending on the initial phase difference but is conserved within one full period of the breathing.

$\delta_r - \delta_l$	θ_l^i	θ_r^i	$\theta_l^i + \theta_r^i$	θ_l^f	θ_r^f	$\theta_l^f + \theta_r^f$
45°	45°	45°	90°	$33^\circ 48'$	$56^\circ 12'$	90°
90°	45°	45°	90°	$24^\circ 06'$	$65^\circ 54'$	90°
0°	20°	20°	40°	$20^\circ 00'$	$20^\circ 00'$	40°
90°	20°	20°	40°	$28^\circ 48'$	$2^\circ 02'$	$30^\circ 50'$
0°	36°	36°	72°	$36^\circ 00'$	$36^\circ 00'$	72°
90°	36°	36°	72°	$53^\circ 00'$	$13^\circ 20'$	$66^\circ 20'$
0°	10°	80°	90°	$21^\circ 05'$	$68^\circ 54'$	$89^\circ 59'$
90°	10°	80°	90°	$9^\circ 27'$	$80^\circ 30'$	$89^\circ 57'$

To solve the main problem numerically, we use an implicit conservative scheme in complex arithmetic.

$$
\begin{aligned}
\mathrm{i}\frac{\psi_i^{n+1}-\psi_i^n}{\tau} &= \frac{\beta}{2h^2}\left(\psi_{i-1}^{n+1}-2\psi_i^{n+1}+\psi_{i+1}^{n+1}+\psi_{i-1}^n-2\psi_i^n+\psi_{i+1}^n\right)\\
&+\frac{\psi_i^{n+1}+\psi_i^n}{4}\left[\alpha_1\left(|\psi_i^{n+1}|^2+|\psi_i^n|^2\right)+(\alpha_1+2\alpha_2)\left(|\phi_i^{n+1}|^2+|\phi_i^n|^2\right)\right]\\
&-\tfrac{1}{2}\Gamma\left(\phi_i^{n+1}+\phi_i^n\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{i}\frac{\phi_i^{n+1}-\phi_i^n}{\tau} &= \frac{\beta}{2h^2}\left(\phi_{i-1}^{n+1}-2\phi_i^{n+1}+\phi_{i+1}^{n+1}+\phi_{i-1}^n-2\phi_i^n+\phi_{i+1}^n\right)\\
&+\frac{\phi_i^{n+1}+\phi_i^n}{4}\left[\alpha_1\left(|\phi_i^{n+1}|^2+|\phi_i^n|^2\right)+(\alpha_1+2\alpha_2)\left(|\psi_i^{n+1}|^2+|\psi_i^n|^2\right)\right]\\
&-\tfrac{1}{2}\Gamma\left(\psi_i^{n+1}+\psi_i^n\right).
\end{aligned}
$$

Numerical Method: Internal Iterations

$$
\begin{aligned}
\mathrm{i}\frac{\psi_i^{n+1,k+1}-\psi_i^n}{\tau} =&\frac{\beta}{2h^2}\Big(\psi_{i-1}^{n+1,k+1}-2\psi_i^{n+1,k+1}+\psi_{i+1}^{n+1,k+1} \\
&+\psi_{i-1}^{n}-2\psi_i^{n}+\psi_{i+1}^{n}\Big) \\
+\frac{\psi_i^{n+1,k}+\psi_i^n}{4}&\Big[\alpha_1\big(|\psi_i^{n+1,k+1}||\psi_i^{n+1,k}|+|\psi_i^n|^2\big) \\
&+(\alpha_1+2\alpha_2)\big(|\phi_i^{n+1,k+1}||\phi_i^{n+1,k}|+|\phi_i^n|^2\big)\Big] \\
\mathrm{i}\frac{\phi_i^{n+1,k+1}-\phi_i^n}{\tau} =&\frac{\beta}{2h^2}\big(\phi_{i-1}^{n+1,k+1}-2\phi_i^{n+1,k+1}+\phi_{i+1}^{n+1,k+1} \\
&+\phi_{i-1}^{n}-2\phi_i^{n}+\phi_{i+1}^{n}\big) \\
+\frac{\phi_i^{n+1,k}+\phi_i^n}{4}&\Big[\alpha_1\big(|\phi_i^{n+1,k+1}||\phi_i^{n+1,k}|+|\phi_i^n|^2\big) \\
&+(\alpha_1+2\alpha_2)\big(|\psi_i^{n+1,k+1}||\psi_i^{n+1,k}|+|\psi_i^n|^2\big)\Big].
\end{aligned}
$$

It is not only convergent (consistent and stable), but also conserves mass and energy, i.e., there exist discrete analogs for (11), which arise from the scheme.

$$M^n = \sum_{i=2}^{N-1} \left(|\psi_i^n|^2 + |\phi_i^n|^2 \right) = \mathrm{const},$$

$$E^n = \sum_{i=2}^{N-1} \frac{-\beta}{2h^2} \left(|\psi_{i+1}^n - \psi_i{}^n|^2 + |\phi_{i+1}^n - \phi_i{}^n|^2 \right) + \frac{\alpha_1}{4} \left(|\psi_i^n|^4 + |\phi_i^n|^4 \right)$$
$$+ \tfrac{1}{2}(\alpha_1 + 2\alpha_2) \left(|\psi_i^n|^2 |\phi_i^n|^2 \right) - \Gamma \Re[\bar{\phi}_i^n \psi_i^n] = \mathrm{const},$$
$$\text{for} \quad \text{all} \quad n \geq 0.$$

These values are kept constant during the time stepping. The above scheme is of Crank-Nicolson type for the linear terms and we employ internal iterations to achieve implicit approximation of the nonlinear terms, i.e., we use its linearized implementation.

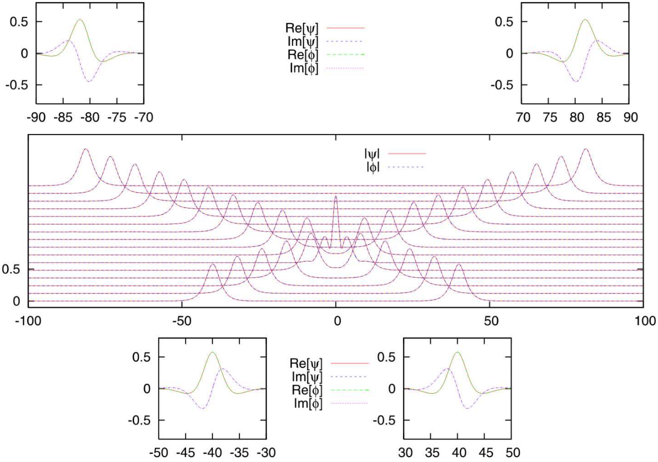


Рис.: 9. $\delta_l = 0^\circ$, $\delta_r = 0^\circ$

Results and Discussion: Initial Circular Polarizations of 45°, $\alpha_2 = 0$

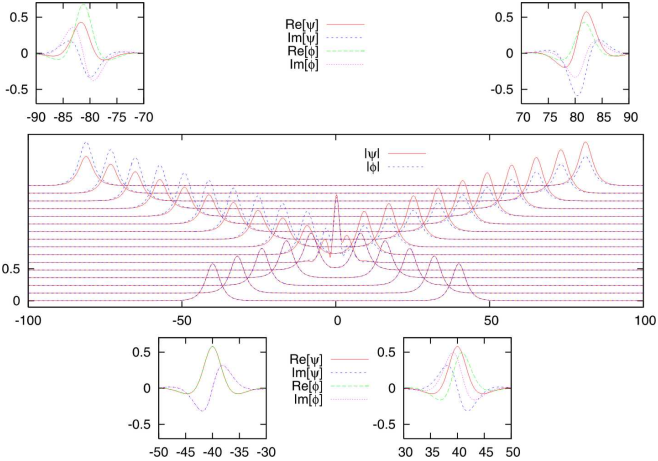


Рис.: 10. $\delta_l = 0^\circ$, $\delta_r = 45^\circ$

Results and Discussion: Initial Circular Polarizations of 45°, $\alpha_2 = 0$

—When both of QPs have zero phases (Fig. 9), the interaction perfectly follows the analytical Manakov two-soliton solution.
—The surprise comes in Fig. 10 where is presented an interaction of two QPs, the right one of which has a nonzero phase $\delta_r = 45^\circ$. After the interaction, the two QPs become different Manakov solitons than the original two that entered the collision. The outgoing QPs have polarizations $33^\circ 48'$ and $56^\circ 12'$. Something that can be called a 'shock in polarization' takes place. All the solutions are perfectly smooth, but because the property called polarization cannot be defined in the cross-section of interaction and for this reason, it appears as undergoing a shock.

Results and Discussion: Initial Circular Polarizations of 45°, $\alpha_2 = 0$

Here is to be mentioned that when rescaled the moduli of ψ and ϕ from Fig. 10 perfectly match each other which means that the resulting solitons have circular polarization (see left panel of Fig. 11 below). The Manakov solution is not unique. There exists a class of Manakov solution and in the place of interaction becomes a bifurcation between them.

Results and Discussion: Initial Circular Polarization and Nonuniqueness of the Manakov Solution

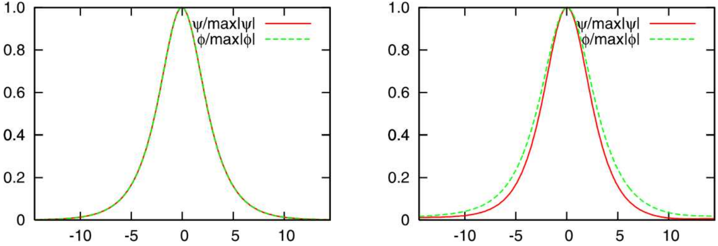


Рис.: 11. Circular polarization (left); Elliptic Polarization (right).

Equal Elliptic Initial Polarizations of $50°08'$ for $\alpha_2 = 2$

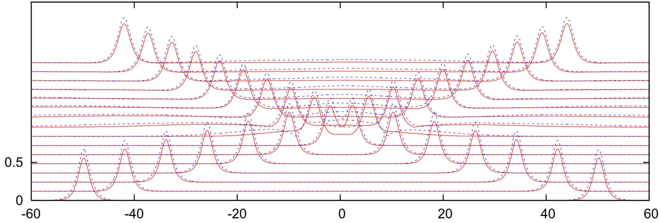


Рис.: 12. $\delta_l = 0°, \delta_r = 0°$

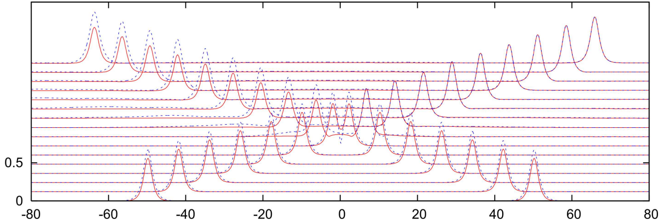


Рис.: 13. $\delta_l = 0°, \delta_r = 180°$

Equal Elliptic Initial Polarizations of $50°08'$ for $\alpha_2 = 2$

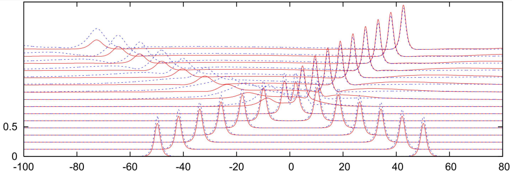


Рис.: 14. $\delta_l = 0°, \delta_r = 130°$

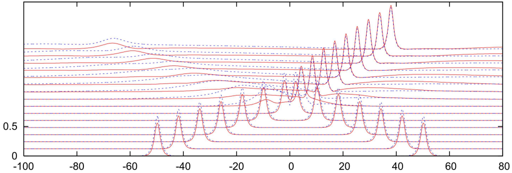


Рис.: 15. $\delta_l = 0°, \delta_r = 135°$

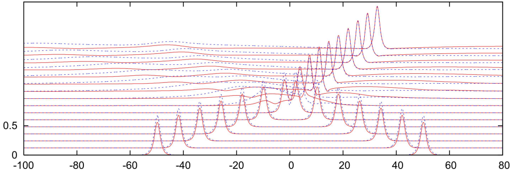


Рис.: 16. $\delta_l = 0°, \delta_r = 140°$

Equal Elliptic Initial Polarizations of $50^\circ 08'$ for $\alpha_2 = 2$.

We choose $n_{l\psi} = n_{r\psi} = -1.5$, $n_{l\phi} = n_{r\phi} = -1.1$, $c_l = -c_r = 1$, $\alpha_1 = 0.75$, and focus on the effects of α_2 and $\vec{\delta}$.

One sees that the desynchronisations of the phases leads in the final stage to a superposition of two one-soliton solutions but with different polarizations from the initial polarization. Yet, for $\delta_r = 130^\circ \div 140^\circ$ one of the QPs loses its energy contributing it to the other QP during the collision and then virtually disappears: kind of energy trapping (Figs.14, 15, 16).

For $\delta_r = 180^\circ$ another interesting effect is seen, when the right outgoing QP is circularly polarized (Fig. 13).

All these interactions are accompanied by changes of phase speeds. The total polarization exhibits some kind of conservation.

Strong Nonlinear Interaction: $\alpha_2 = 10$

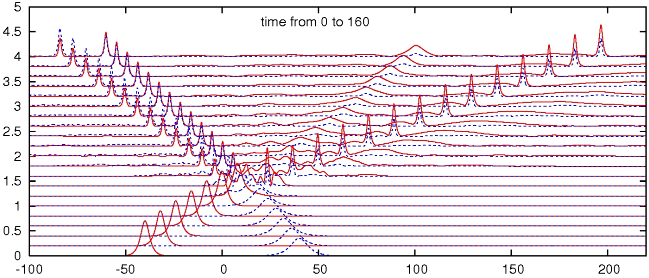


Рис.: 17. $\alpha_2 = 10$, $c_l = 1$, $c_r = -0.5$.

Strong Nonlinear Interaction: $\alpha_2 = 10$

- Two new solitons are born after the collision.
- The kinetic energies of the newly created solitons correspond their phase speeds and masses, but the internal energy is very different for the different QP.
- the total energy of the QPs is radically different from the total energy of the initial wave profile. The differences are so drastic that the sum of QPs energies can even become negative. This means that the energy was carried away by the radiation.
- The predominant part of the energy is concentrated in the left and right forerunners because of the kinetic energies of the latter are very large. This is due to the fact that the forerunners propagate with very large phase speeds, and span large portions of the region.
- All four QPs have elliptic polarizations.
- Energy transformation is a specific trait of the coupled system considered here.

Linear Coupling: Initial Elliptic Polarizations: $\theta = 23°44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

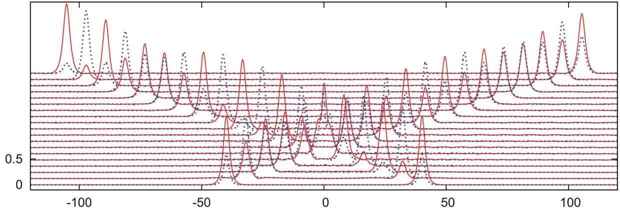


Рис.: 18. $\delta_l = 0°$, $\delta_r = 0°$

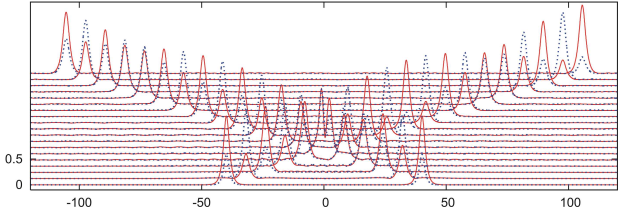


Рис.: 19. $\delta_l = 0°$, $\delta_r = 90°$

Linear Coupling: Initial Elliptic Polarizations: $\theta = 23^\circ 44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

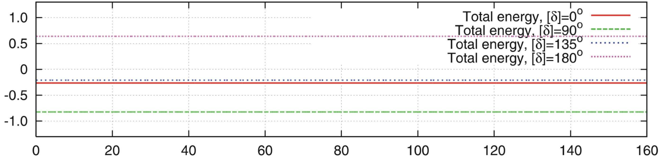


Рис.: 20. Influence of the initial phase difference on the total energy:

$\delta_l = 0^\circ$, $\delta_r = 0^\circ$ – $E = -0.262$;

$\delta_l = 0^\circ$, $\delta_r = 90^\circ$ – $E = -0.821$;

$\delta_l = 0^\circ$, $\delta_r = 135^\circ$ – $E = -0.206$;

$\delta_l = 0^\circ$, $\delta_r = 180^\circ$ – $E = 0.640$

Linear Coupling: Initial Elliptic Polarizations: $\theta = 23^\circ 44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

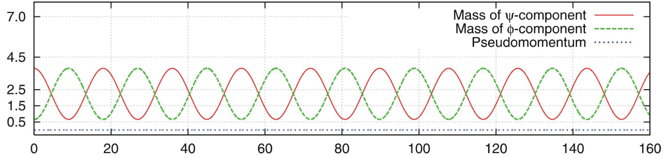


Рис.: 21. $\delta_l = 0^\circ$, $\delta_r = 90^\circ$, $P = 10^{-3} \div 10^{-5}$

Linear Coupling: Initial Elliptic Polarizations: $\theta = 23^\circ 44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

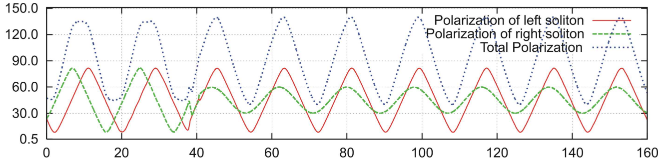


Рис.: 22. $\delta_l = 0^\circ$, $\delta_r = 0^\circ$

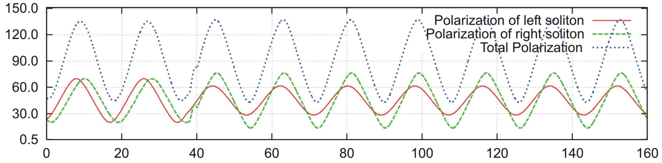
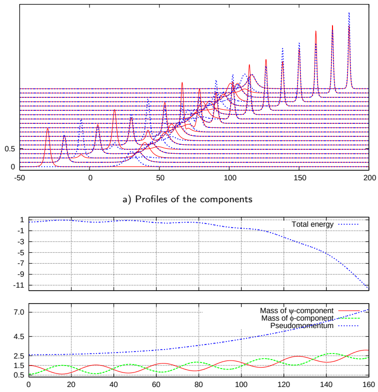


Рис.: 23. $\delta_l = 0^\circ$, $\delta_r = 90^\circ$

Linear Coupling: Initial Linear Polarizations: $\theta_l = 0^\circ$, $\theta_r = 90^\circ$, $\alpha_2 = 0$, $c_l = 1.5$, $c_r = 0.6$, $\Gamma = 0.175 + 0.005i$



b) Masses, psedomomentum, and total energy as functions of time.

Рис.: 24. $\delta_l = 0^\circ$, $\delta_r = 90^\circ$

Linear Coupling: Initial Elliptic Polarizations: $\theta = 23^\circ 44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

- We have found that the phases of the components play an essential role on the full energy of QPs. The magnitude of the latter essentially depends on the choice of initial phase difference (Figure 20);
- The pseudomomentum is also conserved and it is trivial due to the symmetry (Figure 21);
- The individual masses, however, breathe together with the individual (rotational) polarizations. Their amplitude and period do not influenced from the initial phase difference (Figure 21);

Linear Coupling: Initial Elliptic Polarizations: $\theta = 23°44'$, $\alpha_2 = 0$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = -1.1$, $n_{l\phi} = n_{r\phi} = -1.5$, $\Gamma = 0.175$

- The total mass is constant while the total polarization oscillates and suffers a 'shock in polarization ' when QPs enter the collision. The polarization amplitude evidently depends on the initial phase difference (Figures 22,23);
- Due to the real linear coupling the polarization angle of QPs can change independently of the collision.
- Complex parameter of linear coupling: Along with the oscillations of the energy and masses the (negative) energy decreases very fast, while the masses M_ψ and M_ϕ increase all of them oscillating. The pseudomomentum P increases without appreciable oscillation (Figure 24).

Благодарности

Благодаря за вниманието!