Comparison of Some Finite Difference Schemes for Boussinesq Paradigm Equation

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Outline

- Problem formulation
- Families of Finite Difference Schemes
- Numerical Method for 1D Boussinesq Paradigm Equation
- Numerical tests
- Discussion and open problems

Introduction

We consider the Cauchy problem for the Boussinesq Paradigm Equation (BPE)

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \beta_1 \Delta \frac{\partial^2 u}{\partial t^2} - \beta_2 \Delta^2 u + \alpha \Delta f(u), \; x \in \mathbb{R}^n, \; 0 < t \leq T < \infty,$$

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x),$$

$$u(x,t) \to 0, \quad \Delta u(x,t) \to 0 \;\text{ as }\; |x| \to \infty.$$

Here u is surface elevation, $\beta_1, \beta_2 \geq 0$, $\beta_1 + \beta_2 \neq 0$ are two dispersion coefficients, and α is an amplitude parameter. The nonlinear term $f(u)$ has a form $f(u) = u^p$, $p > 1$.

BPE first appears in the modeling of surface waves in shallow waters.

Numerical results for 1D BPE ($\beta_1 = 0$ or $\beta_2 = 0$)

- Bogolubsky, I.L., *Comput. Phys. Commun.* (1977)
- Manoranjan, V.S., Mitchell, A.R., Morris, J.Ll., *SIAM J. Sci. Stat. Comput.* (1984)
- Christov, C.I.and M. Velarde, *Int J. of Bifurcation and Chaos* (1994), *Proc. ICFD* V (1995)
- Bratsos, A.G., Physics Letter A (2002) *Numer. Algor.* (2007), *Chaos, Solitons & Fractals* (2009),
- Lin Q. , Y. H. Wu, R. Loxtona and S.Laib, *J. Comput. Appl. Math.* (2009)
- A.Shokri, M.Dehghan, *Comput. Phys. Comun.* (2010)

Numerical results for 2D BPE

- Chertock, A., Christov, C. I., Kurganov, A., *Computational Science and High Performance Computing IV*, NNFM (2011)
- Christov, C.I., Kolkovska, N., Vasileva, D., *LNCS* (2011)

Families of Finite Difference Schemes

We propose the following families of Finite Difference Schemes (FDS) for the BPE:

$$B\left(\frac{v_{ij}^{n+1} - 2v_{ij}^{n} + v_{ij}^{n-1}}{\tau^2}\right) - \Lambda v_{ij}^{n} + \beta_2 \Lambda^2 v_{ij}^{n} = \alpha \Lambda g(v^{n+1}, v^{n}, v^{n-1}),$$

$$B = I - (\beta_1 + \theta\tau^2)\Lambda + \theta\tau^2\beta_2\Lambda^2.$$

- v_{ij}^{n} – a discrete approximation to u at (x_i, y_j, t_n), τ is a time-step,
- $\Lambda = \Lambda^{xx} + \Lambda^{yy}$ – the standard five-point discrete Laplacian,
- $\Lambda^2 = (\Lambda^{xxxx} + 2\Lambda^{xxyy} + \Lambda^{yyyy})$ – the discrete biLaplacian,
- In approximations to Λv and $\Lambda^2 v$ we use the symmetric θ-weighted approximation to v_{ij}^{n}: $v_{ij}^{\theta,n} = \theta v_{ij}^{n+1} + (1-2\theta)v_{ij}^{n} + \theta v_{ij}^{n-1}$, $\theta \in \mathbb{R}$.

$$B\left(\frac{v^{n+1}-2v^n+v^{n-1}}{\tau^2}\right)-\Lambda v^n+\beta_2\Lambda^2 v^n=\alpha\Lambda g(v^{n+1},v^n,v^{n-1}), \qquad (1)$$

$g(v^{n+1},v^n,v^{n-1})$ – an approximation to the nonlinear term $f(u)$

Family 1: $g(v^{n+1},v^n,v^{n-1})=f(v^n), \qquad (2)$

Family 2: $g(v^{n+1},v^n,v^{n-1})=\dfrac{F(v^{n+1})-F(v^{n-1})}{v^{n+1}-v^{n-1}}, \qquad (3)$

Family 3: $g(v^{n+1},v^n,v^{n-1})=2\dfrac{F(0.5(v^{n+1}+v^n))-F(0.5(v^n+v^{n-1}))}{v^{n+1}-v^{n-1}}, \qquad (4)$

$$F(u)=\int_0^u f(s)\,ds, \qquad f(u)=u^p$$

- explicit with respect to the nonlinearity – **Family 1:** (1), (2)
- implicit with respect to the nonlinearity – **Family 2:** (1), (3) and **Family 3:** (1), (4)

Properties of the Families 1–3

• **Convergence:** We have proved that all schemes considered above have second order of convergence in space and time $O(h^2 + \tau^2)$.

• **Stability:** The schemes are unconditionally stable for $\theta > 1/4$. For $\theta < 1/4$ the schemes are conditionally stable. Thus, for $\theta = 0$ the schemes are stable provided $\tau^2 < \frac{4}{9}\frac{\beta_1}{\beta_2}h^2$.

• **Conservativeness:** For Families 2 and 3 it is proven that the discrete energy is conserved in time , i.e. $E_h(v^{(n)}) = E_h(v^{(0)}), \quad n = 1, 2, \ldots .$

$$\begin{aligned} E_h(v^n) = \quad & -\left\langle \Lambda^{-1} v_t^n, v_t^n \right\rangle + \beta_1 \left\langle v_t^n, v_t^n \right\rangle + \tau^2(\theta - 1/4) \left\langle (I - \beta_2\Lambda) v_t^n, v_t^n \right\rangle \\ & +1/4 \left\langle v^n + v^{n+1} + \beta_2 \Lambda (v^n + v^{n+1}), v^n + v^{n+1} \right\rangle + \tilde{E}_h, \end{aligned}$$

$$\tilde{E}_h(v^n) = \begin{cases} \alpha \left\langle F(v^{n+1}) + F(v^n), 1) \right\rangle & \text{for Family 2} \\ 2\alpha \left\langle F(0.5(v^{n+1} + v^n)), 1) \right\rangle & \text{for Family 3} \end{cases}$$

$v_t^n = (v^{n+1} - v^n)/\tau$

Samarsky, A.: The Theory of Difference Schemes. Marcel Dekker Inc., New York (2001)

$$B = I - (\beta_1 + \theta\tau^2)\Lambda + \theta\tau^2\beta_2\Lambda^2$$

$\tilde{B}$ - **factorized operator**

$$\tilde{B} = (I - \theta\tau^2\Lambda^{xx} + \theta\tau^2\beta_2\Lambda^{xxxx})(I - \theta\tau^2\Lambda^{yy} + \theta\tau^2\beta_2\Lambda^{yyyy})(I - \beta_1\Lambda)$$

- the factorized scheme can be reduced to a sequence of three simpler schemes provided appropriate boundary conditions
- the factorized scheme leads to an economic algorithm, i.e. algorithm with a linear complexity with respect to the number of nodes
- the first two discrete operators of $\tilde{B}$ depend only on one spatial variable

This justifies the need to study the properties of the proposed schemes in the one dimensional case. Our aim is to analyze the proposed FDS for 1D BPE in terms of their rate of convergence, accuracy as well as the energy preservation of the conservative schemes.

Numerical method for 1D BPE

For 1D BPE the families of FDS read

$$D\left(\frac{v_i^{n+1}-2v_i^n+v_i^{n-1}}{\tau^2}\right)-Qv_i^n=\alpha\Lambda g(v_i^{n+1},v_i^n,v_i^{n-1}),\quad i=1,N-1,\quad (5)$$

$$D=I-\beta_1\Lambda^{xx}-\theta\tau^2\Lambda^{xx}+\beta_2\theta\tau^2\Lambda^{xxxx},\qquad Q=\Lambda^{xx}-\beta_2\Lambda^{xxxx},$$

$$x\in[-L_1,L_2],\ x_i=-L_1+ih,\ h=(L_1+L_2)/N,\ i=0,\dots,N.$$

An $O(h^2+\tau^2)$ approximation to the initial conditions is given by

$$v_i^0=u_0(x_i),$$
$$v_i^1=u_0(x_i)+\tau u_1(x_i)+0.5\tau^2D^{-1}\left(Q(u_0)+\alpha(f(u_0))_{xx}\right)(x_i).$$

We consider (5) subject to the following boundary conditions

$$v_0^{n+1}=v_N^{n+1}=0,\qquad v_{xx,0}^{n+1}=v_{xx,N}^{n+1}=0.$$

The nonlinear **Families 2 and 3** are linearized using "successive iterations".

Algorithm:

- Evaluate $v^{(0)}$, $v^{(1)}$ from the initial conditions
- For $n = 2, 3, \ldots$
  - take $v^{(n+1)[0]} = v^{(n)}$
  - for $k = 1, 2, \ldots$ obtain $v^{(n+1)[k+1]}$ from

$$D\left(\frac{v^{(n+1)[k+1]} - 2v^{(n)} + v^{(n-1)}}{\tau^2}\right) - Qv^{(n)} = \alpha\Lambda g(v^{(n+1)[k]}, v^{(n)}, v^{(n-1)})$$

  - if $\left|\frac{v^{(n+1)[k+1]} - v^{(n+1)[k]}}{v^{(n+1)[k]}}\right| < \varepsilon$ then set $v^{(n+1)} = v^{(n+1)[k+1]}$

The resulting systems of linear algebraic equations are five-diagonal with constant matrix coefficients. To solve them we apply a special kind of non monotonic Gaussian elimination with pivoting. (Samarsky, A.A., Nikolaev, E.: Methods for Solving Grid Equations.(1978) (in Russian)

Numerical tests

The aim of the numerical tests are to study:

- convergence
- accuracy
- stability for different nonlinearities $f(u) = u^p$, $p > 1$

Yan,Z., Bluman, G.: Comp. Phys. Commun. 149, 11–18 (2002)

It can be shown that the 1D BPE admits a one parameter family of soliton solutions given by

$$u^s(x,t;c) = \left[\frac{(c^2-1)(p+1)}{2\alpha}\operatorname{sech}^2\left(\frac{1-p}{2}\sqrt{\frac{c^2-1}{\beta_1 c^2-\beta_2}}(x-ct)\right)\right]^{\frac{1}{p-1}}, \quad p \neq 1,$$

where c is a phase speed of the localized wave.

(i) Propagation of a Solitary Wave: In this case we consider the following initial data

$$u(x,0) = u^s(x,0;c) \qquad u_t(x,0) = u_t^s(x,0;c).$$

It represents a single soliton at the initial time moment located at $x = 0$ that is then allowed to evolve according to the BPE.

(ii) Interaction of Two Solitary Waves: The following initial conditions:

$$u(x,0) = u^s(x + x_0^1, 0; c_1) + u^s(x + x_0^2, 0; c_2)$$
$$u_t(x,0) = u_t^s(x + x_0^1, 0; c_1) + u_t^s(x + x_0^2, 0; c_2),$$

are used. At the initial time we take the superposition of two solitons initially located at points $x = x_0^1$ and $x = x_0^2$ and traveling one against another with speeds c_1 and c_2.

Numerical experiments are conducted for $p = 2, 3, 4$,and 5 $(f(u) = u^p)$ and for variety values of the parameters β_1, β_2, α, and c.

Propagation of a Solitary Wave

$$u^s(x,t;c) = \sqrt{\frac{2(c^2-1)}{\alpha}}\,\text{sech}\left(\sqrt{\frac{c^2-1}{\beta_1 c^2-\beta_2}}(x-ct)\right), \qquad f(u) = u^3$$

$$u(x,0) = u^s(x,0;c), \qquad u_t(x,0) = u^s_t(x,0;c)$$

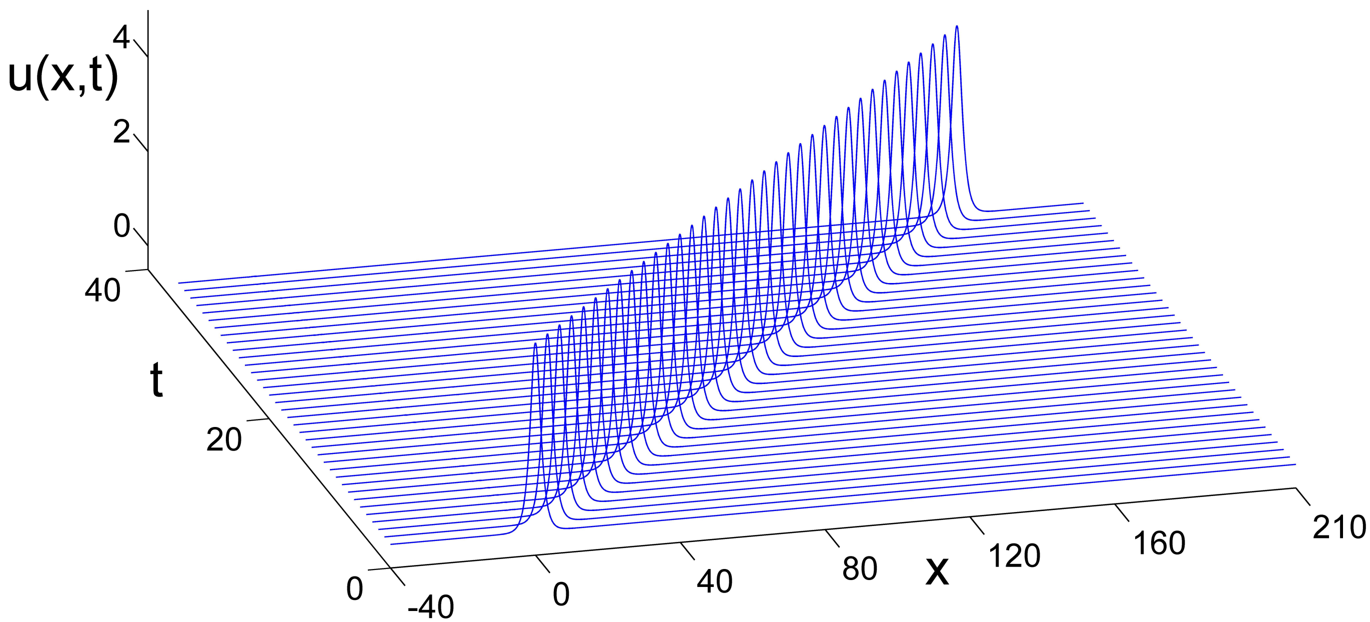

Figure 1: One solitary wave, $f(u) = u^3$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $c = 5$, $0 \le t \le 40$.

Table 1: One solitary wave solution, $f(u) = u^3$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $c = 2$, $x \in [-40, 120]$, $\theta = 0.5$, $\varepsilon = 10^{-13}$, $T = 40$.

$h = \tau$	Rate κ			Error R		
	Family 1	Family 2	Family 3	Family 1	Family 2	Family 3
0.2	–	–	–	0.0944117	0.6079979	0.3483821
0.1	2.05	1.72	1.88	0.0227430	0.1841944	0.0949054
0.05	2.01	1.94	1.97	0.0056362	0.0480465	0.0242024
0.025	2.00	1.99	1.99	0.0014057	0.0121329	0.0060802
0.0125	2.01	1.99	2.00	0.0003487	0.0030431	0.0015243
0.00625	2.89	1.93	1.86	0.0000472	0.0008006	0.0004205

$$\kappa = \log_2\left(\frac{||u^s - v_{[h]}||}{||u^s - v_{[h/2]}||}\right), \quad R = ||u^s - v_{[h]}||, \quad E_{rel} = \max_{0\le k\le n} \frac{|E_h(v^k) - E_h(v^0)|}{E_h(v^0)}$$

- The calculations confirm the second rate of convergence $O(h^2 + \tau^2)$
- Family 1 is almost 9 times more precise than Family 2, and 4.5 times more precise than Family 3.
- The conservative Family 3 is about 2 times more precise than Family 2.
- $h = \tau = 0.025$, $E_{rel} \approx 1 \times 10^{-9}$ at $T = 40$ for both conservative Family 2 and Family 3

Interaction of Two Solitary Waves

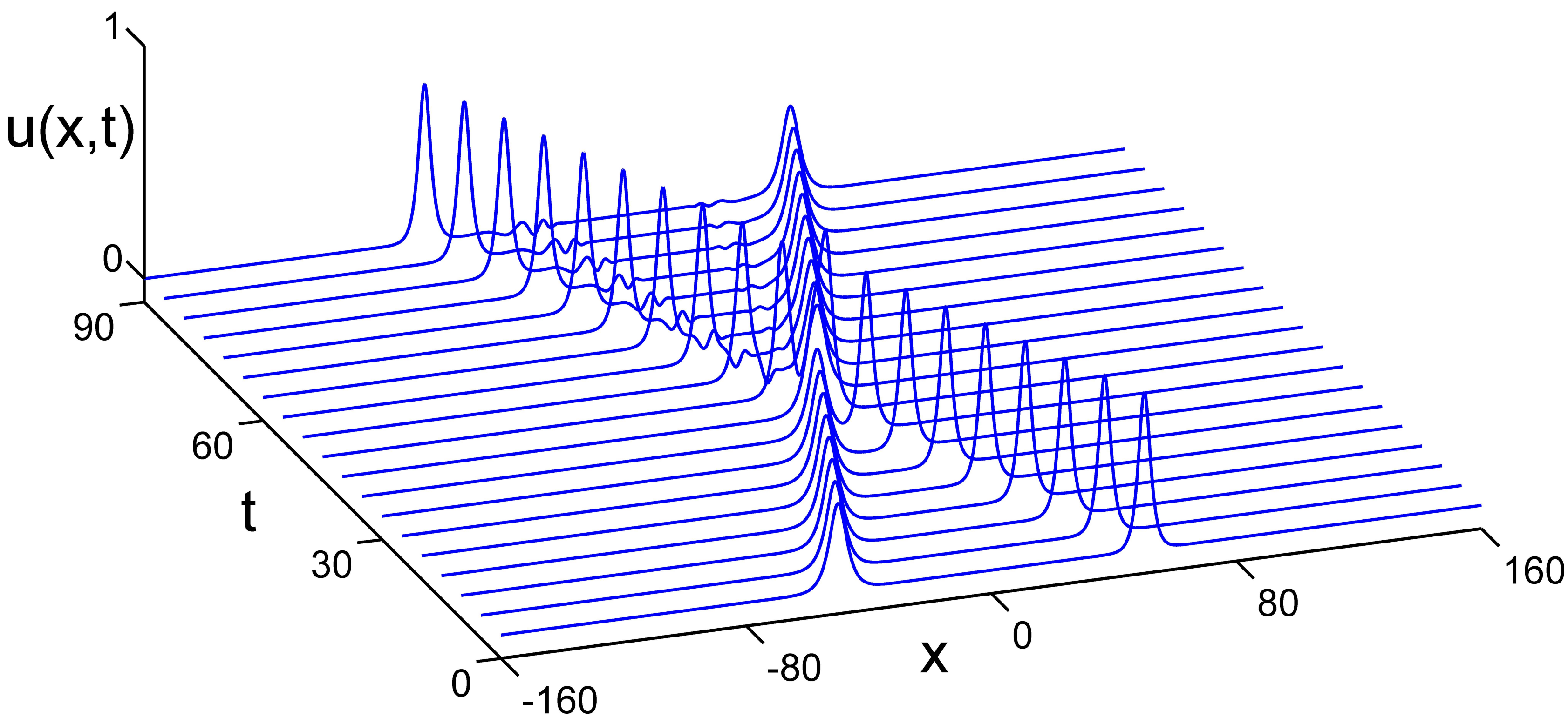

Figure 2: Interaction of two solitary waves $f(u) = u^3$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $c_1 = 1.1$, $c_2 = -1.3$, $0 \leq t \leq 80$.

Table 2: Interaction of two solitary waves, $f(u) = u^3$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $c_1 = 1.1$, $c_2 = -1.3$, $x \in [-160, 160]$, $\theta = 0.5$, $\varepsilon = 10^{-13}$, $T = 80$.

$h = \tau$	Rate κ			Error R		
	Family 1	Family 2	Family 3	Family 1	Family 2	Family 3
0.2						
0.1	1.97	1.90	1.94	0.0197330	0.1038638	0.0663246
0.05	1.99	1.98	1.99	0.0050042	0.0273100	0.0170770
0.025	2.04	2.00	2.01	0.0012437	0.0069027	0.0042895

$$\kappa = \log_2\left(\frac{||u_{[h]} - u_{[h/2]}||}{||u_{[h/2]} - u_{[h/4]}||}\right), \quad R = \frac{(||u_{[h]} - u_{[h/2]}||)^2}{||u_{[h]} - u_{[h/2]}|| - ||u_{[h/2]} - u_{[h/4]}||}$$

$$E_{rel} = \max_{0 \le k \le n} \frac{|E_h(v^k) - E_h(v^0)|}{E_h(v^0)}$$

- The calculations confirm the second rate of convergence $O(h^2 + \tau^2)$
- Family 1 is about 5 times more precise than Family 2 and 3 times more precise than Family 3
- The conservative Family 3 is about 1.6 times more precise than Family 2
- $h = \tau = 0.025$, $E_{rel} \approx 1 \times 10^{-8}$ at $T = 80$ for both conservative Family 2 and Family 3

Discussion and open problems

- The calculations confirm that all families of schemes are of order $O(h^2 + \tau^2)$.

- With respect to the error magnitude Family 1 performs much better than Family 2 and Family 3. One may expect that this is due to the smoothness of the initial data. If the initial data are not smooth enough, then the conservative family of schemes should be chosen. Family 3 is preferable since it has higher accuracy than Family 2.

- The numerical tests demonstrate a high reliability of the proposed schemes for the most studied case of quadratic nonlinearity $f(u) = u^2$. Our experience dealing with various nonlinearities confirms expectations of different phenomenology in the cases $f(u) = u^{2p+1}$ and $f(u) = u^{2p}$, $p \geq 1$ (global existence and blow-up for a finite time of the solution).

- The studied families of FDS could be efficiently applied for the 2D Boussinesq Paradigm Equation.

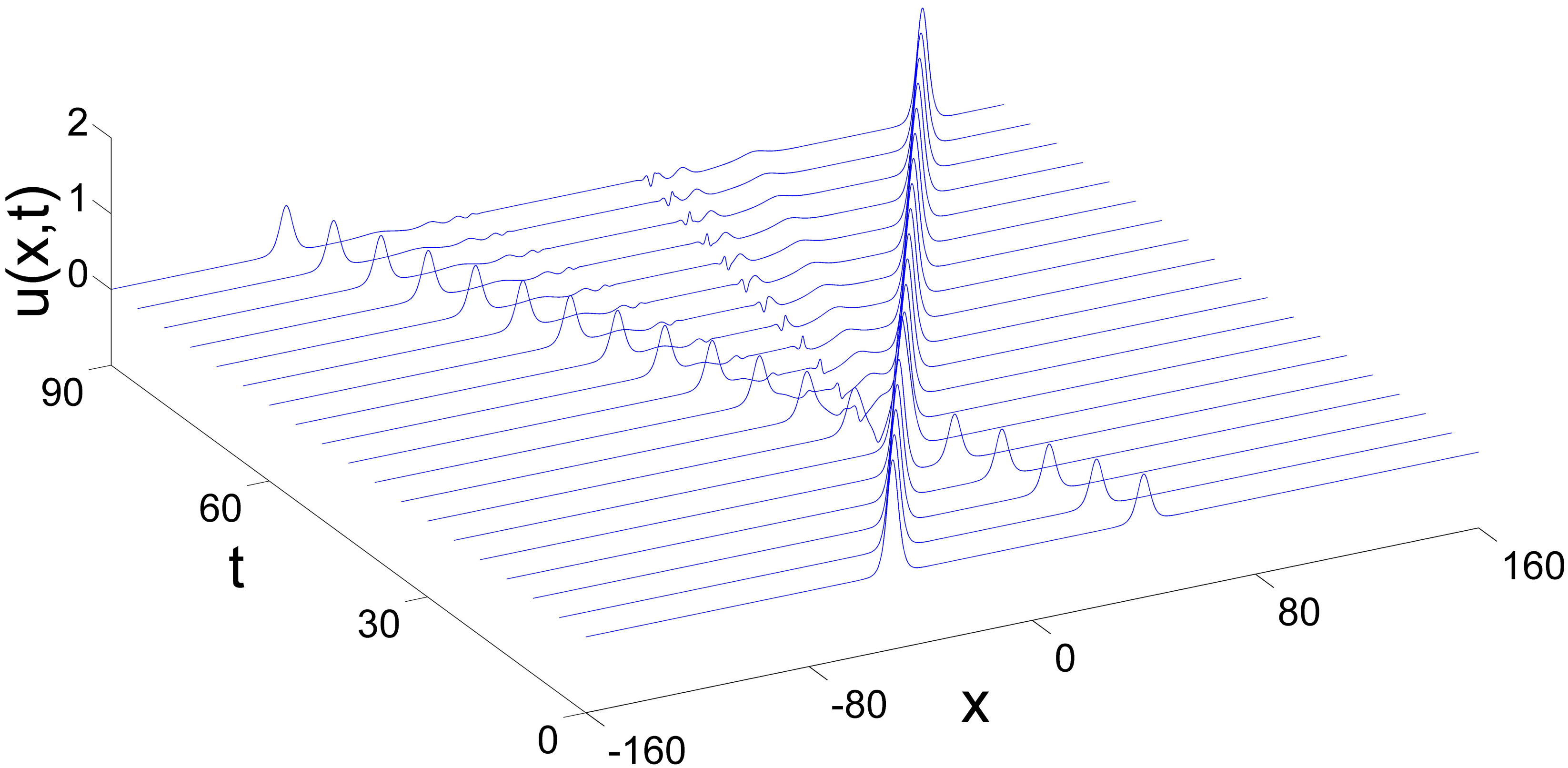

Figure 3: Iteraction of two solitons: $f(u) = u^2$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $c_1 = 2$, $c_2 = -1.5$.

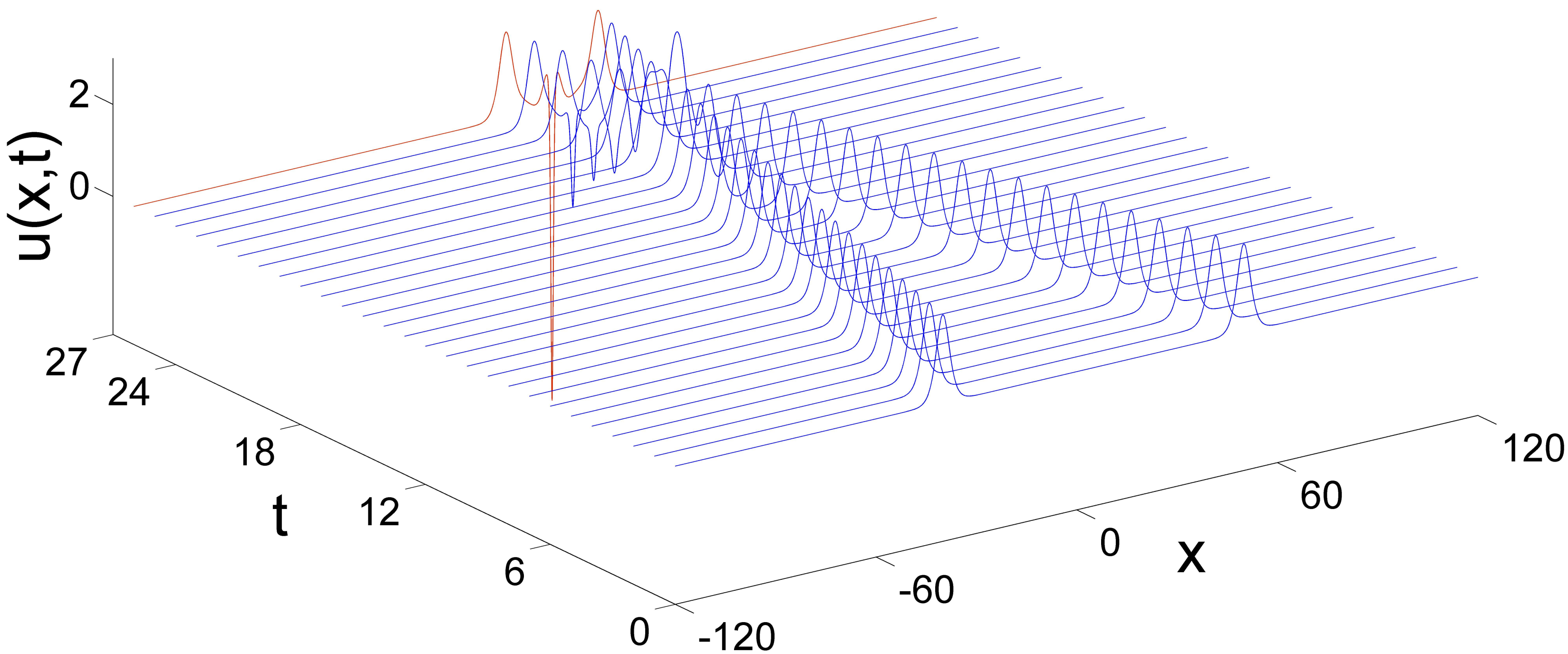

Figure 4: Iteraction of two solitons: $f(u) = u^2$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $c_1 = -c_2 = -2.2$, $t^* \approx 27$, t^* - blow up time.

Theorem: [Convergence of the Schemes] *Let $f(u) = u^n, n \geq 2$ and the parameter θ satisfies*

$$\theta > \frac{1}{4} - \frac{\beta_1}{\tau^2||I - \beta_2\Lambda||}. \tag{6}$$

Assume that the solution u of BPE obeys $u \in C^{4,4}\left(\mathbb{R}^2 \times (0,T)\right)$ and the solution v of the finite difference scheme is bounded in the maximal norm. Let M be a constant such that

$$M \geq \max_{i,j,s \leq k}\left(|u(x_i, y_j, t_s)|, \left|\frac{\partial^2 u}{\partial t^2}(x_i, y_j, t_s)\right|, |v_{i,j}^{(s)}|\right)$$

and τ be sufficiently small, $\tau < (2M)^{-1}$. Then v converges to the exact solution u as $|h|, \tau \to 0$ and the following estimate in the uniform norm holds for the error $z = v - u$:

$$\max_i |z_i^{(k)}| < Ce^{Mt_k}\left(|h|^2 + \tau^2\right), \qquad d = 1;$$

$$\max_{i,j} |z_{i,j}^{(k)}| < Ce^{Mt_k}\sqrt{\ln N}\left(|h|^2 + \tau^2\right), \qquad d = 2.$$