Review

On a competition for an academic position "professor", in the area of higher education 4. Natural sciences, mathematics and informatics, professional field 4.5 Mathematics, scientific specialty "Mathematical analysis" (Applications of fractional calculus).

by Prof. DSc. Tsviatko Rangelov, Member of the Scientific Jury, appointed by order N:206 /16.07.2024 of the Director of Institute of Mathematics and Informatics, BAS, approved by the Scientific Council on 31.05.2024 (Protocol N: 7).

1) The competition, with a term of 2 months, is announced in the State Gazette no. 43 of 17.05.2024, for the needs of the Institute of Mathematics and Informatics (IMI), BAS. Associate Prof. DSc. Emilia Grigorova Bazhlekova has submitted documents for participation in it. She graduated from the Faculty of Mathematics and Informatics, Sofia University, with MSC in mathematics in 1986. In 2001 she defended her PhD thesis "Fractional Evolution Equations in Banach Spaces" at Technical University of Eindhoven, the Netherlands. In 2022 she defended her thesis "Subordination principle for generalized fractional evolution equations" for Doctor of Mathematical Sciences at IMI, BAS. From 1989 to 1993 she was PhD student and a mathematician (1995 - 2004) at the Department of Complex Analysis, IMI-BAS, a mathematician and Ass. Prof. at the Department of Analysis, Geometry and Topology, IMI-BAS (2011 - 2014) and Associate Professor at the same department since 2014.

2) The scientific activity of Assoc. Prof. E. Bazhlekova is in the field of differential equations with fractional derivatives, special functions of fractional calculus, convolutional calculus and applications. There are presented a total list of 57 publications, 22 of which for participation in the competition.

All articles for participation in the competition are published since 2014, in renowned mathematics and applied mathematics journals and in proceedings of international conferences as follows:

Fract. Calc. Appl. Anal. - 3; J. Comput. Appl. Math. - 1; Int. J. Appl. Math. - 1; Fractal Fract. - 4; AIP Conf. Proc. - 6; Comp. Math. Appl. - 1; Numer. Math. -1; Biomath -1; J. Theor. Appl. Mech. -1; J. Ineq. Spec. Funct. - 1; Serdica Math. J. -1; Compt. Rend. Acad. Bulg. Sci. -1.

With Impact Factor (IF) are 11 publications [1, 4, 7, 10, 15, 16, 18 - 22], with SJR are 7 publications [3, 5, 11 - 14, 17] and 4 publications are without IF/SJR [2, 6, 8, 9]. 18 of the publications are co-authored with: I. Bazhlekov, S. Pchenichnov, D. Vasileva, B. Jin, Z. Zhou, R. Lazarov, K. Tsocheva and 4 are independent. Considering the results, obtained in the papers, I am accepting that the contribution of Acoss. Prof. E. Bazhlekova, in joint publications, is essential. Also, the results of none of the above 22 publications, are included in the DSc thesis of Assoc. Prof. E. Bazhlekova in 2022 or in the materials for her procedure for associate professor in 2014.

The publications of Assoc. Prof. E. Bazhlekova are cited more than 460 times (Scopus Datebase) since 2014. Among them a list of 220 citations (without self-citations) of the publications for the competition is presented. It is clear that the requirements of Art. 3 (1), 4. of IMI Rules (for at least 12 publications with IF or SJR) are met.

In connection with Art. 2 of the IMI Rules for the "minimum required score by set of indicators" for the candidate Assoc. Prof. E. Bazhlekova the following is obtained: A - 50 points; B - 134 points; G - 552 points; D - 336 points; E - 205 points, which means that the requirement is fulfilled.

3) The author's report correctly reflects the content and contributions in the works of Assoc. Prof. E. Bazhlekova, presented for the competition.

The submitted works for participation in the competition could be distributed in the following groups:

3a) Investigation of solutions of fractional differential equations and Duhamel-type representations.

3b) Linear visco-elastic models with fractional derivatives.

3c) Inverse problems for equations with fractional derivatives.

3d) Application of fractional calculus in complex physics processes and development of numerical methods for their solution.

Before I analyze the works grouped in the above groups, a would like to share few words about the theme of the competition. Problems with fractional derivatives and applications are a contemporary area of analysis and differential equations. In the last years this domain is a subject of investigation not only by mathematicians, but also by scientists in the field of mechanics and physics, due to their applications in continuum media, viscoelasticity, system stability etc. Although the operators with fractional derivatives of order α with positive integer α are differential operators, the processes being modeled with them as well as the methods for solution of Cauchy problems and of boundary-value problems are completely different, mainly because of the nonlocal character of the equations with fractional derivatives. This demands the development of new methods, approaches and principles in their research.

3a) In this group are the works [1 - 3, 9, 12, 13], where [1] is in Q1.

In [1] the following initial boundary value problem is studied

$$\begin{cases} (1 + aD_t^\alpha)u_t = \mu(1 + bD_t^\beta)\Delta u + F(x,t), \quad x \in \Omega, t \in (0,T], \\ u(x,t) = 0, \quad x \in \partial\Omega, t \in [0,T], \\ u(t,x) = f(x), \quad x \in \bar{\Omega}, \end{cases} \tag{1}$$

where $0 < \alpha, \beta < 1$, $a, b \geq 0$, $\mu > 0$ and D_t^γ is Riemann-Liouville fractional derivative of order γ:

$$D_t^\gamma f(t) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\gamma} \, d\tau, \quad 0 < \gamma < 1.$$

Such problem describes the distribution of the velocity for visco-elastic flow with Oldroyd-B fractional constitutive model. This model covers large class of fluids as Newtonian, Maxwell model with fractional derivatives etc. New results are obtained by using the proposed of I. Dimovski (1990) convolution calculus, as well as the Duhamel-type representation for solutions of (1). Presented are also numerical examples and the method is compared with the applied by other authors solutions of (1) with finite difference method. This paper is cited 51 times, that shows the interest in the research of Assoc. Prof. E. Bazhlekova.

Works [3, 12, 13] are on the study of the equation in (1), as in [3, 12] a more general abstract form is studied, in [13] at $\alpha = \beta$ and $F = 0$ properties of its fundamental solution are investigated.

An interesting work is [9], in which a one-dimensional initial-boundary problem for distributed order diffusion equation is considered

$$\begin{cases} \partial_t^{|\mu|} u(x,t) = u_{xx}(x,t), \quad x \in (0,1), t > 0, \\ u(0,t) = u(1,t) = 0, \quad t \geq 0, \\ u(x,0) = f(x), \quad x \in [0,1], \end{cases} \tag{2}$$

where $f \in C^2([0,1]), f(0) = f(1)$ and $\partial_t^{|\mu|}$ is distributed order fractional derivative, defined as

$$\partial_t^{|\mu|} = \int_0^1 \mu(\beta)\partial_t^\beta d\beta, \quad \partial_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} f'(\tau) d\tau, \quad \beta \in (0,1),$$

is Caputo fractional derivative and $\mu \in C([0,1]), \mu \geq 0$. The solution of (2) is found by applying Duhamel‘s representation as convolution of a particular solution and the initial function.

It is worth mentioning that in works [1], [2] and [9] the convolutional calculus of I. Dimovski (1990) is applied, when obtaining new results for initial boundary value problems with fractional derivatives.

3b) Works [7, 8, 10, 19, 20] belong to this group, of which [10, 19, 20] are in Q1.

Characteristic of this group of publications is the examination of various linear constitutive laws, as generalizations of classical ones. The viscoelastic models are determined by the dependence between stress $\sigma(x,t)$ and strain $\varepsilon(x,t)$.

In the mechanics of continuous media, very important are constitutive laws of Maxwell, Jeffreys, Burgers, Zener. In recent years, due to the numerous applications, their fractional generalizations are studied. For example:

- fractional model of Maxwell $(1 + aD_t^\alpha)\sigma(x,t) = bD_t^\beta\varepsilon(x,t), 0 < \alpha \leq \beta \leq 1, a, b > 0$, to which work [8] is devoted;
- fractional model of Jeffreys $(1 + aD_t^\alpha)\sigma(x,t) = (1 + bD_t^\beta)\dot{\varepsilon}(x,t), 0 < \alpha, \beta < 1, a, b > 0$ to which work [10] is devoted;
- fractional Burgers‘ constitutive equation $(1 + a_1D_t^\alpha + a_2D_t^{2\alpha})\sigma(x,t) = (1 + b_1D_t^\beta + b_2D_t^{2\beta})\dot{\varepsilon}(x,t), 0 < \alpha, \beta \leq 1, a_j, b_j > 0$, to which work [7] is devoted;
- generalized fractional Zener model $(1+aD_t^{(k)})\sigma(x,t) = (1+bD_t^{(k)})\varepsilon(x,t), a, b > 0$ to which works [19, 20] are devoted. Here $D_t^{(k)}$ is a generalized convolutional derivative in the Riemann-Liouville sense with kernel $k(t)$.

In the mentioned works is studied the question of the conditions, that satisfies the modulus of relaxation $G(t)$ defined by the equality

$$\sigma(x,t) = \int_0^t G(t-\tau)\dot{\varepsilon}(x,\tau)d\tau, \quad t > 0.$$

For the fractional derivative model to make physical sense, the conditions $G(t) \geq 0, G'(t) \leq 0$, $t > 0$. must be fulfilled. In the works of this group necessary conditions are found for the fulfillment of this property, as well as sufficient conditions for a stronger property $(-1)^nG^{(n)}(t) \geq 0$, $t > 0, n \in \mathbb{N}$.

3c) To this group belong works [15, 16, 22], all three being in Q1.

In works [15] and [16] an inverse problem for determining the function $h(x)$ and the solution $u(x,t)$ is investigated for a given function $g(x)$ in the initial-boundary value problem

$$\begin{cases} D_t^{(k)} u(x,t) = u_{xx}(x,t) + h(x), \quad x \in (0,1), t \in (0,T), \\ u_x(0,t) = u_x(1,t) = 0, u(1,t) = 0, \quad t \in (0,T], \\ u(x,0) = 0, u(x,T) = g(x), \quad x \in [0,1], \end{cases} \tag{3}$$

where $g(x), h(x) \in L^2[0,1]$ and

$$(D_t^{(k)}) f(t) = \frac{d}{dt} \int_0^t k(t-\tau) f(\tau) - k(t) f(0), \quad t > 0, k(t) \geq 0,$$

is Caputo‘s generalized fractional derivative. In [15] problem (3) is studied in the space of continuous functions, while in [16] - in Sobolev spaces. In [22], more complex problem from (1) is investigated, with the principle of subordination, developed by Assoc. Prof. Bazhlekova in her DSc dissertation in 2022.

It should be noted that problems of the type (3) are complex, subject to intensive research, because of their connection with anomalous diffusion processes. The studies of Assoc. Prof. E. Bazhlekova on problem (3) are recognized by colleagues, which is evident from the fact, that paper [15] published in 2021 has already been cited 26 times.

3d) Works [4 - 6, 14, 17, 18, 21] belong to this group, with [4, 21] being in Q1.

In general, the publications are on the analysis of numerical methods for solving the complex boundary value problems, modeling by the use of fractional derivatives processes from physics and mechanics. For example, a generalized Rayleigh-Stokes model is considered in [4] for generalized fluid with fractional derivatives, set with the initial-boundary problem

$$\begin{cases} \partial_t u - (1 + \gamma \partial_t^\alpha) \Delta u = f, \quad x \in \Omega, t \in (0,T), \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T], \\ u(x,0) = v(x), \quad x \in \Omega, \end{cases} \tag{4}$$

where $\gamma > 0$, ∂_t^α, $\alpha \in (0,1)$ is the Riemann-Liouville fractional derivative of order α, defined in 3a). For problem (4), a semi-discrete Finite Element Method of Galerkin-type and an optimal error estimate is obtained. This is the most cited work - 98 times.

In [21] for a generalization of the Jeffreys model with fractional derivatives, in a more complex form than the equation in (4), the principle of subordination is applied and a result is established for the subordination with respect to suitable evolution equations, with integer order derivatives.

Various applications of fractional calculus, for modeling complex processes in natural science, are the subject of the remaining articles [5, 6, 14, 17, 18] in this group.

4) After 2014, Assoc. Prof. E. Bazhlekova actively participated in scientific projects: 3 national (DFNI I02/9 (2014-2017); DO1-205/23.11.18 (2018-2021); BG05M2OP001-1.001-00 (2018-2023)) and 5 international (KP-06-Russia/5 (2020-2023); under EBR - BAS-SANI (2012-2016, 2017-2019, 2020-2023), BAS-OIJI (2020-2022).

5) I have no critical notes. I know Assoc. Prof. E. Bazhlekova as a hardworking expert in an contemporary field of mathematics.

6) Conclusion: I believe that Assoc. Prof. DSc. E. Bazhlekova completely satisfies the requirements of ZRASRB as well as the Regulations of BAS and IMI-BAS for the position of professor. Also, there is no plagiarism in the papers submitted for the competition.

I recommend the Scientific Jury to propose to the Scientific Council of the Institute of Mathematics and Informatics, BAS, to select Assoc. Prof. DSc. Emilia Bazhlekova for Professor in the professional field 4.5 Mathematics, scientific specialty "Mathematical Analysis" (Applications of fractional calculus).

September 4, 2024 Signature

T. Rangelov