Числен анализ на някои нелинейни математически модели – парадигматично уравнение на Бусинеск и квазилинейно уравнение на топлопроводността

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Публична лекция за встъпване в академична длъжност „доцент“ по специалност „Математическо моделиране и приложение на математиката“

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Boussinesq Paradigm Equation (**BPE**)

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \beta_1 \Delta \frac{\partial^2 u}{\partial t^2} - \beta_2 \Delta^2 u + \Delta f(u), \quad x \in \mathbb{R}^n,\ t \in [0,T),\ T \le \infty,$$
$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x), \qquad x \in \mathbb{R}^n,\ \beta_1 \ge 0,\ \beta_2 > 0$$

Nonlinearities

$$f(u) = au^2;\ f(u) = au^3 + bu^5;\ f(u) = a|u|^p;\ f(u) = a|u|^p u + b|u|^q u$$
$$a, b = const$$

BPE first appears in the modeling of surface waves in shallow waters.

- C. I. Christov, Wave Motion 34, 161-174 (2001)

"Numerical and Analytical Tools for Localized Solutions of Generalized Wave Equations in Multidimension", BNSF Grant DDVU 02/71

Numerical results for **BPE**, references

- finite difference methods (Bogolubsky, 1977; Ortega, Sanz Serna, 1990; Christov, 1994; Bratsos, 2007)
- finite element methods (Manoranjan, Mitchell, Morris, 1984; Pani, 1997; Lin Q. , Y. H. Wu, R. Loxtona and S.Laib, 2009)
- Godunov-type central-upwind scheme (Chertock, Christov,Kurganov, 2011)
- implicit difference schemes based on efficient iterative solvers for sparse linear systems (Kolkovska, 2010, 2011; Christov, Vasileva, Kolkovska, 2011, Vasileva 2013)
- implicit difference schemes based on specially devised operator splitting for biharmonic operators (Kolkovska, Dimova, 2011, 2012, 2013)
- pseudo-spectral method based on Fast Fourier Transform (Todorov)
- spectral method with Christov functions (Christou, 2010)
- vector additive schemes (multicomponent alternating direction method) (Kolkovska, Angelow, 2013)

$$B\left(\frac{v_{ij}^{n+1}-2v_{ij}^{n}+v_{ij}^{n-1}}{\tau^2}\right)-\Lambda v_{ij}^{n}+\beta_2\Lambda^2 v_{ij}^{n}=\Lambda g(v_{ij}^{n+1},v_{ij}^{n},v_{ij}^{n-1}),$$
$$B=I-(\beta_1+\theta\tau^2)\Lambda+\theta\tau^2\beta_2\Lambda^2$$

- v_{ij}^n – a discrete approximation to u at (x_i, y_j, t_n), τ is a time-step
- $\Lambda = \Lambda^{xx} + \Lambda^{yy}$ – the standard five-point discrete Laplacian
- $\Lambda^2 = (\Lambda^{xxxx} + 2\Lambda^{xxyy} + \Lambda^{yyyy})$ – the discrete biLaplacian
- In approximations to Λv and $\Lambda^2 v$ we use the symmetric θ-weighted approximation to v_{ij}^n: $v_{ij}^{\theta,n} = \theta v_{ij}^{n+1} + (1-2\theta)v_{ij}^n + \theta v_{ij}^{n-1}$, $\theta \in \mathbb{R}$

$$\Omega_{\mathrm{h}} = [-L_1, L_1]\times[-L_2, L_2],\ x_i = jh_x,\ i = 0,\dots,N_x,\ \ y_j = jh_y,\ j = 0,\dots,N_y$$

Initial and boundary conditions

$$v_{ij}^0 = u_0(x_i, y_j),$$
$$v_{ij}^1 = u_0(x_i, y_j) + \tau u_1(x_i, y_j) + \frac{\tau^2}{2(I-\beta_1\Lambda)}\left(\Lambda u_0 - \beta_2\Lambda^2 u_0 + \Lambda f(u_0)\right)(x_i, y_j)$$
$$v_{ij}^{n+1} = 0, \quad \Lambda v_{ij}^{n+1} = 0 \quad \text{for} \quad i = 0, N_x \quad \text{or} \quad j = 0, N_y$$

Families of finite difference schemes

Family 1: $$g(v_{ij}^{n+1}, v_{ij}^{n}, v_{ij}^{n-1}) = \frac{F(v_{ij}^{n+1}) - F(v_{ij}^{n-1})}{v_{ij}^{n+1} - v_{ij}^{n-1}};$$

Family 2: $$g(v_{ij}^{n+1}, v_{ij}^{n}, v_{ij}^{n-1}) = 2\frac{F\left(\frac{(v_{ij}^{n+1}+v_{ij}^{n})}{2}\right) - F\left(\frac{(v_{ij}^{n}+v_{ij}^{n-1})}{2}\right)}{v_{ij}^{n+1} - v_{ij}^{n-1}},$$

$$F(u) = \int_0^u f(s)\, ds$$

Properties of Family 1 and Family 2:

- **Convergence:** The schemes have second order of convergence in space and time $O(|h|^2 + \tau^2)$.
- **Stability:** The schemes are unconditionally stable for $\theta > 1/4$. For $\theta = 0$ the schemes are stable provided $\tau^2 < \frac{4}{9}\frac{\beta_1}{\beta_2}h^2$.
- **Conservativeness:** The discrete energy is conserved in time, i.e. $E_h(v^{(n)}) = E_h(v^{(0)}), \quad n = 1, 2, \ldots$

$$B = I - (\beta_1 + \theta\tau^2)\Lambda + \theta\tau^2\beta_2\Lambda^2$$

Factorized operator

$$\widetilde{B} = B_1 B_2 B_3,$$
$$B_1 = (I - \theta\tau^2\Lambda^{xx} + \theta\tau^2\beta_2\Lambda^{xxxx}), B_2 = (I - \theta\tau^2\Lambda^{yy} + \theta\tau^2\beta_2\Lambda^{yyyy}),$$
$$B_3 = (I - \beta_1\Lambda)$$

Numerical analysis in 1D

The aim of the numerical tests are to study:

- convergence
- accuracy
- stability for different nonlinearities $f(u) = au^p$, $p > 1$

Numerical test, 1D

$$u^s(x,t;c) = \sqrt{\frac{2(c^2-1)}{a}}\,\mathrm{sech}\left(\sqrt{\frac{c^2-1}{\beta_1 c^2-\beta_2}}(x-ct)\right),\quad f(u) = au^3, a = const$$

$$u(x,0) = u^s(x+x_0^1, 0; c_1) + u^s(x+x_0^2, 0; c_2)$$
$$u_t(x,0) = u_t^s(x+x_0^1, 0; c_1) + u_t^s(x+x_0^2, 0; c_2),$$

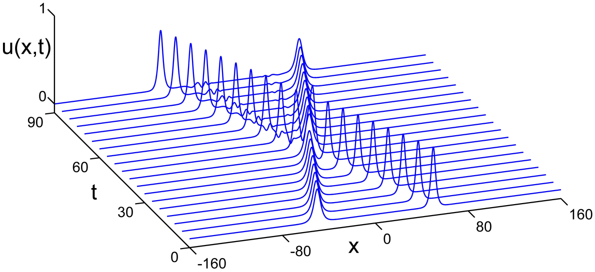


Figure: Interaction of two solitary waves $f(u) = au^3$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $a = 3$, $c_1 = 1.1$, $c_2 = -1.3$, $x_0^1 = -50$, $x_0^2 = 50$, $0 \leq t \leq 80$.

Table: Interaction of two solitary waves, $f(u) = au^3$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $a = 3$, $c_1 = 1.1$, $c_2 = -1.3$, $x \in [-160, 160]$, $\theta = 0.5$, $\varepsilon = 10^{-13}$, $T = 80$.

$h = \tau$	Rate κ		Error R	
	Family 1	Family 2	Family 1	Family 2
0.1	1.90	1.94	0.1038638	0.0663246
0.05	1.98	1.99	0.0273100	0.0170770
0.025	2.00	2.01	0.0069027	0.0042895
0.0125	2.01	2.00	0.0016247	0.0009503

$$\kappa = \log_2\left(\frac{||v_{[h]} - v_{[h/2]}||}{||v_{[h/2]} - v_{[h/4]}||}\right), \quad R = \frac{(||v_{[h]} - v_{[h/2]}||)^2}{||v_{[h]} - v_{[h/2]}|| - ||v_{[h/2]} - v_{[h/4]}||}$$

$$E_{rel} = \max_{0 \le k \le n} \frac{|E_h(v^k) - E_h(v^0)|}{E_h(v^0)}$$

- The calculations confirm the second rate of convergence $O(h^2 + \tau^2)$
- Family 2 is about 1.6 times more precise than Family 1
- $h = \tau = 0.025$, $E_{rel} \approx 1 \times 10^{-8}$ at $T = 80$ for both families Family 1 and Family 2

$$B_1 B_2 B_3 \left(\frac{v_{ij}^{n+1} - 2v_{ij}^{n} + v_{ij}^{n-1}}{\tau^2} \right) = \Lambda v_{ij}^n - \beta_2 \Lambda^2 v_{ij}^n + \Lambda g(v_{ij}^{n+1}, v_{ij}^n, v_{ij}^{n-1})$$

$$\begin{aligned}
\textbf{Step1}: \quad & B_1 w_{ij}^{(1)} = \Lambda v_{ij}^n - \beta_2 \Lambda^2 v_{ij}^n + \Lambda g(v_{ij}^{n+1}, v_{ij}^n, v_{ij}^{n-1}), && i \neq 0, N_x, \\
& w_{ij}^{(1)} = 0, \quad \Lambda^{xx} w_{ij}^{(1)} = 0, && i = 0, N_x \\
\textbf{Step2}: \quad & B_2 w_{ij}^{(2)} = w_{ij}^{(1)}, && j \neq 0, N_y, \\
& w_{ij}^{(2)} = 0, \quad \Lambda^{yy} w_{ij}^{(2)} = 0, && j = 0, N_y \\
\textbf{Step3}: \quad & B_3 w_{ij}^{(3)} = w_{ij}^{(2)}, && i \neq 0, N_x, \;\; j \neq 0, N_y, \\
& w_{ij}^{(3)} = 0, && i = 0, N_x \;\text{ or }\; j = 0, N_y \\
\textbf{Step4}: \quad & v_{ij}^{n+1} = 2v_{ij}^n - v_{ij}^{n-1} + \tau^2 w_{ij}^{(3)}
\end{aligned}$$

- the linearization is based on the Pickard method (method of successive iterations)
- 5-diagonal linear system of algebraic equations
- nonmonotonic Gaussian elimination with pivoting; Conjugate gradient type method designed for the discrete Laplacian equation

Numerical experiments, 2D

Let $u^s(x, y; c)$ be the best-fit approximation to the stationary translating with velocity c solution to **BPE**, obtained by perturbation techniques in

- C. I. Christov, J. Choudhury, Mech. Res. Commun. 38 (2011) 274–281.
- C.I. Christov, M.T. Todorov, M.A. Christou, AIP Conference Proceedings 1404 (2011), 49–56.

Initial conditions

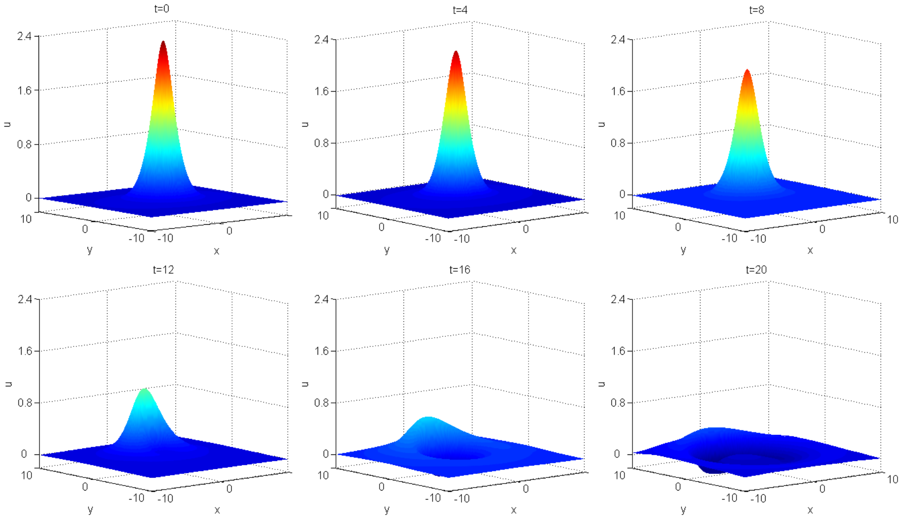
$$u_0(x, y) := u^s(x, y; c), \qquad u_1(x, y) := -cu^s_y(x, y; c),$$

Aim: Investigation of structural stability of the localized solutions propagating stationary with a prescribed phase velocity

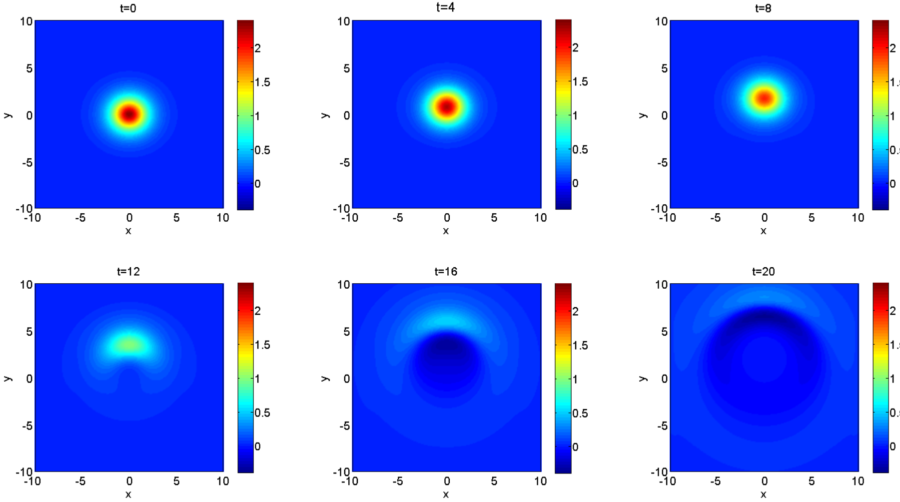
Conclusions:

- $f(u) = au^2$ – the solution preserves its shape for small times, but for larger times it either transforms into a diverging propagating wave or blows-up
- $f(u) = au^3 + bu^5$ – the solution does not blow-up even for relatively large values of c, but it is much less stable and transforms into a diverging propagating wave.

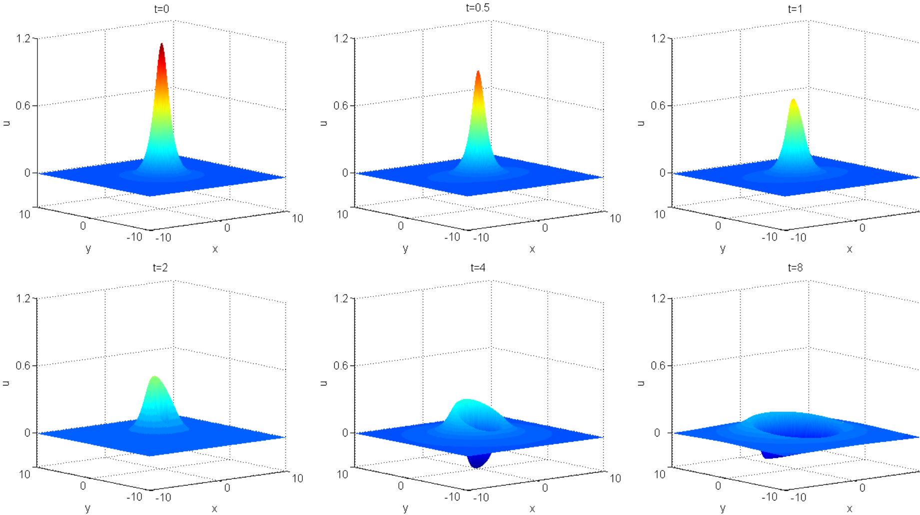
$f(u) = au^2$, $\beta_1 = 3$, $\beta_2 = 1$, $a = -1$, $c = 0.2$



$f(u) = au^2$, $\beta_1 = 3$, $\beta_2 = 1$, $a = -1$, $c = 0.2$



$f(u) = au^3 + bu^5$, $\beta_1 = 3$, $\beta_2 = 1$, $a = -1$, $b = 3/16$, $c = 0.3$



Generalized Boussinesq equation with Bernoulli type nonlinearities

$$\beta_2 u_{tt} - u_{xx} - \beta_1 u_{ttxx} + u_{xxxx} = (f(u))_{xx} \quad \text{for } x \in \mathbb{R},\ t \in [0,T),\ T \le \infty,$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{for } x \in \mathbb{R},\ \beta_1 \ge 0,\ \beta_2 > 0,$$

$$u_0 \in \mathrm{H}^1(\mathbb{R}),\ u_1 \in \mathrm{L}^2(\mathbb{R}),\ (-\Delta)^{-1/2} u_1 \in \mathrm{L}^2(\mathbb{R}),$$

$$(-\Delta)^{-s} u = \mathcal{F}^{-1}\left(|\xi|^{-2s}\mathcal{F}(u)\right),\ s > 0$$

Generalized Bernoulli nonlinearities

$$f(u) = a|u|^p u + b|u|^{2p} u, \qquad a, b = cost \ne 0, \quad p > 0$$

$f(u) = au^3 + bu^5$ – cubic-quintic

Generalized Bernoulli equation (stationary to **BPE**):

$$\psi''(x) = \psi(x) + a|\psi(x)|^p \psi(x) + b|\psi(x)|^{2p}\psi(x)$$

$$f(u) = a|u|^p, \qquad f(u) = a|u|^{p-1}u, \quad a = const, \quad p > 1$$

- F. Linares, J. Differential Equations 106 (1993) 257–293
- S. Wang, G. Chen, Nonlinear Anal. 64 (2006) 159–173
- Y. Liu, R. Xu , Physica D. 237 (2008) 721–731
- Y. Liu, R. Xu , J. Math. Anal. Appl. 338 (2008) 1169–1187
- R. Xu, Y. Liu, J. Math. Anal. Appl. 359 (2009) 739–751

$$f(u) = \sum_{k=1}^{m} a_k|u|^{p_k-1}u, \qquad 1 < p_1 < \ldots < p_m,$$

$$\exists\, \overline{s}:\ \overline{s} \in [1, m-1]: \begin{cases} a_i \geq 0 & \text{for} \quad i \in [1, \overline{s}]; \\ a_i \leq 0 & \text{for} \quad i \in [\overline{s}+1, m-1]; \qquad a_m < 0. \end{cases}$$

- R. Xu, Math. Methods in Appl. Sci. 34 (2011) 2318–2328.

Generalized Bernoulli nonlinearities

$$f(u) = a|u|^p u + b|u|^{2p} u, \qquad a, b = cost \neq 0, \quad p > 0$$

$b < 0$	Potential well method
$b > 0,\ a < 0,\ a^2 - \frac{(p+2)^2}{p+1} b > 0$	Nonstandard potential well method

$$\left.\begin{array}{l} b > 0,\ a < 0,\ a^2 - \frac{(p+2)^2}{p+1} b = 0 \\ b > 0,\ a < 0,\ a^2 - \frac{(p+2)^2}{p+1} b < 0 \\ b > 0,\ a > 0 \end{array}\right\} \text{Method of conservation law}$$

- R. Xu, Y. Lin, 2009
- Kutev, Kolkovska, Dimova 2011, 2012, 2013

Important functionals

Conservation low: $E(t) = E(0) \quad \text{for every} \quad t \in [0, T)$,

$$E(t) = E(u(\cdot,t), u_t(\cdot,t)) =$$
$$\frac{1}{2}\left(\beta_2 \left\|(-\Delta)^{-1/2} u_t(\cdot,t)\right\|^2 + \beta_1 \|u_t(\cdot,t)\|^2 + \|u(\cdot,t)\|^2_{\mathrm{H}^1}\right) + \int_{\mathbb{R}} F(u(x,t))\,dx$$

Potential energy functional $J(u)$:

$$J(u) = \frac{1}{2}\|u\|^2_{\mathrm{H}^1} + \frac{a}{p+2}\int_{\mathbb{R}} |u|^{p+2}\,dx + \frac{b}{2(p+1)}\int_{\mathbb{R}} |u|^{2p+2}\,dx$$

Nehari functional $I(u)$:

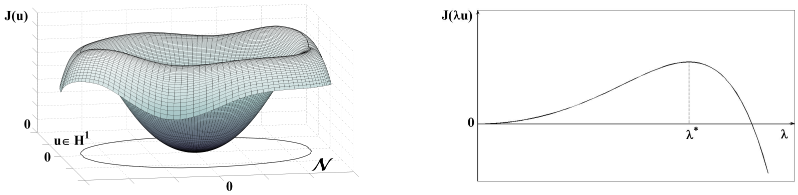
$$I(u) = J'(u)u = \|u\|^2_{\mathrm{H}^1} + a\int_{\mathbb{R}} |u|^{p+2}\,dx + b\int_{\mathbb{R}} |u|^{2p+2}\,dx$$

Nehari manifold $\mathcal{N}$: $\mathcal{N} = \{u \in \mathrm{H}^1 : \ I(u) = 0, \ \|u\|_{\mathrm{H}^1} \neq 0\}$

Critical energy constant d: $d = \inf\limits_{u \in \mathcal{N}} J(u)$

Depth D of the potential well: $D = \inf\limits_{u \in \mathrm{H}^1 \setminus \{0\}} \sup\limits_{\lambda \geq 0} J(\lambda u) > 0$

$d = D$



(a) (b)

Figure: (a) Schematic illustration of $J(u)$ as a function of $u \in \mathrm{H}^1$; (b) Cross section of $J(\lambda u)$ as a function of λ for a fixed $u \in \mathrm{H}^1$.

Theorem

Let $f(u) = a|u|^p u + b|u|^{2p} u$, $b < 0$, and $u_0 \in \mathrm{H}^1$, $u_1 \in \mathrm{L}^2$, and $(-\Delta)^{-1/2} u_1 \in \mathrm{L}^2$.

- If $E(0) < d$ and $I(u_0) > 0$ or $\|u_0\|_{\mathrm{H}^1} = 0$, then problem **BPE** has a unique global solution defined for every $t \in [0, \infty)$;
- If $E(0) < d$ and $I(u_0) < 0$, then the weak solution of **BPE** blows up in a finite time.

$$d = D = \inf_{u \in \mathrm{H}^1 \setminus \{0\}} \sup_{\lambda \geq 0} J(\lambda u) = J(\psi)$$

Ground state solution:

$$\psi(x) = (p+2)^{1/p} \left(\sqrt{a^2 - \frac{(p+2)^2}{p+1} b} \cosh(px) - a \right)^{-1/p}$$

Explicitly evaluation of the critical energy constant d

Theorem

Let $f(u) = a|u|^p u + b|u|^{2p} u$ and $b < 0$. Then the critical energy constant d is given by:

$$d = -\frac{a}{2}(p+2)^{2/p} \int_{\mathbb{R}} \left(\sqrt{a^2 - \frac{(p+2)^2}{p+1} b} \cosh(y) - a \right)^{-(p+2)/p} dy$$

$$- b\frac{(p+2)^{2(p+1)/p}}{2(p+1)} \int_{\mathbb{R}} \left(\sqrt{a^2 - \frac{(p+2)^2}{p+1} b} \cosh(y) - a \right)^{-2(p+1)/p} dy.$$

Corollary

Let $f(u) = au^3 + bu^5$ and $b < 0$. Then the critical energy constant d is equal to:

$$d\Big|_{p=2} = -\frac{12a}{(3a^2 - 16b)} - \frac{384ab}{(3a^2 - 16b)^2} - \frac{3a}{8b(3a^2 - 16b)^2}(9a^4 - 256b^2 - 192a^2 b)$$

$$+ \frac{\sqrt{3}(3a^2 - 16b)}{32(-b)^{3/2}} \left(\frac{\pi}{2} + \arctan \frac{a}{4}\sqrt{\frac{3}{-b}} \right).$$

Initial data:

$$u_0(x) = -\delta\psi(x), \qquad u_1(x) = -\varepsilon\delta\widetilde{\psi}(x),$$
$$\widetilde{\psi}(x) = \psi(x) - \frac{\nu}{2}\psi(\nu(x-M)) - \frac{\nu}{2}\psi(\nu(x+M)),$$
$$\psi(x) = 2\left(\sqrt{a^2 - \tfrac{16}{3}b}\,\cosh(2x) - a\right)^{-1/2},$$

$\psi(x)$ – ground state solution; δ, ε, ν and M – constants, $\delta > 0$

$$\int_{-\infty}^{\infty}\widetilde{\psi}(x)\,dx = 0 \Rightarrow u_1 \in \mathrm{L}^2 \text{ and } (-\Delta)^{-1/2}u_1 \in \mathrm{L}^2$$

Example: $\beta_1 = 1.5$, $\beta_2 = 0.5$, $a = 1$, $b = -3$;
$\varepsilon = 0.1$, $\nu = 0.1$, $M = 10$

A regular mesh defined in [-250,250] with space step $h = 0.01$ and time step $\tau = 0.01$ is used. In addition mesh refinement analysis is performed.

Numerical experiments, $f(u) = au^3 + bu^5$, $a = 1$, $b = -3$, $d \approx 1.08963046$

- $\delta = 0.8$, $\widetilde{E}(0) \approx 0.92808881 < d$, $\widetilde{I}(u_0) \approx 1.31204620 > 0$

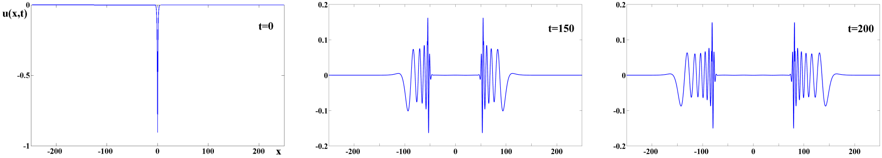


Figure: Profiles of the numerical solution $u(x,t)$ of **BPE** computed for $a = 1$, $b = -3$, $\delta = 0.8$ at different evolution times: (a) t=0; (b) t=150; (c) t=200.

Numerical experiments, $f(u) = au^3 + bu^5$, $a = 1$, $b = -3$, $d \approx 1.08963046$

- $\delta = 1.2$, $\quad \widetilde{E}(0) \approx 0.76106892 < d$, $\quad \widetilde{I}(u_0) \approx -5.82783925 < 0$

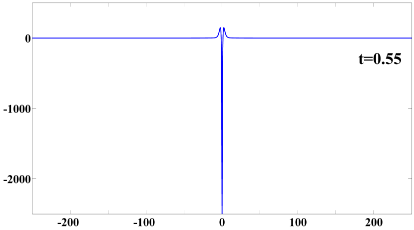


Figure: Profiles of the numerical solution $u(x,t)$ of **BPE** computed for $a = 1$, $b = -3$, $\delta = 1.2$ at evolution time t=0.55; $t^* \approx 0.56$ – blow up time

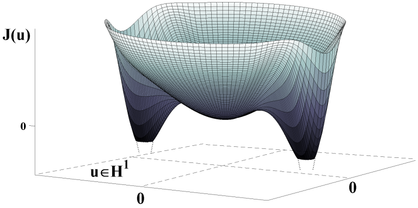
Nonstandard potential well method

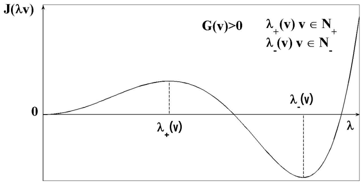
	Potential well method $b < 0$	Nonstandard potential well method $b > 0, a < 0, a^2 - \frac{(p+2)^2}{p+1} b > 0$
d	$d > 0$	$d = -\infty$
D	$D = d$	$D = +\infty$
$\mathcal{N}$	simply connected set	unbounded set with complicated structure

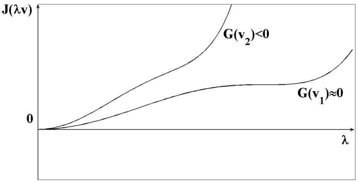
$$I(\lambda u) = 0,$$

$$I(\lambda u) = \lambda^2 \left(\|u\|^2_{\mathrm{H}^1} + a\lambda^p \int_{\mathbb{R}} |u|^{p+2}\, dx + b\lambda^{2p} \int_{\mathbb{R}} |u|^{2p+2}\, dx \right)$$

$$G(u) = \|u\|^{2(p+2)}_{\mathrm{L}^{p+2}} - \frac{4b}{a^2} \|u\|^2_{\mathrm{H}^1} \|u\|^{2p+2}_{\mathrm{L}^{2p+2}} - \text{discriminant}$$







Nehari manifold $\mathcal{N}$: $\quad \mathcal{N} = \mathcal{N}_+ \cup \mathcal{N}_- \cup \mathcal{N}_0,$

$$\mathcal{N}_\pm = \{\lambda_\pm(v)v : \; v \in \mathrm{H}^1, \|v\|_{\mathrm{H}^1} = 1, \; G(v) > 0, \; I(\lambda_\pm(v)v) = 0\}$$

$$\mathcal{N}_0 = \{\lambda_0(v)v : \; v \in \mathrm{H}^1, \|v\|_{\mathrm{H}^1} = 1, \; G(v) = 0, \; I(\lambda_0(v)v) = 0\}$$

New energy constant d_+ (analog of d)

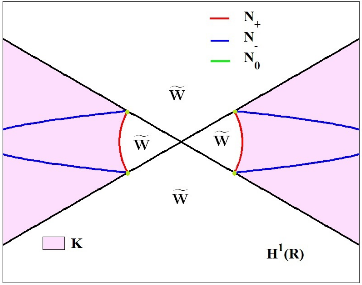
$$d_+ = \inf_{u \in \mathcal{N}_+ \cup \mathcal{N}_0} J(u)$$

$$\begin{aligned} d_+ = & J(\psi) = -\frac{a}{2}(p+2)^{2/p} \int_{\mathbb{R}} \left(\sqrt{a^2 - \frac{(p+2)^2}{p+1} b} \cosh(y) - a \right)^{-(p+2)/p} dy \\ & - b \frac{(p+2)^{2(p+1)/p}}{2(p+1)} \int_{\mathbb{R}} \left(\sqrt{a^2 - \frac{(p+2)^2}{p+1} b} \cosh(y) - a \right)^{-2(p+2)/p} dy > 0 \end{aligned}$$

$$\begin{aligned} d_+ \Big|_{p=2} = & -\frac{12a}{(3a^2 - 16b)} - \frac{384ab}{(3a^2 - 16b)^2} - \frac{3a}{8b(3a^2 - 16b)^2}(9a^4 - 256b^2 - 192a^2 b) \\ & + \frac{\sqrt{3}(3a^2 - 16b)}{64 b^{3/2}} \ln \frac{\sqrt{3}a + 4\sqrt{b}}{\sqrt{3}a - 4\sqrt{b}} \end{aligned}$$

$$\widetilde{W} = \mathrm{H}^1 \setminus \overline{K},$$

$$K = \{\lambda v : \; v \in \mathrm{H}^1, \; \|v\|_{\mathrm{H}^1} = 1, \; G(v) > 0 \text{ and } \lambda > \lambda_+(v)\}$$



Theorem, Global existence

Let $f(u) = a|u|^p u + b|u|^{2p} u$, $b > 0$, $a < 0$, $a^2 - \frac{(p+2)^2}{p+1} b > 0)$. Suppose $u_0 \in \mathrm{H}^1$, $u_1 \in \mathrm{L}^2$, and $(-\Delta)^{-1/2} u_1 \in \mathrm{L}^2$. If $E(0) < d_+$ and $u_0 \in \widetilde{W}$ then **BPE** has a unique global solution $u(x,t)$ defined for every $t \in [0, \infty)$.

Initial data:

$$u_0(x) = -\delta\overline{\psi}(x), \quad \overline{\psi}(x) = 2\left(\sqrt{a^2 - \frac{16}{3}b}\,\cosh(2x) - a(1+\varepsilon)\right)^{-1/2},$$

$$u_1(x) = 0, \qquad \varepsilon = 5, \qquad E(0) \equiv J(u_0)$$

$$G(\overline{\psi}) = \|\overline{\psi}\|^8_{\mathrm{L}^4} - \frac{4b}{a^2}\|\overline{\psi}\|^2_{\mathrm{H}^1}\|\overline{\psi}\|^6_{\mathrm{L}^6} > 0, \qquad \lambda_+\overline{\psi} \in \mathcal{N}_+, \qquad \lambda_-\overline{\psi} \in \mathcal{N}_-$$

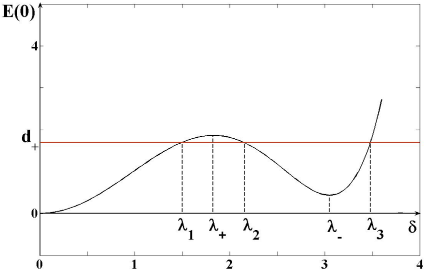
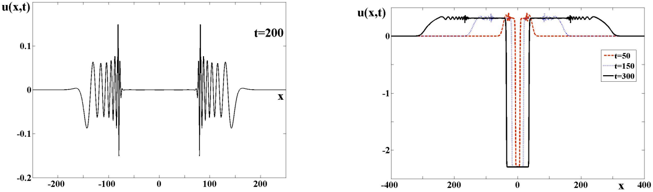


Figure: $\lambda_+ \approx 1.83$, $\lambda_- \approx 3.05$, $\lambda_1 \approx 1.51$, $\lambda_2 \approx 2.16$, $\lambda_3 \approx 3.48$

For $\forall\, \delta \in (0, \lambda_1)$ both conditions of the Theorem are satisfied: $E(0) < d_+$ and $u_0 \in \widetilde{W}$.

- $\delta = 1.4$, $\widetilde{E}(0) \approx 1.59242338 < d_+$, $u_0 \in \widetilde{W}$ – Figure (a)
- $\delta = 3$, $\widetilde{E}(0) \approx 0.44503829 < d_+$, $u_0 \notin \widetilde{W}$ – Figure (b)



(a) (b)

Figure: Profiles of the numerical solution $u(x,t)$: (a) when the conditions of the Theorem are satisfied; (b) when the conditions of the Theorem fail.

Conjecture

Let $f(u) = a|u|^p u + b|u|^{2p}u$, $b > 0$, $a < 0$, $a^2 - \frac{(p+2)^2}{p+1} b > 0$). If the initial data satisfy either $E(0) > d_+$ or $E(0) < d_+$ and $u_0 \in \overline{K}$ then solutions of **BPE** are globally defined for $t \in [0, \infty)$ and possibly blow up at $t = \infty$.

Model of heat structures

$$u_t = \nabla \cdot (k(u)\nabla u) + Q(u), \qquad t > 0,\ x \in \mathbb{R}^N$$

- A.A. Samarskii, S.P. Kurdyumov, M.I. Bakirova, V.A. Galaktionov, V.A. Dorodnicyn, G.G. Elenin, N.V. Zmitrenko, E.S. Kurkina, A.P. Mihailov, Y.P. Popov, A.B. Potapov, M.N. LeRoux, S. Svirshchevskii, H. Wilhelmsson,...

$$k(u) = u^{\sigma}, \qquad Q(u) = u^{\beta}, \qquad \sigma > 0,\ \beta > 1$$

- S.N. Dimova, M. Kaschiev, D. Vasileva, T. Chernogorova, M. Dimova

$\beta > \sigma + 1$	$\beta = \sigma + 1$	$\beta < \sigma + 1$
(LS regime)	(S regime)	(HS regime)

$$u_t = \frac{1}{r}\frac{\partial}{\partial r}\left(ru^\sigma \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \varphi}\left(u^\sigma \frac{\partial u}{\partial \varphi}\right) + u^\beta, \quad t > 0,$$
$$u(0, r, \varphi) = u_0(r, \varphi) \geq 0, \quad \sup u_0 < \infty$$

- S.R. Svirshchevskii 1985

Self-similar solution

$$u(t, r, \varphi) = g(t)\,\theta(\xi, \phi), \quad g(t) = (1 - \frac{t}{T_0})^{-\frac{1}{(\beta-1)}},$$
$$\xi = r(1 - \frac{t}{T_0})^{-\frac{\beta-\sigma-1}{2(\beta-1)}}, \quad \phi = \varphi + \frac{C_0}{\beta-1}\ln(1 - \frac{t}{T_0})$$

$T_0 > 0$ - blow-up time, C_0 - parameter of the family of solutions

$$r(t)e^{s\varphi(t)} = r(0)e^{s\varphi(0)} = \xi e^{s\phi} = \text{const}, \qquad s = \frac{\beta-\sigma-1}{2C_0}, \; C_0 \neq 0$$

$\theta(\xi, \phi) \geq 0$ – **self-similar function**

Self-similar problem

$$\mathcal{L}(\theta) \equiv -\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\theta^{\sigma}\frac{\partial\theta}{\partial\xi}\right) - \frac{1}{\xi^2}\frac{\partial}{\partial\phi}\left(\theta^{\sigma}\frac{\partial\theta}{\partial\phi}\right) + \frac{\beta-\sigma-1}{2}\xi\frac{\partial\theta}{\partial\xi} - C_0\frac{\partial\theta}{\partial\phi}$$
$$+\theta - \theta^{\beta} = 0, \qquad 0 < \xi < \infty,\ 0 \le \phi < 2\pi \quad T_0 = (\beta-1)^{-1}$$

$\theta_H^0 \equiv 0,\ \theta_H^1 \equiv 1$

$$\lim_{\xi\to 0} \xi\theta^{\sigma}\frac{\partial\theta}{\partial\xi} = 0, \qquad \frac{\partial\theta}{\partial\phi}(\xi,0) = \frac{\partial\theta}{\partial\phi}(\xi,2\pi) = 0,$$
$$\lim_{\xi\to\infty} \theta(\xi,\phi) = \theta_H^1 \equiv 1 \qquad \left(\lim_{\xi\to\infty} \theta(\xi,\phi) = \theta_H^0 \equiv 0\right)$$

Tasks:

- find a ”good” initial approximation to each of the solutions
- find asymptotics and pose suitable boundary conditions at ”computational infinity”
- construct a reliable iteration process, fast converging to the desired solution (corresponding to the initial approximation)

Linear approximations

$$\tilde{\theta}(\xi,\phi) = 1 + \alpha y(\xi,\phi), \quad |\alpha y| \ll 1, \quad \alpha = const$$

$$-\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial y}{\partial\xi}\right) - \frac{1}{\xi^2}\frac{\partial^2 y}{\partial\phi^2} + \frac{\beta-\sigma-1}{2}\xi\frac{\partial y}{\partial\xi} - C_0\frac{\partial y}{\partial\phi} + (1-\beta)y = 0$$

Particular solutions

$\beta = \sigma + 1$ $\quad y_k(\xi,\phi) = J_k(\sqrt{\sigma}\xi)\cos(k\phi), \quad k \in \mathbb{N}$

$\beta < \sigma + 1$ $\quad y_k(\xi,\phi) = \xi^k |{}_1F_1(a,b;z)| \cos(arg({}_1F_1(a,b;z)) + k\phi),$

$$a = -\frac{\beta-1+C_0 k\mathrm{i}}{\beta-\sigma-1} + \frac{k}{2}, \; b = 1+k, \; z = \frac{\beta-\sigma-1}{4}\xi^2, \quad k \in \mathbb{N}$$

$y_k(\xi,\phi)$ are $2\pi/k$ periodic functions and have k axes of symmetry

Asymptotics and boundary conditions

$$\tilde{\theta}_k(\xi,\phi) = 1 + \alpha y_k(\xi,\phi) - \text{initial approximations}$$

$\beta = \sigma + 1$

$$\theta_k(\xi,\phi) \sim 1 + \nu\sqrt{\frac{2}{\pi\sqrt{\sigma}\xi}}\cos\left(\sqrt{\sigma}\xi - \frac{k\pi}{2} - \frac{\pi}{4}\right)\cos(k\phi), \quad \xi \to \infty,$$

$$\nu = const$$

$$\frac{\partial\theta_k}{\partial\xi} = \frac{1-\theta_k}{2\xi} - \nu\sqrt{\frac{2}{\pi\sqrt{\sigma}\xi}}\sin\left(\sqrt{\sigma}\xi - \frac{k\pi}{2} - \frac{\pi}{4}\right)\cos(k\phi), \quad \xi = l \gg 1$$

$\beta < \sigma + 1$

$$\theta_k(\xi,\phi) \sim 1 + \gamma\xi^{1/m}\cos(k\phi + \frac{k}{s}\ln\xi + \mu), \qquad \xi \to \infty,\ \gamma,\mu = const$$

$$\frac{\partial\theta_k}{\partial\xi} = \frac{\theta_k - 1}{m\xi} - \frac{\gamma k}{s\xi^{(m-1)/m}}\sin(k\phi + \frac{k}{s}\ln\xi + \mu), \qquad \xi = l \gg 1$$

Numerical method

Continuous analog of Newton's method: $\mathcal{L}(\theta) = 0$

$$\mathcal{L}'(\theta(t))\frac{\partial\theta(t)}{\partial t} = -\mathcal{L}(\theta(t)), \quad \theta(\xi, 0) = \theta_0(\xi), \quad 0 < t < \infty$$

$$v(\xi, \phi, t) = \frac{\partial\theta(\xi, \phi, t)}{\partial t}$$

$$\mathcal{L}'(\theta_n)v_n = -\mathcal{L}(\theta_n),$$

$$\theta_{n+1} = \theta_n + \tau_n v_n, \qquad 0 < \tau_n \leq 1, \quad n = 0, 1, \ldots$$

$$\theta_n = \theta(\xi, \phi, t_n), \quad v_n = v(\xi, \phi, t_n)$$

$$\theta_0 = \tilde{\theta}_k(\xi, \phi) - \text{initial approximation}$$

- Galerkin FEM
- bilinear finite elements
- $A(\theta_n)\overline{V}_n = -B(\theta_n)\overline{\Theta}_n$; $A = LU$

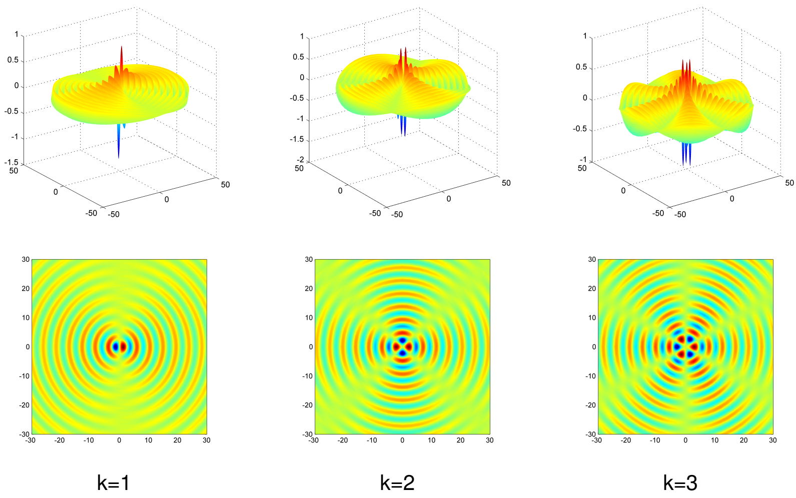
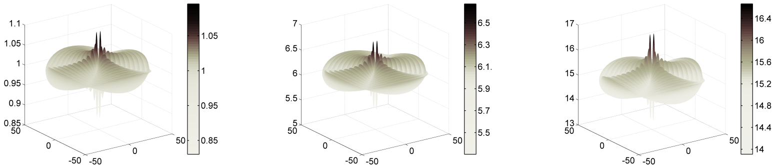
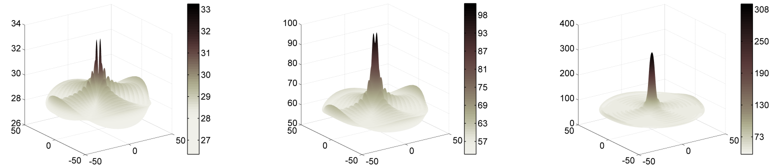


Figure: Self-similar functions $\theta(\xi, \phi)$, $\sigma = 2$, $k = 1, 2, 3$



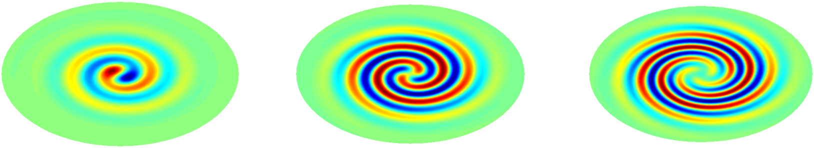
$t = 0$ $t = 0.486748$ $t = 0.497776$



$t = 0.499367$ $t = 0.499856$ $t = 0.499914$

Figure: Evolution of a complex wave in S-regime: $\sigma = 2$, $\beta = 3$, $C_0 = 0$, $k = 2$, $T_0 = 0.5$.

$\beta < \sigma + 1$, self-similar functions $\theta(\xi, \phi)$



$k = 1$ $k = 2$ $k = 3$

Figure: $\sigma = 3$, $\beta = 3.6$, $C_0 = 1$: One-armed spiral solution ($k = 1$); two-armed spiral solution ($k = 2$); tree-armed spiral solution ($k = 3$).

$\beta < \sigma + 1$, self-similar solution $u(t, r, \varphi)$

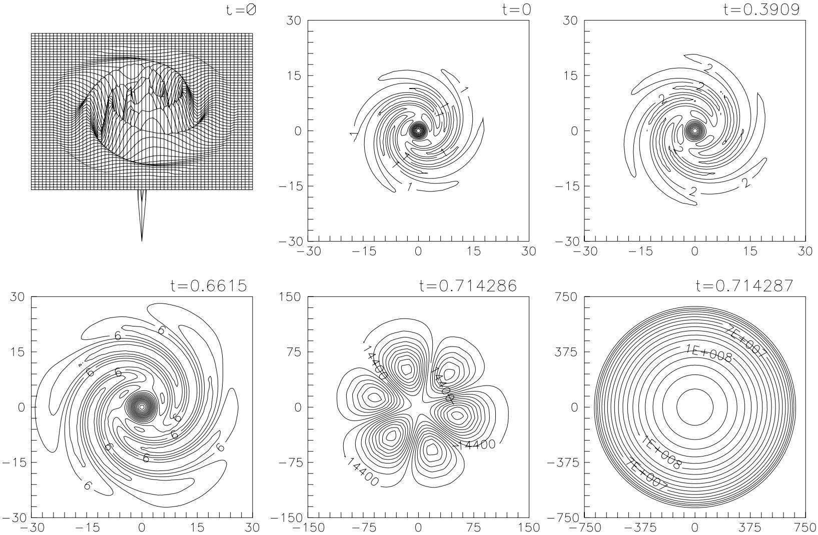


Figure: Evolution of three-armed spiral wave in HS-regime: σ = 2, $\beta = 2.4$, $C_0 = 1$, $k = 3$, $T_0 = 0.(714285)$.

Thank you
for your attention!