Institute of Mathematics and Informatics
Bulgarian Academy of Sciences

Functions Holomorphic over Finite-Dimensional Commutative Associative Unital $\mathbb{C}$-Algebras – Some Local, Global, and Universal Aspects

by

Marin Vladimirov Genov

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To my Mom, whom I dearly miss, and
to my Dad for his unconditional support.

Declaration of Originality

I hereby declare that this thesis was written in its entirety by myself, that the work contained herein is my own, except where explicitly indicated otherwise, and that this work has not been submitted for any other degree.

Acknowledgements

In memory of my mother, who shaped my relationship with mathematics.

Part of the results presented in this thesis originated in one form or another during my M.Sc. studies at the Faculty of Mathematics and Informatics of Sofia University some years ago under the supervision of Corresponding Member of BAS Emil Horozov, who sadly passed away in January this year. He gave me the opportunity and freedom to dug into this topic at a key time for me, and without his wisdom and encouragement this undertaking would likely not have seen fruition past its infancy. The present thesis is thus also dedicated to his memory. I am also thankful to Prof. A. Kasparian and Acad. S. Ivanov for the numerous stimulating conversations during that time at FMI and I am deeply grateful to Acad. V. Drensky for his continuous support and indispensable guidance over the last few years at IMI-BAS. Last but not least, I am indebted to Prof. Velichka Milousheva for her never ceasing moral support and level-headedness in all matters and for helping me gracefully navigate the administrative side of this endeavour quite literally since the very first moment I set foot in IMI-BAS till this very day. All mistakes are mine and mine alone.

Abstract

Let $\mathcal{A}$ be a finite-dimensional commutative associative unital $\mathbb{K}$-algebra, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and ${}_{\mathcal{A}}M$ a finitely generated $\mathcal{A}$-module. In the present thesis we generalize the *core theory* of (vector-valued) *holomorphic* functions of a single variable to the setting of functions $f\colon U \to {}_{\mathcal{A}}M$, $U \subseteq \mathcal{A}$ open, having an $\mathcal{A}$-linear differential Df, where the role of $\mathbb{C}$ is thus now assumed by $\mathcal{A}$ and the complex vector space in the codomain is accordingly replaced by the module ${}_{\mathcal{A}}M$. We make a point of distinguishing between two main groups of foundational results – those that do not actually require the presence of a "full" complex structure and follow already from partial properties such as commutativity, and those for which the presence of an underlying complex structure is essential (e.g. requiring path-connectedness of $\mathbb{K}^{\times}$). For $\mathbb{K} = \mathbb{C}$ this yields a function theory of several complex variables that is nonetheless very similar to single-variable complex analysis in more ways than one. Where such (partial) generalizations already exist in the research literature, our versions are not only more general but oftentimes also much stronger or even maximal – in some precise narrow sense – for the respective special cases, in part due to our introduction of a *spectral* point of view into the general theory. Furthermore, an organic part of our investigation is the study of the interplay between analysis on the one hand and elementary algebraic and categorical aspects of the involved objects on the other as they come naturally organized in various distinct categories. Finally, of some special interest is the case of representations arising from morphisms $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ as this gives a "functorial-like" behavior of the relevant regularity classes.

Contents

Notations & Conventions

Given a direct sum decomposition, say $V \oplus W$, we will write $v \oplus w$ to emphasize that the sum $v + w$ is in fact with respect to said decomposition, in particular $v \in V$ and $w \in W$. In the same vein we will write $\bigoplus_R$ and $\oplus_R$ to specify over which ring the direct sum holds (for example, the direct sum decomposition may hold only after a restriction of scalars). This is actually completely consistent with the usage $\bigotimes_R$ and $\otimes_R$.

The spectral norm on matrices will be denoted by $\|\cdot\|_{\mathrm{sp}}$ instead of the rather standard $\|\cdot\|_2$ to avoid confusion with the Euclidean norm (matrices are vectors after all).

By abuse of notation, we will use the same symbol to mean both a curve $\gamma\colon [0,1] \to X$ as a continuous mapping and its image. More generally, we will use the same symbol to denote both a cycle Γ and its support, for example $\int_\Gamma \omega$. Furthermore, loops $\gamma\colon \mathbb{S}^1 \to X$ will always be parametrized as $\gamma(t)$ with $t \in [0,1]$ for the sake of notational simplicity.

In the presence of tensors, we will try to always denote free indices by i, j, k, ℓ and summation indices by r, s, t, unless we forget to do so.

For open sets $U_i, U_j, U_k, \dots$ we put $U_{ij} =: U_i \cap U_j$, $U_{ijk} =: U_i \cap U_j \cap U_k$ etc.

All rings are assumed associative unless explicitly stated otherwise.

$\mathbb{N}$: **the natural numbers** $\{1, 2, 3, \dots\}$.

$\mathbb{N}_0 \coloneqq \mathbb{N} \cup \{0\}$.

$[n] \coloneqq \{1, \dots, n\} \subset \mathbb{N}$ for some $n \in \mathbb{N}$.

$\mathcal{P}(S)$: power set of some $S \in \mathsf{Set}$.

$\mathcal{P}X$: path space

$\mathcal{L}X$: loop space

$\mathcal{L}^k_R(M, N)$: space of k-times R-multilinear maps $M^k \to N$ for $M, N \in \mathsf{Mod}(R)$, $R \in \mathsf{CRing}$.

$\mathrm{Sym}^k_R(M, N)$: space of *symmetric* k-times R-multilinear maps $M^k \to N$ for $M, N \in \mathsf{Mod}(R)$, $R \in \mathsf{CRing}$.

$\subset\subset$: compactly contained.

$\mathrm{Dom}(f)$: domain of a map or a morphism $f\colon X \to Y$, i.e. $\mathrm{Dom}(f) \stackrel{\text{def}}{=} X$.

$\mathrm{Codom}(f)$: codomain of a map or a morphism $f\colon X \to Y$, i.e. $\mathrm{Codom}(f) \stackrel{\text{def}}{=} Y$.

$\mathrm{Im}(f)$: image of a map $f\colon X \to Y$, i.e. $\mathrm{Im}(f) \stackrel{\text{def}}{=} f(X)$.

$\Re(z)$: real part of a complex quantity $z \in \mathbb{C}$, $\mathbb{C}^n$, $\mathrm{M}_n(\mathbb{C})$ etc.

$\Im(z)$: imaginary part of a complex quantity $z \in \mathbb{C}$, $\mathbb{C}^n$, $\mathrm{M}_n(\mathbb{C})$ etc.

$\times_{i=1}^n x_i \stackrel{\text{def}}{=} (x_1, \dots, x_n)$: a shorter notation for an element of the Cartesian product $\prod_{i=1}^n X_i$.

$\times_{i\in I} f_i\colon \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$: the Cartesian product of maps $f_i\colon X_i \to Y_i$, $i \in I$.

$R \in \mathsf{Ring}$: a ring, usually unital and commutative, object of Ring (see below).

$R > 0$: some positive radius.

$R^\times$: the group of units (multiplicative group) of $R \in \mathsf{Ring}$, also denoted as $\mathrm{U}(R) := R^\times$.

$\mathrm{Sing}(R) := R \setminus R^\times$: the singular set of $R \in \mathsf{Ring}$.

$\mathfrak{J}(R)$: the Jacobson radical of R.

$J_{\mathbb{K}}F$: the ($\mathbb{K}$-)Jacobian of a map F.

$\mathrm{nil}(\mathcal{A})$: set of nilpotent elements of $\mathcal{A} \in \mathsf{Alg}_R$, e.g. the nilradical of $\mathcal{A}$ if $\mathcal{A} \in \mathsf{CAlg}_R$.

$\mathrm{Kdim}(R)$: the Krull dimension of R.

K: arbitrary field, usually $\operatorname{char} K = 0$ or perfect, unless explicitly stated otherwise.

κ: a residue field, usually $\operatorname{char} \kappa = 0$, unless explicitly stated otherwise.

$\mathbb{K}$: the field $\mathbb{R}$ or $\mathbb{C}$.

$\mathbb{C}^\times$: the multiplicative group of the complex numbers.

$\mathbb{C}^+$: the additive group of the complex numbers.

$\mathbb{G}_a$: the additive group scheme.

$\mathbb{G}_m$: the multiplicative group scheme.

$\mathfrak{S}_n$: the symmetric group of $[n]$.

$\mathbb{S}^n$: the n-sphere.

$\mathbb{T}^n := (\mathbb{S}^1)^n$: the standard n-torus.

CAT: the category of all categories.

Cat: the category of all *locally* small categories, $\mathsf{Cat} \in \mathsf{CAT}$.

Set: the category of sets.

Ab: the category of abelian groups.

$\mathsf{Vec}(K) = \mathsf{Vec}_K$: the category of K-vector spaces.

$\mathsf{FdVec}(K) = \mathsf{FdVec}_K$: the subcategory of Vec_K of finite-dimensional K-vector spaces.

Grp: the category of groups.

Ring: the category of non-commutative unital rings.

CRing: the category of commutative unital rings.

$\mathsf{LocCRing}$: the category of commutative unital *local* rings with *local* homomorphisms.

$\mathsf{Mod}(R) \stackrel{\text{def}}{=} \mathsf{Mod}_R$: the category of R-modules over some $R \in \mathsf{CRing}$.

$\mathsf{LMod}(R)$: the category of left R-modules over some $R \in \mathsf{Ring}$.

$\mathsf{RMod}(R)$: the category of right R-modules over some $R \in \mathsf{Ring}$.

$\mathsf{BMod}(R,S) = \mathsf{BMod}_{R,S}$: the category of R-S-bimodules over some $R, S \in \mathsf{Ring}$.

$\mathsf{FdMod}(\mathcal{A})$: the category of modules over the $\mathbb{K}$-algebra $\mathcal{A}$ that are finite-dimensional over the ground field $\mathbb{K}$.

$\mathrm{End}_{\mathcal{A}}(M)$: the algebra of $\mathcal{A}$-linear endomorphisms of $M \in \mathsf{LMod}(\mathcal{A}), \mathsf{RMod}(\mathcal{A}), \mathsf{Mod}(\mathcal{A})$ for $\mathcal{A} \in \mathsf{Alg}_R$, $R \in \mathsf{CRing}$.

$\mathrm{GL}_{\mathcal{A}}(M) \coloneqq \mathrm{Aut}_{\mathcal{A}}(M) \coloneqq \mathrm{U}(\mathrm{End}_{\mathcal{A}}(M)) \stackrel{\text{def}}{=} \mathrm{End}_{\mathcal{A}}(M)^{\times}$: the general linear group of the $\mathcal{A}$-module M, also written as $\mathrm{GL}({}_{\mathcal{A}}M)$ or $\mathrm{GL}(M_{\mathcal{A}})$ etc.

Sch/S: the category of schemes over a base scheme S.

$\mathsf{Sch}_{\mathcal{A}}$: the category $\mathsf{Sch}/\operatorname{Spec}\mathcal{A}$.

$\widehat{\mathbb{A}}^1_R \coloneqq \operatorname{Spf} R[t]$: the formal spectrum of $R[t]$, completion of the affine line over $R \in \mathsf{CRing}$.

$\mathsf{Alg}(R) = \mathsf{Alg}_R$: the category of unital associative algebras over $R \in \mathsf{CRing}$.

Alg^*_R: the category of *non-unital* associative algebras over $R \in \mathsf{CRing}$...

$\mathsf{CAlg}(R) = \mathsf{CAlg}_R$: the category of unital commutative associative algebras over $R \in \mathsf{CRing}$.

$\mathsf{LocCAlg}(R) = \mathsf{LocCAlg}_R$: the category of unital commutative associative *local* algebras over $R \in \mathsf{CRing}$.

$\mathsf{FdAlg}(K) = \mathsf{FdAlg}_K$: the category of finite-dimensional unital associative K-algebras.

$\mathsf{FdCAlg}(K) = \mathsf{FdCAlg}_K$: the category of finite-dimensional unital commutative associative K-algebras.

Art_R: the category of commutative Artinian R-algebras over $R \in \mathsf{CRing}$.

LocArt_R: the subcategory of Art_R of commutative *local* Artinian R-algebras.

$\mathsf{NrmVec}_{\mathbb{K}}$: category of normed $\mathbb{K}$-vector spaces.

$\mathsf{NrmAlg}_{\mathbb{K}}$: category of normed unital associative $\mathbb{K}$-algebras.

$\mathsf{NrmCAlg}_{\mathbb{K}}$: category of normed unital commutative associative $\mathbb{K}$-algebras.

$\mathsf{NrmFdCAlg}_{\mathbb{K}}$: category of normed finite-dimensional commutative associative unital $\mathbb{K}$-algebras.

$\mathsf{NrmLMod}(\mathcal{A})$: category of normed left $\mathcal{A}$-modules over $\mathcal{A} \in \mathsf{NrmAlg}_{\mathbb{K}}$.

$\mathsf{NrmRMod}(\mathcal{A})$: category of normed right $\mathcal{A}$-modules over $\mathcal{A} \in \mathsf{NrmAlg}_{\mathbb{K}}$.

$\mathsf{NrmBMod}$: category of normed $\mathcal{A}$-bimodules over $\mathcal{A} \in \mathsf{NrmAlg}_{\mathbb{K}}$.

NrmMod: category of normed $\mathcal{A}$-modules over $\mathcal{A} \in \mathsf{NrmCAlg}_{\mathbb{K}}$.

$\mathsf{NrmFdMod}$: category of normed finite-dimensional $\mathcal{A}$-modules over $\mathcal{A} \in \mathsf{NrmFdCAlg}_{\mathbb{K}}$.

$\mathsf{NrmFdVec}$: category of normed finite-dimensional $\mathbb{K}$-vector spaces.

$\mathsf{Ban}(\mathbb{K}) = \mathsf{Ban}_{\mathbb{K}}$: the category of Banach spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ (with bounded linear maps as morphisms).

$\mathsf{BanAlg}(\mathbb{K}) = \mathsf{BanAlg}_{\mathbb{K}}$: the category of unital Banach algebras over $\mathbb{K}$.

$\mathsf{CBanAlg}(\mathbb{K}) = \mathsf{CBanAlg}_{\mathbb{K}}$: the subcategory of commutative unital Banach algebras over $\mathbb{K}$.

$\mathsf{BanAlg}(\mathcal{A}) = \mathsf{BanAlg}_{\mathcal{A}}$: the category of Banach $\mathcal{A}$-algebras over $\mathcal{A} \in \mathsf{CBanAlg}_{\mathbb{K}}$.

$\mathsf{BanMod}(\mathcal{A})$: the category of Banach $\mathcal{A}$-modules over $\mathcal{A} \in \mathsf{CBanAlg}_{\mathbb{K}}$.

$\mathsf{LBanMod}(\mathcal{A})$: the category of left Banach $\mathcal{A}$-modules over $\mathcal{A} \in \mathsf{BanAlg}_{\mathbb{K}}$.

$\mathsf{RBanMod}(\mathcal{A})$: the category of right Banach $\mathcal{A}$-modules over $\mathcal{A} \in \mathsf{BanAlg}_{\mathbb{K}}$.

$\mathscr{B}(E_1, E_2)$: the Banach space of bounded linear operators between the Banach spaces E_1 and E_2.

$\mathscr{B}(E) \coloneqq \mathscr{B}(E, E)$: the Banach $\mathbb{K}$-algebra of bounded linear operators on a Banach $\mathbb{K}$-space E.

$\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$: (possibly finite-dimensional) unital (possibly commutative) associative algebras over R, K or $\mathbb{K}$.

$\mathrm{ZD}(R)$: set of zero-divisors of $R \in \mathsf{CRing}$.

$\mathcal{Z}(\mathcal{A})$: the center of an associative R-algebra $\mathcal{A}$.

$\mathrm{C}_{\mathcal{A}}(X)$: the centralizer of a subset $X \subseteq \mathcal{A}$ in $\mathcal{A} \in \mathsf{Alg}_R$.

$\mathcal{Z}(f)$: the zero set of a function f whenever that makes sense.

$\mathrm{Mat}_{n\times m}(R) = R^{n\times m}$: $n \times m$-matrices over some ring R.

$\mathrm{M}_n(R) \coloneqq \mathrm{Mat}_n(R) \coloneqq R^{n\times n}$: the ring of $n \times n$-matrices over $R \in \mathsf{Ring}$.

$I_n \in \mathrm{M}_n(R)$: the identity matrix of size $n \times n$.

$I \coloneqq [0, 1]$: the unit interval.

$A^T \in R^{m\times n}$: transpose of $A \in R^{n\times m}$.

$s_{\max}(A)$: largest singular value of $A \in \mathrm{M}_n(\mathbb{K})$.

$\mathrm{Int}(\gamma)$: the interior of a Jordan curve $\gamma \subseteq \mathbb{C}$.

$\mathrm{supp}(f)$, $\mathrm{supp}(\Gamma)$: support of a function f or of a (topological) cycle Γ.

$\mathrm{Supp}(M)$: support of $M \in \mathsf{Mod}(R)$.

$\mu_n(\mathbb{C})$: the group of complex n-th roots of unity.

$\mathbb{D}_r(z_0)$: the open disc with radius r around $z_0 \in \mathbb{C}$.

$\mathbb{D}^*_r(z_0)$: the punctured open disc with radius r around $z_0 \in \mathbb{C}$.

$\Delta_r(z)$: open polydisc of radius r around $z \in \mathbb{K}^n$.

$\overline{\Delta}_r(z)$: closed polydisc of radius r around $z \in \mathbb{K}^n$.

$\mathbb{B}_r(z)$: open ball of radius r around $z \in \mathbb{K}^n$.

$\overline{\mathbb{B}}_r(z)$: closed ball of radius r around $z \in \mathbb{K}^n$.

$\mathscr{C}^k(X, Y)$: space of k-times continuously differentiable maps $U \to V$, $k \in \mathbb{N}_0 \cup \{\infty, \omega\}$.

$\mathscr{C}^k_{\mathrm{pw}}(X, Y)$: space of *piece-wise* k-times continuously differentiable maps $X \to Y$, $k \in \mathbb{N}_0 \cup \{\infty, \omega\}$.

$H^n_{\mathscr{C}^k}(U, G)$: k-times continuously differentiable singular homology with coefficients in $G \in \mathsf{Ab}$, i.e. using k-times continuously differentiable singular chains, in particular $H^n_{\mathscr{C}^0}(U, G) \overset{\text{def}}{=} H^n(U, G)$.

$\|f\|_K \coloneqq \sup_{x\in K} \|f(x)\|$.

pt: the one-point space.

$\triangle$: a solid triangle with boundary $\partial\triangle$.

$\square$: two solid triangles glued along one side.

$\star$: a star-shaped open set.

$\Omega^\bullet_{\mathrm{dR}}$: de Rham complex.

$\Omega^\bullet \coloneqq \Omega^\bullet_{\mathrm{hol}}$: holomorphic de Rham complex.

$\mathrm{Crit}(f)$: the critical locus of a differentiable map f.

$\mathrm{Reg}(f)$: the regular locus of a differentiable map f.

🍃: a leaf from a tree or a bush.

Introduction & Overview

This introduction needs a short introduction on its own. Namely, it is organized in the following parts:

- **The Landscape:** here we paint the general landscape of *function theories* over algebras and how they came to be historically.
- **The Finite-Dimensional Commutative Case:** following the previous point, we focus attention to our case of interest, motivating its study and describing what has already been done.
- **Motivation & Goals of the Present Thesis:** in the context of the previous point, we motivate our place of the present thesis in it.
- **Differentiability with Respect to a Representation:** here we discuss and motivate our main definition(s) and provide necessary context for it, incl. from neighbouring fields.
- **The Special Case of a Morphism:** here we give a discussion of a special, important subcase of interest that we motivate and again provide some context for.
- **Overview of the Present Thesis & New Results:** self-explanatory.
- **The Algebraic Classification Problem:** here we touch upon the problem of classification of the main objects we are working with.
- **Things Omitted:** here we give a shortlist of important topics not included in the present thesis for one or another reason.
- **Applications:** here we mention some applications of the theory.
- **A Note on Terminology:** here we add to the reader's confusion already created by the preceeding parts of the introduction.

The Landscape. Starting with the deceptively simple-looking definition of complex-differentiability given by the well familiar difference quotient limit

$$f'(z_0) \coloneqq \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

for a function $f : U \to \mathbb{C}$ with $U \subseteq \mathbb{C}$ open and $z_0 \in U$, it is a completely *natural* question whether one can repeat the *beauty and success*[1] of Complex Analysis & Geometry of single and several variables by replacing $\mathbb{C}$ by another $\mathbb{K}$-algebra $\mathcal{A}$ for, say, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ now. Then one would call such a function $\mathcal{A}$-differentiable or $\mathcal{A}$-regular or $\mathcal{A}$-holomorphic etc., depending on the given context. In line with tradition, we will call *Funktionentheorie*[2] (or *function theory*) a one-variable theory of functions that exhibits at least some (but ideally all) of the following broadly defined properties:

- Presence of generalized Cauchy–Riemann equations;
- Presence of generalized Cauchy's and Morera's Integral Theorems;
- Presence of generalized Cauchy's Integral Formula;
- Presence of Analyticity.

It should come as no surprise that this question in its simplest form already goes back to the late 19th century amidst the vigorous developments complex analysis had been witnessing at the time at the hands of giants like Hurwitz, Klein, Picard, and Poincare (standing of course on the shoulders of the earlier generation of giants of 19th century complex analysis such as Cauchy, Riemann, and Weierstraß), and amidst all the active attempts by many others to generalize its spectacular results

[1]yes, some of us mathematicians are shallow like that;

[2]borrowed from German, it literally means *function theory* and traditionally refers exclusively to the area of study of holomorphic functions;

in all directions imaginable (see for example Hurwitz's survey [Hur32] at the turn of the century). It is fair to say that to an extent this is still the case even to this day. More precisely, the question at hand was considered for the first time and studied in some detail by Georg Scheffers in 1893 in his two consecutive papers [Sch93a] and [Sch93b], where, among other things, he stated a corresponding system of generalized Cauchy–Riemann equations (cf. Section 1.3) under the assumption of (complex) analyticity of the components of f and then later also considered appropriate groups of coordinate transformations to go with that. In the following years it became abundantly clear that it is conclusively necessary to distinguish *at a minimum* between the following fundamentally different cases and subcases:

(1) $\mathcal{A}$ is a *commutative associative* (usually unital) $\mathbb{K}$-algebra, i.e. $\mathcal{A} \in \mathsf{CAlg}_{\mathbb{K}}$:
  - (i) $\mathcal{A}$ is a *finite-dimensional* commutative associative $\mathbb{K}$-algebra, i.e. $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$:
    - (a) $\mathcal{A}$ is a finite-dimensional commutative associative $\mathbb{R}$-algebra, i.e. $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{R}}$;
    - (b) $\mathcal{A}$ is a finite-dimensional commutative associative $\mathbb{C}$-algebra, i.e. $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{C}}$;
  - (ii) $\mathcal{A}$ is an *infinite-dimensional* commutative associative $\mathbb{K}$-algebra;

(2) $\mathcal{A}$ is a finite-dimensional, *non-commutative*, associative (usually unital) or non-associative $\mathbb{K}$-algebra;

For a rather extensive bibliography of all the developments up to the late '30s we refer the reader to [War40]. As far as we know, there are no published attempts of a working definition in the case of $\mathcal{A}$ being an *infinite-dimensional non-associative* $\mathbb{K}$-algebra. The main reason for the distinction between (1) and (2) is quite simple: the few naive definition choices available in the commutative theory fail dramatically in the non-commutative setting as they all tend to produce rather trivial *function theories* (see for example [Ros00]). Nevertheless, it is also fair to say that this history of failures[3] eventually led to the birth of the areas of Quaternionic and, more generally, Clifford Analysis as well as other function theories for more general non-commutative $\mathbb{K}$-algebras $\mathcal{A}$. Very briefly and very roughly, there are three groups of approaches to *non-commutative $\mathcal{A}$-analysis*:

- generalized Cauchy–Riemann–Fueter[4] and Dirac(–Fueter) *operators* (and/or, dually, differential form conditions, ensuring the vanishing of the corresponding integrals on some class of submanifolds in the spirit of Cauchy's Integral Theorem);
- Banach-Algebra-theoretical (incl. matrix algebras) methods (e.g. spectral methods) and non-commutative analyticity (= non-commutative power series) (for starters, we refer the interested reader in chronological order to [Die30], [Sch31], [Wag42], [AC55], [Bru58; Bru59], [Rin55; Rin57; Rin60; Rin61; RW62; RW64], [CH66; CH67], [Den67; Den70; Den71], [Sze78], [Sti87]);
- slice regularity/differentiability (e.g. requiring holomorphy of the restrictions of f to complex lines spanned by the various complex structures of $\mathcal{A}$) – this obviously has some overlap with the previous point.

In fact, the Dirac–Fueter approach also turns out to be well applicable to the *finite-dimensional non-associative* case as well (see [Kra07]). We will discuss the group (1)(ii) as we go.

The Finite-Dimensional Commutative Case. Barring this introduction, the present thesis deals exclusively with the setting of (1)(i), with a nod to (1)(ii) as the reader may observe in Chapter 1, the endgame being (1)(i)(b) though. We would argue that the finite-dimensional commutative case is in a sense also the most fundamental one for several reasons, some less obvious than others:

- it is the next most immediate and straightforward generalization of $\mathbb{C}$ as it encompasses all finite commutative ring extensions of $\mathbb{R}$;

[3] see for example the historical account given in [DKM01];

[4] these should not be confused with the Cauchy–Riemann–Scheffers *equations* for *finite-dimensional commutative* $\mathbb{K}$-algebras (cf. Section 1.3), which is why we rather prefer the terminology of (generalized) Dirac and Dirac–Fueter operators;

- it is the "baby" case among the different setups and should serve as a (stronger than $\mathbb{C}$) prototype in the following sense: any general finite-dimensional non-commutative (understood as not necessarily commutative) function theory over $\mathcal{A} \in \mathsf{FdAlg}_{\mathbb{K}}$ ought to degenerate to it for $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$ and any infinite-dimensional (understood as not necessarily finite-dimensional) commutative function theory over $\mathcal{A} \in \mathsf{CAlg}_{\mathbb{K}}$ ought to also degenerate to it for $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$;
- as the reader may have already guessed, the slice differentiability approach appears well amenable to restrictions to more general classes of commutative subalgebras of a given finite-dimensional non-commutative algebra;
- Wilkerson [Wil82] showed that on finite-dimensional commutative $\mathbb{R}$-algebras one can produce "Dirac–Fueter-regular" functions on $\mathcal{A}$ from $\mathcal{A}$-differentiable functions in the "standard" sense[5], *provided* that $\mathcal{A}$ is itself *not local*, that is, that $\mathcal{A}$ is a *non-trivial product* of local Artinian $\mathbb{R}$-algebras (as per the decomposition theorem for commutative Artinian rings), which we found a little surprising;
- in the spirit of the previous point, if one is willing to drop finite-dimensionality, Krasnov showed in [Kra03] and [Kra07] that many *finite-dimensional non-commutative* function theories (incl. Quaternionic and Clifford Analysis) correspond in some narrow sense to (infinite-dimensional) *commutative* (associative) function theories *on operators*.

Given the naturality of the finite-dimensional commutative setting and the accompanying *surprise factor*, what should not come as a surprise though is that many of its core results have been independently rediscovered in one form or another several times over throughout history. Thus, for instance, the elementary treatment in [Fox49] appears to be the first systematic study of the case $\mathbb{K} = \mathbb{R}$ and in particular states a version of Cauchy's Integral *Theorem*[6] (CIT) *in the absence of a complex structure*. We speculate that it must have been known even earlier, but we were not able to pinpoint a precise reference in support of this. The *real* Cauchy's Integral Theorem was also rediscovered in [Ped92], but it was clearly already known to [Hab76] as well. In that time span there were likely other accounts of it as well. Further aspects of the theory over $\mathbb{R}$ (partially included in earlier treatments too) are discussed in [Wat92a; Wat92b], [GM96], and [Pog07]. Some low-dimensional and special subcases of the real setting also appear in varying forms for example in [Hor63], [BR73], [MR98], [KS05], [RS04], and [Di +23]. On the other hand, we are quite certain that the first concise treatment of the case $\mathbb{K} = \mathbb{C}$ was given already in 1928 by P. Ketchum in [Ket28a]. His paper can be seen as a proof of concept, laying the groundwork for the whole subject over $\mathbb{C}$: he proves that the main pillars of single-variable complex analysis (Cauchy-Riemann equations, Cauchy's Integral Theorem, Cauchy's Integral *Formula*, Analyticity, Laurent Series Expansion etc.) have corresponding analogoues for every $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{C}}$. So, the remaining question was not whether but how far one can take this analogy, how strong the corresponding *core* theorems can be, and what are the eventual, inevitable differences to the complex plane. In that sense, the next key result in the history of the finite-dimensional commutative complex $\mathcal{A}$-analysis came from G.B. Rizza, who proved a homological version of Cauchy's Integral Formula in [Riz52a] and [Riz52b]. He later gave a more systematic treatment of the core theory in [Riz54b], based on his previous results. Unsurprisingly though, the complex case suffers from the same fate as the real case in that the core theory has been rediscovered independently several times over in various forms, for example in [Ede73; Ede76b]. A proof of a generalized Cauchy's Integral Formula (and some of its usual consequences) is also given in [Shp15], starting from a Gateaux-type $\mathcal{A}$-differentiability, which, however, can be shown from "first principles" to be equivalent to the standard definition(s). Low-dimensional and special subcases of the

[5] he calls the first kind "F-regular" (after Fueter) and the second kind "S-regular" (after Scheffers) for short, a terminology that appears to go back to [Kri52];

[6] not to be confused with Cauchy's Integral *Formula*;

complex theory have been investigated in [Ril53], [NT55; NT56], [Goo88], [Pri91], [Ola02], [Rön08], [Alp+13], [Lun+15], [RS04], and probably others. We remark that all these function theories for low-dimensional $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$ have of course natural classical-geometric counterparts, starting with [Stu03][7], but then also [Grü06], [Tak40a; Tak41a], [Tak40b; Tak41b], [Tak40c; Tak41c], and others. We refer the interested reader to the books [Ben73], [Yag66], [Yag79], or to the nice, very short exposition in [HH04], or to [Cer17] for some of the history of the classical-type geometry over $\mathcal{A}$. Moving on, a more recent exciting development in the low-dimensional theory is the construction of the analogue of Sato's hyperfunctions in the bi-complex setting, see [VV12], [Col+11], and [MS20; MS21; Mat22; Mat25]. But despite some such isolated advancements, the sheer number of historical "independent" rediscoveries only attest to the rather sorry state the subject of finite-dimensional commutative $\mathcal{A}$-analysis is in, which in and of itself should be taken as a negative *development*, considering its lively origins. Indeed, with some notable exceptions, the area shows a sad tendency of authors picking (oftentimes the same) lowest of low-hanging fruits. We aim to fix that by picking all of the remaining low-hanging fruits. An unfortunate but not unexpected byproduct of the aforementioned development is also the scattared nature of the subject. Waterhouse even writes in [Wat92b]: *"The theory has been developed only in fits and starts, and some of the simple results we need seem not to be on record at all, while others are only recorded in language that is now hard to understand"*. One might even go as far as saying that the topology of the subject itself in the ambient space-time has a bit too many connected components.

Motivation & Goals of the Present Thesis. But our desire to reopen a closer look into the *finite-dimensional commutative associative theory* is not only dictated by the somewhat unsatisfactory, outdated, and fragmented state of the subject. The purpose of the present work is *manifold*:

- to separate the wheat from the chaff, in particular to clearly delineate the most trivial aspects of the theory from the less trivial ones, thus identifying possible core issues in need of a conceptually satisfactory answer;
- and in doing so, to conceptually clear up and elucidate various aspects of the theory and give conceptually clean descriptions of its objects;
- in order to consolidate the main theory and streamline its proofs;
- by using proper "modern" mathematical language (really, mostly just the basic language of commutative algebra, algebraic topology, and differential geometry by today's standards, but as opposed to the notational and mathematical inbreeding often seen otherwise);
- thereby providing a concise but comprehensive systematic and somewhat in-depth study of the subject (with one eye open towards the infinite-dimensional commutative case);
- in a way accessible to a broad audience of mathematicians from different areas since the subject is not nearly widely enough known and deserves more attention;
- but ultimately to provide solid foundations to develop geometry over $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{C}}$ in the spirit of the beautiful theory of Riemann Surfaces and more general Complex-Analytic Geometry.

Indeed, while smooth manifolds over the "bi-real" numbers[8] and over Weil algebras[9] have been extensively studied, the latter class notably by the Kazan school of geometry[10] in the east and in the setting of Weil bundles and Weil functors in the west[11], and are seeing some recent resurgence due

[7] this is the same E. Study, after whom the *Fubini–Study* metric is called;

[8] see for instance the survey [CFG96] on manifolds with (almost) $\mathcal{A}$-structure, where $\mathcal{A} = \mathbb{R}[x]/(x^2-1) \overset{\text{def}}{=} \mathbb{R}[j] \cong \mathbb{R}\times\mathbb{R}$, $j^2 = 1$, as well as [GGM02] and [GGM03];

[9] commutative local Artin $\mathbb{R}$-algebras with residue field $\mathbb{R}$;

[10] see for example the book [VSS85] or Shurygin's more recent survey [Shu02], incl. the references therein, or Salimov's book [Sal23] for some more up-to-date developments;

[11] see for example the corresponding chapters in [KMS93] and the references therein, but also independently of Weil's ideas [BC74] on integrable almost-tangent (i.e. $\mathbb{R}[x]/(x^2)$) structures;

to newfound connections to information geometry, Frobenius manifolds, and Manin's pre-Frobenius manifolds (see [CM20], [CM22], as well as [MS22]), in contrast the complex counterpart remains to our best knowledge largely unexplored[12]. Circling back to the infinite-dimensional commutative case, the usefulness of manifolds modelled on commutative Banach algebras was recognized by Kobayashi [Kob87] in the setting of mapping spaces and later by H. Yagisita [Yag19a; Yag19b] in the setting of continuous families[13] of complex manifolds. Roughly speaking, this is because the corresponding Banach manifold becomes finite-dimensional-like over a suitable (infinite-dimensional) Banach algebra $\mathcal{A}$. In the finite-dimensional case, complex $\mathcal{A}$-manifolds (and more generally, $\mathcal{A}$-analytic spaces) arise naturally in several ways, for example:

- complex tori $\mathbb{C}^n/\Lambda$ automatically become $\mathcal{A}$-curves for free via coverings $\mathcal{A} \twoheadrightarrow \mathcal{A}/\Lambda$;
- analytifications of projective spaces over $\mathcal{A}$;
- interesting group actions $G \curvearrowright \mathcal{A}$ as a result of $\mathcal{A} \in \mathsf{LocArt}_{\mathbb{C}}$ usually having highly non-trivial positively dimensional automorphism group;

etc.

Differentiability with Respect to a Representation. Rather than just an algebra $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, the starting point of the present thesis is a finite-dimensional representation $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$, or equivalently, $M \in \mathsf{FgMod}(\mathcal{A})$. Then, given $U \subseteq \mathcal{A}$ open and $f\colon U \to M$ a function, we call $f(Z)$ ρ-differentiable (representation perspective), or equivalently, $\mathcal{A}$-differentiable (module perspective) at $Z_0 \in U$ if there exists an element (and our to-be-derivative) $f'(Z_0) \in M$ such that

$$f(Z_0 + H) = f(Z_0) + H \cdot f'(Z_0) + r(H) \overset{\text{def}}{=} f(Z_0) + \rho(H)(f'(Z_0)) + r(H),$$

where we distinguish between three modes of *basic* ρ- (or $\mathcal{A}$-) regularity, yielding three different *basic* structure sheaves on $|\mathcal{A}|$ (the underlying topological space of $\mathcal{A}$ for the analytic topology), depending on what condition we put on the remainder term $r(H)$:

(1) *Classical* or *limit-wise* ρ-differentiability $\mathscr{L}_\rho$:

$$\lim_{H\to 0} \rho(H)^{-1}(r(H)) = 0,$$

corresponding to the *classical limit* definition via a difference quotient (roughly speaking, so see Section 1.1 for the precise details);

(2) *Strict* $\mathcal{A}$-equivariant Frechét differentiability $\mathscr{F}_\rho^{\mathrm{str}}$:

$$\|r(H)\|_M = o(\|\rho(H)\|_{\mathrm{End}_{\mathbb{K}}(M)})$$

as $H \to 0$;

(3) $\mathcal{A}$-equivariant[14] Frechét differentiability $\mathscr{F}_\rho$:

$$\|r(H)\|_M = o(\|H\|_{\mathcal{A}})$$

as $H \to 0$.

We will use the terms ρ-differentiable and $\mathcal{A}$-differentiable interchangeably, depending on which perspective is the most convenient one. Since ρ is bounded, it is moreover clear that $\mathscr{F}_\rho^{\mathrm{str}} \subseteq \mathscr{F}_\rho$. We also introduce the sheaves $\mathscr{L}_\rho^k$, $\mathscr{F}_\rho^{\mathrm{str},k}$, $\mathscr{F}_\rho^k$ of k-times (not necessarily continuously) ρ-differentiable func-

[12] modulo some isolated exceptions treated in [Sal23] – the reader should not be confused by Salimov's usage of the term "holomorphic", it is (somewhat generously) deployed to mean $\mathcal{A}$-differentiable even when $\mathcal{A}$ itself does not carry a complex structure;

[13] as opposed to the complex-analytic families in Kodaira-Spencer deformation theory

[14] If one prefers group representations over algebra representations, then by openness $\mathcal{A}$-equivariance is equivalent to $\mathcal{A}^\times \cong \mathbb{G}_{\mathrm{m}}^k \times \mathbb{G}_{\mathrm{a}}^\ell$-equivariance for some $k \in \mathbb{N}$, $\ell \in \mathbb{N}_0$. Of course, the non-trivial part is describing the action of the vector group $\mathbb{G}_{\mathrm{a}}^\ell$.

tions, $k \in \mathbb{N}\cup\{\infty\}$, as well as the sheaves $\mathscr{C}^k_\rho$, $k \in \mathbb{N}\cup\{\infty, \omega\}$, of k-times *continuously* ρ-differentiable functions on $\mathcal{A}$. We remark that imposing continuity of the derivative removes the distinction between $\mathscr{L}_\rho$, $\mathscr{F}^{\mathrm{str}}_\rho$, and $\mathscr{F}_\rho$-regularity (see Proposition 1.1.14), hence the unified notation in that case. Nevertheless, in the spirit of the previously outlined goals of the thesis, a substantial amount of it is devoted to the precise relationship between these lower-regularity classes of ρ-differentiability, their properties, as well as their interaction with basic algebraic constructions in $\mathsf{FdCAlg}_{\mathbb{K}}$ and $\mathsf{FdMod}(\mathcal{A})$. The motivation for carefully studying low ρ-differentiability classes is quite natural: it is a standard but non-trivial fact in complex analysis that continuity of the derivative of holomorphic functions is a superflous assumption. This raises the inevitable question how far the $\mathcal{A}$-equivariant condition alone can take us without necessarily assuming an underlying (almost) complex structure of $\mathcal{A}$. While the regularity class $\mathscr{L}_\rho$ behaves well with respect to pretty much every algebraic construction (see Section 1.1), we remark that it is an unnatural definition in the Banach setting since infinite-dimensional commutative Banach $\mathbb{K}$-algebras need not have dense groups of units. On the other hand, $\mathscr{F}_\rho$ is the "gold standard" in the Banach setting, but unfortunately it does not appear to behave well with respect to some crucial algebraic operations (notably pre-compositional factorizations with respect to epimorphisms, see Proposition 1.2.1). This is where the *auxiliary* notion of $\mathscr{F}^{\mathrm{str}}_\rho$ enters as it is both more natural to the Banach setting and behaves well with respect to pre-compositional factorizations. Since we cannot have nice things, $\mathscr{F}^{\mathrm{str}}_\rho$, however, does not appear to behave well with respect to *post-compositional* operations without further assumptions (see for instance Proposition 1.1.23 and afterwards). The $\mathscr{F}^{\mathrm{str}}_\rho$-condition is not a strange one, it naturally arises in the ρ-analytic setting

$$f(Z+H) = f(Z) + \sum_{k=1}^{\infty} H^k \cdot \frac{f^{(k)}(Z)}{k!} \overset{\text{def}}{=} f(Z) + \sum_{k=1}^{\infty} \frac{1}{k!}\rho(H)^k\big(f^{(k)}(Z)\big)$$

in the self-evident algebraic way, but then again it also concides with $\mathscr{L}_\rho$ and $\mathscr{F}_\rho$ under such high regularity. As far as we know, [Wat92a] is the only meaningful treatment of low $\mathcal{A}$-regularity issues and only for the case ${}_{\mathcal{A}}M = \mathcal{A}$, proving $\mathscr{L}_{\mathcal{A}} \subseteq \mathscr{F}_{\mathcal{A}}$ among a few other things that we will come back to in a jiffy. The idea to look at representations of $\mathcal{A}$ instead of $\mathcal{A}$ itself is not new, not only in the sense of reconstruction theorems from categories of representations but also in our modest analytic setup, and in that regard the most natural generalization would be to replace $\mathcal{A}$ in both the domain and codomain by ${}_{\mathcal{A}}M, {}_{\mathcal{A}}N \in \mathsf{FdMod}(\mathcal{A})$ and then study functions $f\colon U \to N$, $U \subseteq M$ open, with an $\mathcal{A}$-linear differential instead of only functions of the type $\mathcal{A} \to N$. Indeed this setting naturally arises for example in the study of finite-dimensional real $\mathcal{A}$-manifolds, see for example [Shu99], [SS01], [Mik01], [KM03], [Kur06] and later papers of Mikulski, Kureš, and others, but these all assume the standard differential-geometric $\mathscr{C}^\infty$-setting from the get-go and do not concern themselves with subtler regularity questions, that have no bearing on the differential-geometric content anyway. In the infinite-dimensional setting, topological (projective, finitely-generated) $\mathcal{A}$-modules for topological algebras $\mathcal{A} \in \mathsf{CAlg}_{\mathbb{K}}$ and $\mathcal{A}$-differentiable[15] maps between them also arise naturally, for example starting with the aforementioned work of Kobayashi [Kob87] and Yagisita [Yag19a; Yag19b] (Banach modules over $\mathcal{A} \in \mathsf{CBanAlg}_{\mathbb{K}}$) and in the context of non-linear Serre-Swan theorems (see [Man92; Man95; Man98; Man17] and [Pap92a]) and $\mathcal{A}$-vector bundles (see [Pap92b; Pap94; Pap98; Pap99b; Pap99a] as well as the references therein to the beautiful work of Mallios). Indeed it is no secret that many problems in finite-dimensional differential geometry give rise to objects in infinite-dimensional differential geometry. As far as we know, this is **one of two** research directions within the group (1)(ii) (differentiability with respect to infinite-dimensional commutative associative $\mathbb{K}$-algebras) outlined at the beginning of the introduction. We also use the occasion to

[15]for a suitable definition, depending on the exact infinite-dimensional context;

mention the beautiful odd-duck paper [Ped97], proving Cauchy's Integral *Theorem* over all finitely generated commutative associative $\mathbb{R}$-algebras, suitably topologized of course. Together with the cited work of Manoharan and Papatriantafillou on $\mathcal{A}$-differentiability in topological $\mathcal{A}$-modules we take this as evidence that the concept of $\mathcal{A}$-differentiability and $\mathcal{A}$-holomorphy should be possible in very general suitably topologized formal algebraic setting, conceivably even globalized to Noetherian schemes over $\mathbb{K}$ in place of a single Noetherian $\mathbb{K}$-algebra. But we digress! Observe moreover that the "two-sided" representation setting is also "intrinsically" natural: $\mathcal{A}$-differentiability involves a (non-canonical) choice of an $\mathcal{A}$-structure on a $V \in \mathsf{FdVec}_{\mathbb{K}}$, which is the same as a faithful representation of $\mathcal{A}$, of which there are usually very many. Furthermore, it also encompasses "several $\mathcal{A}$-variables" as a special important case (free $\mathcal{A}$-modules of finite rank). It is prudent to mention at this point that "several $\mathcal{A}$-variables" also has another line of generalization, namely to functions of several different $\mathcal{A}$-variables for varying algebras $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$ by means of *multi-modules*, that is, $\mathbb{K}$-vector spaces that admit several simultaneous module structures (over different algebras) that are compatible with each other. For example, $\mathcal{B} \in \mathsf{FdCAlg}_{\mathbb{K}}$ generally admits many $\mathbb{K}$-algebra morphisms into itself (e.g. think of all the $\mathbb{K}$-subalgebras contained in $\mathcal{B}$). While some of our intended applications later on would in principle indeed require the development of a theory of functions over more general $\mathcal{A}$-modules, we will focus in the present work only on the special case of $\mathcal{A}$-differentiable functions of the form $f\colon U \to {}_{\mathcal{A}}M$, $U \subseteq \mathcal{A}$ open. There are several reasons for that:

- first and foremost, they have the special property that the ρ-derivative (when existent) is again a function $f'\colon U \to M$;
- speaking in terms of relations and generators, to understand the more general case of an $\mathcal{A}$-module domain of f, one first needs to understand the case of several *free* $\mathcal{A}$-variables; but to understand that case well, one first needs to understand well enough the case of a single $\mathcal{A}$-variable;
- this case is comparable to vector-valued holomorphic functions on $\mathbb{C}$ and nothing is lost by replacing $\mathcal{A}$ by ${}_{\mathcal{A}}M$ in the codomain because much of the theory is *additive* anyway;
- to treat the more general case in the way that it deserves would require to include significantly more module theory in a work that is already rather long (i.e. for mundane, *"length"* reasons);
- the hidden agenda of the present thesis is actually *differentiability with respect to a morphism* $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ in $\mathsf{FdCAlg}_{\mathbb{K}}$, but this is treated most naturally in the module language since big portion of the theory is *additive*, as just mentioned.

But, as previously alluded, even in that special case we are already *secretely* doing functions over (multi-)modules since if $f\colon U \to {}_{\mathcal{A}}M$ has an $\mathcal{A}$-equivariant differential Df, then Df is also equivariant with respect to all $\mathbb{K}$-subalgebras of $\mathcal{A}$.

The Special Case of a Morphism. Speaking of morphisms $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ in $\mathsf{FdCAlg}_{\mathbb{K}}$, one can and does define the corresponding basic regularity classes of φ-differentiability by simply replacing ρ by φ in the above definitions. However, since much of the theory is *additive*, it is more convenient to treat both the representation case and the morphism case on equal footing by passing to the corresponding representation $\varphi^{\mathrm{rep}} \overset{\mathrm{def}}{=} \rho_\varphi\colon \mathcal{A} \xrightarrow{\varphi} \mathcal{B} \xrightarrow{\rho_{\mathcal{B}}} \mathrm{End}_{\mathbb{K}}(\mathcal{B})$ instead, where $\rho_{\mathcal{B}}$ is the regular representation of $\mathcal{B}$. On some occasions, however, it will be actually more convenient to treat the representation ρ as a morphism ρ^{mor} in $\mathsf{FdCAlg}_{\mathbb{K}}$ by passing to a *convenient corestriction* of it (see Section 0.1.4). Most prominently, this will be the case in Section 2.2, where we will want to explicitly identify the lattice in $\mathrm{End}_{\mathbb{K}}(M)$, which the generalized analytic index (generalized winding number) $\mathrm{Ind}_\rho(\Gamma, Z_0)$ of a singular 1-cycle Γ with respect to a point $Z_0 \in \mathcal{A}$ is taking values in. The main reasons why one would want to build morphisms $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ into the definition of $\mathcal{A}$-differentiability are rather simple. First and foremost, the concept of differentiability is inherently relative (proof: by looking at the definition).

It should then come as no surprise that under mild regularity assumptions such differentiability behaves functorially: if f is φ-differentiable and g is ψ-differentiable, then $g \circ f$ turns out to be $\psi \circ \varphi$-differentiable (provided of course that the compositions are well-defined). Once the possibility of *a function theory* over every $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$ is established – thus replacing the once canonical choice of $\mathbb{C}$ – it becomes abundantly clear that the theory is not just categorically flavoured, but the categorical nature of the subject is inescapable, as we shall see in Section 1.1. The only remaining question then is how far this perspective can be taken[16]. While φ being part of the definition of $\mathcal{A}$-differentiability is arguably only a cosmetic modification, it is conceptually also an important one with its own subtleties and encompasses even standard relevant examples such as anti-holomorphic functions and $\mathcal{B}$-valued holomorphic functions. In fact, the necessity of such a setup was recognized by [Shu99] in the context of constructing a category of all smooth manifolds over Weil algebras. We remark that some of our results for the special case $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ were announced (without proofs) in [Gen24]. Here we will include of course full proofs (in the more general setting of representations). Recall also that there is a well established relative notion in algebraic geometry, in particular on the level of coefficients/scalars. We want to introduce a similar relative notion beyond $\mathbb{C}/\mathbb{R}$ in complex analysis as well. In the same vein, Weil restriction is a well established procedure/method in algebraic geometry, especially in arithmetic geometry, e.g. algebraic groups/abelian varieties over a number field extension L/K. Indeed, $\mathcal{A}$-holomorphic functions can be interpreted as holomorphic $\mathcal{A}$-operator calculus for holomorphic ${}_{\mathcal{A}}M$-valued functions, then passing to a $\mathbb{C}$-basis of $\mathcal{A}$ yields a special class of holomorphic mappings of several complex variables.

Now consider the special case $\varphi = \mathrm{id}_{\mathcal{A}}$ and put $\mathscr{F}_{\mathcal{A}} := \mathscr{F}_{\mathrm{id}_{\mathcal{A}}}$. If $\mathcal{A} \in \mathsf{CBanAlg}_{\mathbb{C}}$, the definition $\mathscr{F}_{\mathcal{A}}$ actually goes back to [Lor43], who proved Cauchy's Integral Theorem, Cauchy's Integral Formula, and analyticity under the assumption of convexity of the corresponding domain[17]. Some 12 years later Blum [Blu55] dropped the convexity assumption of Lorch and substantially expanded the core theory. Since then the subject of holomorphy over general infinite-dimensional commutative Banach $\mathbb{C}$-algebras has grown slowly but steadily (see for example [Mib60] and [Gli68; Gli70; Gli71]) and is still an active field of research even to this day, see [MMP09; MP13; MMP16; MP16]. As the reader might have guessed, this is the second line of research within the group (1)(ii) introduced at the beginning. We remark that while the commutative Banach setting is certainly very general and in principle subsumes the finite-dimensional case, the latter also has a lot of special extra structure (starting with the decomposition of commutative Artinian algebras into local ones) not necessarily available in the generality of the Banach setting and has the language and the tools of standard differential geometry and algebraic topology at its disposal.

Overview of the Present Thesis & New Results. Following our philosophy of identifying the right reasons behind any *core* result made it quickly clear upon closer review of the pre-existing literature that a more in-depth look into the *real* theory, that is, over the indeterminate $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, is desperately required. Thus, even though the main objective of the present thesis is the finite-dimensional commutative *complex* theory, we will have to spend more time on the underlying *real* theory than we originally intended. As a *result*, the main body of the present thesis organically splits into the following two chapters / fields of coefficients / reasoning groups:

(1) Differentiation over $\mathcal{A}$ (Chapter 1) covers the theory over $\mathbb{K}$, i.e. the part that is independent of the presence of an underlying complex structure in $\mathcal{A}$, or equivalently, core results that follow for purely *algebraic reasons*, more precisely: given $f\colon U \to \mathcal{A}$ $\mathcal{A}$-differentiable, the *commutativity* of $\mathcal{A}$ implies that the corresponding ${}_{\mathcal{A}}M$-valued differential 1-form $\omega := \mathrm{d}Z \cdot$

[16]we are a little afraid that the answer to this question might be "the sky is the limit";

[17]it is thus tempting to call $\mathscr{F}_{\mathcal{A}}$-regular functions for $\mathcal{A} \in \mathsf{CBanAlg}_{\mathbb{C}}$ *Lorcholomorphic*;

$f(Z) \overset{\text{def}}{=} \rho(\mathrm{d}Z)(f(Z))$ is d-closed, i.e. $\mathrm{d}\omega = 0$, leading to the generalized Cauchy's Integral *Theorem* over $\mathcal{A}$. *In particular, Cauchy's Integral Theorem (even over $\mathbb{C}$) has nothing to do with the complex numbers per se (except historically), but only with their structure of a commutative $\mathbb{R}$-algebra.*

(2) Integration Theory over $\mathcal{A}$ (Chapter 2) covers the theory over $\mathbb{C}$, or equivalently, core results that follow for purely *topological reasons*, more precisely: $\pi_0(\mathbb{C}^\times) = 1$ implies $\pi_0(\mathcal{A}^\times) = 1$, which in turn enables the the existence of (the holomorphic) generalized Cauchy's Reproducing Kernel over $\mathcal{A}$, leading to e generalized analytic index (winding number) over $\mathcal{A}$, to Cauchy's Integral *Formula(s)* over $\mathcal{A}$ etc. This is the most crucial first difference to the *purely* real theory, implying all the expected special consequences for $\mathcal{A}$-analogues from the complex plane.

In more detail, the first fundamental issue we deal with in Section 1.1 is showing that $\mathscr{L}_\rho \subseteq \mathscr{F}_\rho^{\mathrm{str}} \subseteq \mathscr{F}_\rho$. In the case ${}_{\mathcal{A}}M = \mathcal{A}$, it was shown by Waterhouse in [Wat92a] that $\mathscr{L}_{\mathcal{A}} \subseteq \mathscr{F}_{\mathcal{A}}$, however even in that case our approaches differ substantially. His proof strategy goes as follows: given $\mathcal{A} = \prod_{k=1}^n (\mathcal{A}_k, \mathfrak{m}_k, \mathbb{R})$, one shows first that any $f \in \mathscr{L}_{\mathcal{A}}(U)$ can be written as $f(Z) = f(Z_1, \ldots, Z_n) = (f_1(Z_1), \ldots, f_n(Z_n))$, where each $f_k(Z_k)$ is $\mathscr{L}_{\mathcal{A}_k}$-differentiable (this is a special case of our later Proposition 1.2.9) and then uses the rather special algebraic structure of the local $(\mathcal{A}_k, \mathfrak{m}_k, \mathbb{R})$ to conclude that in fact f_k is $\mathscr{F}_{\mathcal{A}_k}$-differentiable, hence in particular continuous, $1 \leq k \leq n$. In contrast, our approach completely avoids appealing to any algebraic facts about $\mathcal{A}$ and shows that the inclusion in question holds for *purely topological / analysis* reasons (as it should!). In fact, we show continuity of $f \in \mathscr{L}_\rho(U)$ (again for purely topological reasons) before we show that it is Frechét $\mathcal{A}$-differentiable (again as one should!). Moreover, Waterhouse is not particularly precise about the domains of definition of the components f_k, $1 \leq k \leq n$. Finally, we are not entirely convinced by his proof of the decomposition of f since he only actually shows it for products of copies of $\mathbb{R}$ and $\mathbb{C}$ by making an essential use of their absolute values and then remarks that the proof for the general case $\mathcal{A} = \prod_{k=1}^n \mathcal{A}_k$ is analogous, but recall that in general $\mathcal{A}_k$, $1 \leq k \leq n$, would only admit *submultiplicative* norms, which appears to break his argument[18]. The remaining part of Section 1.1 is mostly devoted to verifying various "ρ-calculus" rules, incl. various chain rules. In particular, we state a "mixed" Leibniz rule in Proposition 1.1.19 for the product of a "morphism-differentiable" function with a "representation-differentiable" function. As for the chain rules, an important subtlety is that the middle sheaf $\mathscr{F}_\rho^{\mathrm{str}}$ is not unconditionally well-behaved with respect to post-compositions. This even includes post-compositions with $\mathcal{A}$-linear homomorphisms, which causes $\mathscr{F}_\rho^{\mathrm{str}}$ to behave somewhat strangely compared to the other two regularity classes (see Proposition 1.1.23). We also prove a substantial sharpening of a main result of [GM96] in Proposition 1.1.16: in our notation their result reads $\mathscr{C}_{\mathcal{A}}^\omega(U, {}_{\mathcal{A}}\mathcal{A}) = \mathscr{L}_{\mathcal{A}}^\infty(U, {}_{\mathcal{A}}\mathcal{A}) \cap \mathscr{C}^\omega(U, \mathcal{A})$, whereas we show that in fact $\forall\, k \in \mathbb{N} \cup \{\infty, \omega\}\colon \mathscr{C}_{\mathcal{A}}^k(U, {}_{\mathcal{A}}\mathcal{A}) = \mathscr{L}_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathcal{A}) \cap \mathscr{C}^k(U, \mathcal{A})$, that is, one can drop the "*$\mathscr{L}_{\mathcal{A}}$-smoothness*" condition to just the most basic $\mathscr{L}_{\mathcal{A}}$-regularity (the underlying reason being *purely algebraic*, namely that $\mathcal{A}$-linearity propagates to all higher total derivatives due to their symmetry), see the remarks after Proposition 1.1.16 for details.

In Section 1.2 we prove various automatic ρ-differentiable extensions of ρ-differentiable functions and canonical factorizations by studying the interplay between the analysis and basic algebraic/categorical constructions, notably epimorphisms. We remark that canonical factorizations and automatic ρ-differentiable continuations are really two sides of the same medal. This is where $\mathscr{F}_\rho^{\mathrm{str}}$ finally gets to shine after being a disappointment in Section 1.1. Barring $\mathcal{A}$-prolongation and the aforementioned factorization result of [Wat92a], all of these factorization/extension results are to the best of our knowledge new. Naturally, they all rely on the conservation of $\mathcal{A}$-differentiability by pre- and post-

[18] we are even tempted to say that his proof does not hold *water*;

compositions established in Section 1.1. This is also where various **spectral considerations** enter the play for the first time and are there to stay. More precisely, given a finite-dimensional representation $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$, where $\mathcal{A} = \prod_{j=1}^{n}(\mathcal{A}_j, \mathfrak{m}_j, \mathbb{K}_j)$, we have that ${}_{\mathcal{A}}M$ admits an indecomposable decomposition ${}_{\mathcal{A}}M = \bigoplus_{k=1}^{m} {}_{\mathcal{A}}M_k$ and ρ determines an index subset $I_\rho \subseteq [n]$ of "relevant" local factors $\mathcal{A}_j$, or equivalently of characters χ_j of $\mathcal{A}$, $j \in I_\rho$, in such a way that ρ reduces to representations $(\mathcal{A}_j, \mathfrak{m}_j, \mathbb{K}_j) \to \mathrm{End}_{\mathbb{K}}(M_k)$ in a precise narrow sense (see Section 0.1.5 for the precise details). In other words, on the representation side it is completely sufficient to consider only *indecomposable representations of local Artinian* $\mathbb{K}$*-algebras*. It turns out that this picture permeates the analysis side as well, see Proposition 1.2.12, and that for the purposes of analysis it is in fact sufficient to consider only *faithful* indecomposable representations of local Artinian $\mathbb{K}$-algebras.

In Section 1.3 we state the homogeneous generalized Cauchy–Riemann(–Scheffers) equations for ρ-differentiable functions and discuss various cosmetic aspects of them. These are more or less well-known. Then we *derive* generalized Wirtinger operators, *universal* for all $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$. To our surprise, these are completely new despite the fact that the construction is entirely natural. A subtle point is that while the generalized Cauchy–Riemann–Scheffers equations evidently depend on the $\mathcal{A}$-module M (unlike for vector-valued holomorphic functions, that all satisfy the same Cauchy–Riemann equations), more precisely on the structure tensor of the representation ρ, the generalized Wirtinger operators are stated entirely in terms of $\mathcal{A}$ (of course, one still has to know the action $\mathcal{A} \curvearrowright M$), see Definition 1.3.2 and the subsequent paragraphs. They also have natural interpretations as the "components" of the 2-form $\mathrm{d}(\mathrm{d}Z \cdot f(Z))$.

Speaking of which, in Section 1.4 we finally consider $\omega \coloneqq \mathrm{d}Z \cdot f(Z) \stackrel{\text{def}}{=} \rho(\mathrm{d}Z)(f(Z)) \in \Omega^1(U, {}_{\mathcal{A}}M)$ and give a rather short proof, in particular without invoking the generalized Cauchy–Riemann–Scheffers equations, of the key fact that $f(Z)$ is ρ-differentiable if and only if $\mathrm{d}\omega = 0$, see Proposition 1.4.8. In the case ${}_{\mathcal{A}}M = \mathcal{A}$ this result is not new, but as far as we know, its proof is, likely because it makes use of the generalized Wirtinger operators defined previously. There we also consider the reverse question of *ρ-integrability*, i.e. existence of *ρ-primitives* of ρ-differentiable functions $f(Z)$ on various domains and prove this to be the case on *homologically 1-connected* domains. Of course, on the way there we prove various versions of the generalized Cauchy's Integral Theorem, in particular a generalized Cauchy–Goursat Theorem (after Pringsheim) that notably removes the condition of continuity of the derivative, see Proposition 1.4.11. To the best of our knowledge this adaptation of Pringsheim's proof is new in the finite-dimensional commutative literature, especially for modules, but appears in the infinite-dimensional commutative case in [Ped97]. Regardless, it is a natural continuation of the preceeding low regularity discussion of ρ-differentiability.

In Section 2.1 we finally begin our treatment of the complex case. This is where **spectral considerations** enter the picture in a decisive manner. It is prudent to point out at this point that when it comes to the complex case, there are two seemingly competing points of view. The first one is so to speak the *intrinsic* one, reflecting the desire to demonstrate that $\mathcal{A}$-holomorphy is a completely self-contained and self-sufficient body of theory that has earned its independent existence and is on par and equal in rights with classical complex analysis. This is of course philosophically and conceptually important, but it is feasible only to a certain point. The second perspective is namely the **spectral** one, reducing various problems to the complex plane, sometimes quite literally so. But it is more than just an effective way to simplify problems – without it the theory of $\mathcal{A}$-differentiability and $\mathcal{A}$-holomorphy simply remains incomplete. Thus, Section 2.1 introduces all the necessary topological and algebro-topological notions that will support the integration theory of Cauchy's Reproducing Kernel (and related issues) *in its strongest topological forms* later on and they are all intimately to the *spectral notions* introduced in Section 0.1.9. We say "strongest topological form" because **the**

spectral POV clearly demonstrates for many (but not all!) types of problems over and over again that the only obstruction is what is (not) known and/or doable in the complex plane. This applies in full strength to Section 2.2, Section 2.3, and Section 2.4. Very briefly, as previously mentioned, the finite-dimensional representation $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{C}}(M)$ determines a subset of characters $\chi_j \in \mathcal{X}(\mathcal{A})$ of $\mathcal{A}$, $j \in I_\rho$, that assemble to a "multi-character" $\chi := \chi_\rho\colon \mathcal{A} \to \mathbb{C}^{I_\rho}$ (see Section 0.1.13 for the precise details), providing a "relevant" strong deformation retract for homotopical "reduction" of the corresponding integral to the complex plane (more precisely, a Cartesian power thereof), roughly speaking. To the best of our knowledge, **the spectral method**, *both in the real and complex* finite-dimensional commutative theory, is a whole completely new approach as attested by the substantially stronger results we have obtained than what is currently available in the corresponding literature. In particular, this also includes the simple algebro-topological invariants introduced at the end of Section 2.1 that govern the basic algebraic topology of $\mathcal{A}$-holomorphy. On the other hand, of course, spectral arguments are nothing new in infinite-dimensional Banach algebras.

In Section 2.2 we introduce the ρ-analogue of the analytic index (winding number) from the plane for singular 1-cycles Γ with respect to a point $Z_0 \in \mathcal{A}$ and give a complete characterization of it, using the aforementioned spectral-homotopical approach, showing among other things that it is valued in a canonical lattice in $\mathrm{End}_{\mathbb{C}}(M)$. We remark that in the special case $\varphi = \mathrm{id}_{\mathcal{A}}$ with $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C})$ the generalized index appears as an undetermined integer in [Riz52a; Riz52b]. It also appears in [Ede76b] and [Shp15], but neither of the authors has managed to give a satisfactory description (or full calculation) of it.

In Section 2.3 we prove several versions of Cauchy's Integral Formula over $\mathcal{A}$, building from a local "pointed" formulation, through a homotopical "pointed" and local non-pointed formulations, all the way up to the homological version, also being the most general one. Barring the latter one, all of the other formulations are *intrinsic*. This peculiar gradation of *intrinsic* versions has to do with the fact the singularity of Cauchy's Reproducing Kernel is somewhat more *complex* than that in the complex plane, so it all becomes a nice little balancing act between the contours of integration, the points of "reproducing", and the domain of definition of the ρ-holomorphic function $f(Z)$. We remark that a homological version of Cauchy's Integral Formula over $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C})$ also appears in the aformentioned papers of G.B. Rizza [Riz52a; Riz52b], but our topological hypothesis on the integration 1-cycle Γ is much weaker than his (hence our version is much stronger): while Rizza requires Γ to be null-homologous, we require Γ to be only spectrally null-homologous (see Definition 2.1.6), i.e. that only certain spectral projections (more precisely, the images under χ_j, $j \in I_\rho$) of Γ are null-homologous. A byproduct of our *spectral* proof is that this version is in a sense the strongest possible since the statement reduces to the homological Cauchy Integral Formula in the complex plane. We also prove automatic ρ-holomorphic continuation of ρ-holomorphic functions and give a complete characterization of the domains of ρ-holomorphy (Theorem 2.3.5 coupled with Corollary 2.3.9). Then we also derive the $\mathcal{A}$-analogues of various standard consequences of Cauchy's Integral Formula. Finally, we prove a Cauchy Transform over $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C})$ (see Proposition 2.3.16), from which we derive a second, but *intrinsic* proof of the ρ-analyticity of ρ-holomorphic functions (the first one being Proposition 1.1.16), as well as an $\mathcal{A}$-analogue of Morera's Theorem (Proposition 2.3.18). While Morera's theorem over $\mathcal{A}$ is certainly in the literature (except for the fact that our version for ρ-holomorphic functions rather than $\mathcal{A}$-holomorphic ones), the Cauchy Transform over $\mathcal{A}$ is to the best of our knowledge new.

Finally, in Section 2.4 we prove a Cauchy–Pompeiu formula over $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C})$ for sufficiently differentiable ${}_{\mathcal{A}}M$-valued mappings f and orientable compact surfaces $\Sigma \subset \mathcal{A}$ in a general position with respect to $\mathfrak{m}$. This connects back to the "standard" theory of *holomorphic mappings* of several complex variables as follows. Since $\mathcal{A}$-holomorphic functions are in particular holomorphic mappings, this yields for *holomorphic mappings* f a kind of a low-dimensional Bochner–Martinelli formula for f over Σ, but with a non-vanishing surface integral term and *holomorphic kernels* for both the boundary and the surface integral.

The Algebraic Classification Problem. We remark that in $\dim_{\mathbb{K}} \geq 7$ there are uncountably infinitely many isomorphism types of $(\mathcal{A}, \mathfrak{m}, \mathbb{K}) \in \mathsf{LocArt}_{\mathbb{K}}$ (see [ST03; ST68] or [Sup56]). A full classification up to $\dim_{\mathbb{K}} \mathcal{A} \leq 6$ was given by Poonen in [Poo08b]. Finding a classification/explicit parametrization of all such algebras is an important open problem (see [Poo08a] for the state of the art about the underlying moduli space). Furthermore, the representation theory of $(\mathcal{A}, \mathfrak{m}, \mathbb{K})$ with $\mathfrak{m}^2 \neq 0$ but $\mathfrak{m}^3 = 0$ is known to be wild.

Omitted Topics. It is safe to say that the number of important aspects we omitted in the present work is at least as much as the included ones. More precisely, we have not included the following:

- Any even remotely deep representation/module theory for $\mathcal{A}$ as well as any finer structure theory for $\mathcal{A}$ itself (equivalently, "deeper look into the radical");
- ρ-differentiability in the sense of Gateaux;
- A whole chapter on ρ-analyticity and mapping properties of ρ-analytic functions for both $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$;
- Any meromorphic theory and singularities over $\mathcal{A}$. This actually has to do with the appearance of a new singularity phenomenon for Laurent series due to the presence of nilpotents (compared to the complex plane), that we have dubbed "essential poles". This is most easily illustrated by an example: already in $\mathbb{C}[\epsilon]((Z))$, $\epsilon^2 = 0$, we have
$$\left(\frac{\epsilon}{Z} - i\right)\left(\frac{\epsilon}{Z} + i\right) = 1,$$
which gets progressively more complicated with higher-dimensional $\mathcal{A}$. It is an interesting problem to characterize when such singularities appear, but that is well beyond the scope of the present work. So, yes, there are important qualitative differences to the complex plane that cannot be explained away by spectral arguments!
- Solving the *inhomogeneous* Cauchy-Riemann-Scheffers equations and the corresponding analogue of Neumann's $\bar{\partial}$-problem over $\mathcal{A}$;
- Continuing the previous point, we have given no treatment of related cohomology theories. However, we refer the reader to [Mal90; Mal99; Mal08], [Shu96], [SS01], and [MS16] for the case $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{R}}$. With an underlying $\mathbb{C}$-structure the corresponding cohomology theories probably grow as $O(n^2)$.

And several more that would otherwise belong organically to the present thesis.

Applications. Even ignoring Section 2.4 or the natural appearance of $\mathcal{A}$-structures in geometry, the theory of $\mathcal{A}$-differentiable functions for $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$ is certainly not an end in itself. There is a solid historical track record of applications to differential (and integral) equations: for example, [Ket28b], [Lam49], [Wag48], [War53], [Eic61], [Ede76a], [Wat92b], [PRS13], [Lóp22], [LMT23], [AFL23], [DLO25], and many others. Even more so, if we include the non-commutative setting (see for instance [BKK03; Kra07; Kra09; Kra13; BK14; KM14; KT18] and countless others).

A Note on Terminology. In various *function theories* over $\mathcal{A} \in \mathsf{Alg}_{\mathbb{K}}$ the functions that are objects of these theories are often called *monogenic*, meaning that the corresponding notion of a derivative is independent of the choice of direction, and otherwise *polygenic*. Finite ring extensions of $\mathbb{R}$, commutative or not, have historically often been summarily called *hypercomplex numbers* (hence names like *hypercomplex analysis*, *hypercomplex variable*, *hyperholomorphic functions* etc.), but not everyone understands the same under this term. For example, it is a bit strange to call a finite ring extension of $\mathbb{R}$ that is not even a left or right $\mathbb{C}$-vector space *hypercomplex*. The bicomplex numbers $\mathbb{C}[x]/(x^2-1) \cong \mathbb{C} \times \mathbb{C}$ are sometimes also called *motor variables* (mostly by some physicists only). The "bireal" numbers $\mathbb{R}[x]/(x^2-1) \cong \mathbb{R} \times \mathbb{R}$ have many names: *double numbers*, *hyperbolic numbers*, *split-complex numbers*, *paracomplex numbers* (and hence one speaks for example of *paracomplex geometry* and *paraholomorphic functions* etc.). Some authors call $\mathcal{A}$-differentiable functions *holomorphic* regardless of whether $\mathcal{A}$ has an underlying complex structure or not, probably because they satisfy generalized Cauchy-Riemann equations (see for example [Sal23]). But perhaps the top three positions for most abused terms are taken by *analytic*, *regular*, and *function theory*.

0. Preliminaries

In Section 0.1 we give a detailed discussion of various *basic* structural properties of $\mathcal{A} \in \mathsf{FdCAlg}_K$ and its finite-dimensional representations. Even though we are ultimately interested in the case $K = \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, various aspects turn out to hold for very general reasons that can otherwise end up being obscured by superficial concreteness. To this end, the purpose of Section 0.1 is to give the right amount of general context, where each of these hold. As it happens, this does not necessarily require commutativity (especially when considering representations of $\mathcal{A}$) or finite-dimensionality or even an underlying field to well-define the former. We introduce several notions that will be crucial later on for the analysis part.

In Section 0.2 we give a short discussion of the group of units $\mathcal{A}^\times$ of $\mathcal{A} \in \mathsf{FdCAlg}_K$ and some related facts. The topology of $\mathcal{A}^\times$ plays an essential role for establishing the analogue of Cauchy's reproducing kernel over $\mathcal{A}$ in the case $K = \mathbb{C}$.

In Section 0.3 we give an overview of norm theory for $\mathcal{A} \in \mathsf{FdCAlg}_\mathbb{K}$, its (finitely generated) modules, and some related *basic* algebra and module constructions, including the advantages and disadvantages of various norm choices, painting an approximate picture of what is feasible in terms of favourable norm properties. One of the main issues is that algebra norms are generically only submultiplicative, which sometimes presents certain challenges when trying to generalize some of the landmarks of classical one-variable complex analysis. Naturally, various norm estimates, especially those relying on structure properties of finite-dimensional commutative algebras, play later on an important role for obtaining tighter or otherwise meaningful bounds.

0.1. Algebraic Preparations

Since finite-dimensional commutative algebras and their finite-dimensional representations are central to what follows, it is prudent to discuss some facts from both the commutative and non-commutative algebra worlds that will be used in various places later on. For conceptual reasons we spend some time doing so in appropriate generality since little is gained from specialization, as it turns out. Taking such an approach is also consistent with our secondary, ulterior motives of keeping one eye open towards the infinite-dimensional case. Most of the preparatory facts we present here are certainly well-known, sometimes in disguise, but we don't have a precise reference where the necessary material is consolidated in one place and organized both to our liking and in a way that supports the main part of the theory. It's also mostly an excuse to fix various notations and conventions that we are going to follow closely later on and setting a tone for the main part. So, it is the case here that, in a sense, the definitions will be more important than the actual lemmas as the latter will mostly serve for *proofing* the definitions, demonstrating whether we have indeed made the correct choice of definitions (or not). Thus Section 0.1 has more of a (modest) foundational flavour and is about deducing statements for the right reasons in line with the goals we have set ourselves in the introduction. But in reality it is mostly just a careful navigation through a lot of simple observations. One can identify three central themes recurring throughout:

(I) the back-and-forth passage between morphisms and representations of algebras;

(II) *standard* factorizations of morphisms between different products of algebras and of representations of products of algebras, reducing the analysis later on to morphisms and representations of *blocks*;

(III) emphasis on the spectral point of view and *its interplay with products.*

In Section 0.1.1 we give a preliminary discussion relating to (I). The context of that is the desire to eventually put the morphism and representation setting on equal footing for the purposes the analysis later on.

In Section 0.1.2 we have consolidated some well-known material on the relationship between idempotents and direct (product or sum) decompositions, giving it a treatment more to our liking.

In Section 0.1.3 we introduce the perhaps lesser known notion of *external* direct products and connect it to the familiar (internal) direct products. This has to do with the fact that the structure sheaves we introduce in Section 1.1 exhibit naturality with respect to both kinds of products.

In Section 0.1.4 we finally introduce certain *convenient corestrictions* of representations. This is a more in-depth continuation of Section 0.1.1, further addressing (I). Very briefly, *convenient corestrictions* provide a way to pass from the inherently non-commutative world of representations of commutative algebras to the commutative world of morphisms (of commutative algebras) in a way that is tuned to our setting. These will reappear most prominently in Section 2.2, where it will be more *convenient* to work with morphisms instead of representations.

In Section 0.1.5 we tackle (II), relying on Section 0.1.2. A subtle point is that, even though the standard factorizations can be deduced for both morphisms and representations in great generality, surprisingly neither one follows from the other in such generality, which necessitates certain amount of repetitiveness in establishing the validity in both settings. We will examine this at some length. We remark that one is able to deduce these factorizations *from each other* only after sufficient specialization (for example, this is so for our case of interest, commutative Artinian $\mathbb{K}$-algebras).

In Section 0.1.6 we introduce *relative* notions of certain basic algebraic objects. These in turn sort of specialize into "*product* notions" in Section 0.1.7 and "*spectral* notions" in Section 0.1.9, though not entirely strictly.

In Section 0.1.7, as just mentioned, we introduce *relative* versions – with respect to direct products – of certain basic algebraic objects of interest.

In Section 0.1.8 we deal very briefly with the fact of life that preimages of units need not be themselves units (as well as some related issues). This is obviously *related* to our *relative* point of view. Basically, it has to do with our difference quotient definition in Section 1.1 which only requires $\rho(H)$, resp. $\varphi(H)$ to be invertible. On the bright side, this gives more flexibility to the definition of the derivative, which in turn allows for more *natural* proofs of the "functorial" behavior of this regularity class down the lane.

In Section 0.1.9 we introduce the spectral point of view (issue (III)) (characters of algebras) in a general *algebraic* setting and some corresponding notions that will be essential later on. In doing analysis over a general $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$ (instead of a local $(\mathcal{A}, \mathfrak{m}, \mathbb{K})$) and over its representations, it becomes self-evident that this is the conceptually correct point of view, at least for the time being.

In Section 0.1.10 we deal with finite products of local rings as a somewhat more general venue than commutative Artinian rings. This is also where Section 0.1.7 and Section 0.1.9 finally meet and start to interact with each together. To this end, we give, however, the following **notational forewarning**: as special cases of Section 0.1.6 we will have introduced in Section 0.1.7 the notations $\mathcal{A}^{\times}_{(I)}$, $\mathrm{Sing}_{(I)}(\mathcal{A})$, $\mathfrak{J}_{(I)}(\mathcal{A})$, $[a]_{(I)}$, $[a]^{\times}_{(I)}$, and in Section 0.1.9 the notations $\mathcal{A}^{\times}_{\mathcal{I}}$, $\mathrm{Sing}_{\mathcal{I}}(\mathcal{A})$, $\mathfrak{J}_{\mathcal{I}}(\mathcal{A})$, $[a]_{\mathcal{I}}$, $[a]^{\times}_{\mathcal{I}}$, respectively. In Section 0.1.10 we will then have a self-evident natural correspondence $I \leftrightarrow \mathcal{I}$ and then *prove* that the two sets of notions do actually agree in that special setting. In other words, the notation will indeed eventually converge, nevertheless the reader should keep in mind that it ultimately expresses *very different and generically unrelated points of view*. On top of that, given $X \subseteq \mathcal{A}$, we will also have the notations X_I and $X_{(I)}$ (introduced in Definition 0.1.6 in Section 0.1.2), $\widetilde{X}_{(I)}$ (introduced in Definition 0.1.35 in Section 0.1.7), and $\widehat{X}_{\mathcal{I}}$ (introduced in Definition 0.1.42 in Section 0.1.9). The latter two do not have precise matches in the "*relative* notions" from Section 0.1.6, but rather are closures thereof.

In Section 0.1.11 we discuss some key aspects about the residue field κ of a commutative local *ring* $(\mathcal{A}, \mathfrak{m}, \kappa)$, but most importantly under what circumstances it induces a κ-algebra structure on the local ring $\mathcal{A}$. Even then, an important subtlety is that $\mathcal{A} \supset \kappa$ need not imply that the composition $\kappa \hookrightarrow \mathcal{A} \twoheadrightarrow \kappa$ is the identity on κ, which is, however, what we are ultimately after. In other words, one needs to answer the question when the "character" map $\chi\colon \mathcal{A} \twoheadrightarrow \kappa$ admits a *section* $\iota\colon \kappa \hookrightarrow \mathcal{A}$. When that is the case, κ is called a coefficient field for $\mathcal{A}$. We list a few general well-known criteria for this to happen. The problem is most naturally understood in the setting of *complete* local rings. Somewhat out of place, positive characteristic will pop up, but this is implicitly only so we can exclude it[1].

In Section 0.1.12 we finally restrict attention to our special case of interest – commutative Artinian rings and algebras. Its purpose is to accumulate some basic critical mass of concrete commutative algebra that will be needed later on for the analysis part on $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$.

In Section 0.1.13 we essentially summarize the preceeding sections and how all these different sets of notions and perspectives converge together in the special setting of commutative Artinian algebras. More precisely, Section 0.1.13 is a direct continuation of Section 0.1.10, where we *show*, among other things, how the now matching "product" and "spectral" notions tie back to the original "relative" notions from Section 0.1.6 for a given representation $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$ or morphism $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$.

0.1.1. Preamble: Morphisms vs. Representations

Given $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$, we will need to consider two intimately related perspectives that, however, will ultimately give rise to two very different categorical structures, namely working with morphisms $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ in $\mathsf{FdCAlg}_{\mathbb{K}}$ (relative point of view) vs. working with finite-dimensional representations $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$ of $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$ (representation-theoretic point of view for not very short), that is, $M \in \mathsf{FdMod}(\mathcal{A})$ (the equivalent module-theoretic point of view). Worse still, despite the equivalence of the latter two we will need to be somewhat flexible and will occasionally have to vary (of course, in principle interchangably) between the representation-theoretic formulation and the module-theoretic one, depending on which one is more convenient or natural in the given context. Even though $\mathcal{A}$ itself will be eventually assumed commutative, we will also introduce in the following subsections some aspects from non-commutative algebra as a natural setting for the representation-theoretic point of view.

As previously promised, much of it can and will be done in a rather general, ring-theoretic setting. But rings are just $\mathbb{Z}$-algebras, while algebras are just rings with an additional compatibility structure, and so we are faced with a choice. Since the main focus of the present work is on algebras, it is more natural to take the perspective $\mathsf{Ring} = \mathsf{Alg}_{\mathbb{Z}}$ and likewise $\mathsf{CRing} = \mathsf{CAlg}_{\mathbb{Z}}$. Thus we fix a base ring $R \in \mathsf{CRing}$ and we let $\mathcal{A}, \mathcal{B} \in \mathsf{Alg}_R$ (or $\mathcal{A}, \mathcal{B} \in \mathsf{CAlg}_R$). We also make the following conventions. In representation-theoretic context we will use λ to denote representations (corresponding to *left* modules) and ρ to denote *anti*-representations (corresponding to *right* modules). This should not be confused with the occasional usage of λ (together with κ) as a *residue field* (it should be clear

[1] *Know thine enemy.*

from the context anyway). Later on, when we finally restrict our attention to commutative algebras, we will only use ρ to denote representations of $\mathcal{A} \in \mathsf{CAlg}_R$. In the same vein, we will denote by $\lambda_{\mathcal{B}} \colon \mathcal{B} \hookrightarrow \mathrm{End}_R(\mathcal{B})$, $b \mapsto (\ell_b \colon x \mapsto bx)$, the regular representation and by $\rho_{\mathcal{B}} \colon \mathcal{B} \hookrightarrow \mathrm{End}_R(\mathcal{B})$, $b \mapsto (r_b \colon x \mapsto xb)$, the regular anti-representation of $\mathcal{B}$. Both are indeed always injective since $\forall\, b \in \mathcal{B} \colon \lambda_{\mathcal{B}}(b)(1_B) = \rho_{\mathcal{B}}(b)(1_B) = b$.

The astute reader will observe that the relative point of view ought to be treated as a special case of the representation-theoretic one since any $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ induces an $\mathcal{A}$-bimodule[2] structure ${}_{\mathcal{A}}\mathcal{B}_{\mathcal{A}}$ in the usual way via $\forall\, a \in \mathcal{A}\ \forall\, b \in \mathcal{B} \colon a \cdot b \coloneqq \varphi(a)b$ and $b \cdot a \coloneqq b\varphi(a)$. Moreover, since φ is multiplicative, it is in particular also a homomorphism of $\mathcal{A}$-bimodules: indeed we have $\forall\, \alpha, a \in \mathcal{A} \colon \varphi(\alpha \cdot a) = \varphi(\alpha)\varphi(a) \overset{\text{def}}{=} \alpha \cdot \varphi(a)$ and $\varphi(a \cdot \alpha) = \varphi(a)\varphi(\alpha) \overset{\text{def}}{=} \varphi(a) \cdot \alpha$. We will denote the corresponding left, right, and $\mathcal{A}$-bimodule maps by ${}_{\mathcal{A}}\varphi \colon {}_{\mathcal{A}}\mathcal{A} \to {}_{\mathcal{A}}\mathcal{B}$, $\varphi_{\mathcal{A}} \colon \mathcal{A}_{\mathcal{A}} \to \mathcal{B}_{\mathcal{A}}$, and ${}_{\mathcal{A}}\varphi_{\mathcal{A}} \colon {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \to {}_{\mathcal{A}}\mathcal{B}_{\mathcal{A}}$, respectively. In other words, we have very generally a map

$$\begin{aligned} (-)^{\mathrm{rep}} \colon \mathrm{Mor}_{\mathsf{Alg}_R}(\mathcal{A}, \mathcal{B}) &\to \mathrm{Mor}_{\mathsf{Alg}_R}(\mathcal{A}, \mathrm{End}_{\mathsf{Mod}_R}(\mathcal{B})) \\ \varphi &\mapsto (\varphi^{\mathrm{rep}} \colon a \mapsto (\hat{a} \colon b \mapsto \varphi(a)b)). \end{aligned} \tag{0.1.1}$$

More precisely, φ actually induces a representation-anti-representation pair:

$$\varphi \mapsto \varphi^{\mathrm{rep}} \coloneqq (\lambda_{\varphi} \coloneqq \lambda_{\mathcal{B}} \circ \varphi, \rho_{\varphi} \coloneqq \rho_{\mathcal{B}} \circ .\varphi) \tag{0.1.2}$$

Obviously, this distinction disappears for CAlg_R. Having said that, there are two more *practical* issues for us with this passage from morphisms to representations:

(i) it forgets the extra algebra structure on $\mathcal{B}$;
(ii) it is not *functorial* in any suitable sense;
(iii) the treatment of direct products becomes more awkward because (even finite) direct products of modules generally do not match direct products of the corresponding representations as *morphisms* (or equivalently, the endomorphism algebra of a direct sum of modules is not a direct product of the individual endomorphism algebras unless the modules are pair-wise Hom-orthogonal).

One conceivable way around this is to leave CAlg_R altogether and work in the larger category Alg_R since then representations become morphisms. For us this concretely means abandoning $\mathsf{FdCAlg}_{\mathbb{K}}$ for the larger $\mathsf{FdAlg}_{\mathbb{K}}$, which qualitatively changes the venue of doing analysis over algebras to such an extent that it effectively renders our rather direct approach to the subject unviable!

In the opposite direction, an alternative to the above would be to instead consider some *corestriction* of the representation λ to a *nice* commutative subalgebra, for example its image $\lambda(\mathcal{A})$, since the source is already commutative, which clearly turns the corestricted λ into a morphism in CAlg_R again. In summary, we thus have so far generically

$$\begin{aligned} (\mathcal{A} \xrightarrow{\varphi} \mathcal{B}) &\mapsto (\varphi^{\mathrm{rep}} \colon \mathcal{A} \xrightarrow{\varphi} \mathcal{B} \xhookrightarrow{\lambda_{\mathcal{B}}} \mathrm{End}_R(\mathcal{B})) \\ (\mathcal{A} \xrightarrow{\lambda} \mathrm{End}_R(M)) &\mapsto (\lambda^{\mathrm{im}} \colon \mathcal{A} \to \lambda(\mathcal{A}) \subseteq \mathrm{End}_R(M)), \end{aligned} \tag{0.1.3}$$

where λ^{im} denotes the obvious image corestriction. For our purposes, however, it will turn out later on that this definition of a CAlg_R-morphism out of λ is not the most practical one and that there is a more *convenient* one, which we will simply dub *the convenient corestriction* of λ and which for the intents of analysis behaves much better with respect to our other central topic, namely direct products.

Details follow, but for now observe that the assignment $\varphi \mapsto \varphi^{\mathrm{rep}}$ preserves φ since $\lambda_{\mathcal{B}}$ is always injective. In the other direction, even though now $\lambda(\mathcal{A}) \in \mathsf{BMod}(\mathcal{A})$ via $\lambda^{\mathrm{im}} \mapsto (\lambda^{\mathrm{im}})^{\mathrm{rep}}$, the assign-

[2] clearly, we have $\forall\, a_1, a_2 \in \mathcal{A}\ \forall\, b \in \mathcal{B} \colon (a_1 \cdot b) \cdot a_2 = a_1 \cdot (b \cdot a_2)$;

ment $\lambda \mapsto \lambda^{\mathrm{im}}$ is certainly lossy as we lose track of the precise $M \in \mathsf{LMod}(\lambda(\mathcal{A}))$, of which there are many, and λ itself certainly need not be injective. Finally,

$$(\varphi^{\mathrm{rep}})^{\mathrm{im}} \overset{\mathrm{def}}{=} (\lambda_{\mathcal{B}} \circ \varphi)^{\mathrm{im}} = \lambda_{\mathcal{B}}^{\mathrm{im}} \circ \varphi \overset{\mathrm{def}}{=} (\lambda_{\mathcal{B}})^{\mathrm{im}} \circ \varphi = \varphi \tag{0.1.4}$$

since $\lambda_{\mathcal{B}}^{\mathrm{im}} \colon \mathcal{B} \xrightarrow{\cong} \lambda_{\mathcal{B}}(\mathcal{B})$ is a *very* canonical isomorphism. On the other hand,

$$(\lambda^{\mathrm{im}})^{\mathrm{rep}} \overset{\mathrm{def}}{=} \lambda_{\lambda(\mathcal{A})} \circ \lambda^{\mathrm{im}} \colon \mathcal{A} \to \mathrm{End}_R(\lambda(\mathcal{A})) \tag{0.1.5}$$

with $\mathrm{End}_R(\lambda(\mathcal{A}))$ in place of $\mathrm{End}_R(M)$, where $\lambda(\mathcal{A}) \subseteq \mathrm{End}_R(M)$. We will navigate these issues in order to reconcile the morphism-theoretic and the representation-theoretic points of view so as they can be treated on equal footing *at least for our purposes.*

Problem 1: Which representations λ (really, compatible pairs (λ, ρ) of representation and anti-representation in the sense of BMod) arise from morphisms φ?

0.1.2. Direct Decompositions & Idempotents

We gather here some basic stuff about idempotents and decompositions that is certainly well-known, but is unfortunately often scattered throughout the literature in lemmas, exercises, and remarks. We thus use the opportunity to organize it in a somewhat comprehensive way to our liking and, more crucially, *tailored to our exposition later on.* To this end, the terminology might *slightly* vary.

Definition 0.1.1 (Idempotents)**:** Let $e, e_1, e_2, \dots, e_n \in \mathcal{A}$ be idempotents.
1. e is called non-trivial $:\Leftrightarrow e \notin \{0,1\}$.
2. e is called central $:\Leftrightarrow e \in \mathcal{Z}(\mathcal{A})$.
3. e_1, e_2 are called orthogonal $:\Leftrightarrow e_1e_2 = e_2e_1 = 0$.
4. e is called primitive $:\Leftrightarrow e \neq 0$ and $e \neq e_1 + e_2$ for any two non-trivial orthogonal idempotents e_1, e_2.
5. e is called centrally primitive $:\Leftrightarrow e \neq 0$ central and $e \neq e_1 + e_2$ for any two non-trivial orthogonal central idempotents e_1, e_2, i.e. e cannot be further decomposed into elements of the same kind.
6. The system $\{e_1, \dots, e_n\}$ is itself called (centrally) primitive $:\Leftrightarrow \forall\, 1 \leq j \leq n \colon e_j$ is (centrally) primitive.
7. The system $\{e_1, \dots, e_n\}$ is called complete $:\Leftrightarrow \sum_{j=1}^n e_j = 1_{\mathcal{A}}$.

Remarks:
1. The condition for a set $\{e_1, \dots, e_n\} \subset \mathcal{A}$ to be a system of mutually orthogonal idempotents can be succinctly expressed as $\forall\, 1 \leq i,j \leq n \colon e_ie_j = \delta_{ij}e_j$.
2. Any system $\{e_1, \dots, e_n\} \subset \mathcal{A}$ of mutually orthogonal idempotents can be *orthogonally completed*: putting $e \coloneqq \sum_{j=1}^n e_j$ and $e_{n+1} \coloneqq 1 - e$, we have $e^2 \overset{\mathrm{def}}{=} (\sum_{j=1}^n e_j)^2 = \sum_{j=1}^n e_j^2 + \sum_{i\neq j} e_ie_j = \sum_{j=1}^n e_j + \sum_{i \neq j} \delta_{ij}e_j = e$, i.e. e is idempotent, hence so is e_{n+1}, and moreover $\forall\, 1 \leq i \leq n \colon e_ie = ee_i \overset{\mathrm{def}}{=} \sum_{j=1}^n e_ie_j = \sum_{j=1}^n \delta_{ij}e_j = e_i$, hence $\forall\, 1 \leq i \leq n \colon e_ie_{n+1} = e_{n+1}e_i \overset{\mathrm{def}}{=} e_i(1-e) = e_i - e_ie = e_i - e_i = 0$.

Definition 0.1.2: Let $\mathcal{A} \in \mathsf{Alg}_R$ and $M \in \mathsf{LMod}(\mathcal{A})$.
1. M is called finitely decomposable if there exists a finite family $\{M_k\}_{1 \leq k \leq m} \subset \mathsf{LMod}(\mathcal{A})$ of indecomposable $\mathcal{A}$-submodules of M such that $M = \bigoplus_{k=1}^m M_k$.
2. If M is finitely decomposable, then a decomposition $M = \bigoplus_{k=1}^m M_k$ into $\mathcal{A}$-submodules is called complete[3] if M_k is indecomposable for every $1 \leq k \leq m$.

Analogously for $M \in \mathsf{RMod}(\mathcal{A})$ or $\mathsf{BMod}(\mathcal{A})$.

[3]sometimes also called a *Krull-Schmidt decomposition*, e.g. in [Lam01], or an *indecomposable decomposition*;

Remarks:

(1) In particular, any *indecomposable* module is finitely decomposable with $m = 1$.

(2) In general, complete (finite) decompositions need not be unique (up to a choice of an ordering of the indecomposable modules). Nevertheless, if M happens to admit a *strong*[4] complete decomposition $M = \oplus_{k=1}^{m} M_k$, then the Krull–Remak–Schmidt–Azumaya Theorem (e.g. see [Lam01, Thm. 19.21]) tells us that it is unique both in terms of the number of summands m and their isomorphism types $[M_k]$, $1 \leq k \leq m$. This actually encompasses a wide variety of important examples.

Definition 0.1.3 (Decomposability in Alg_R)**:** Let $\mathcal{A} \in \mathsf{Alg}_R$. Then:

(1) $\mathcal{A}$ is called left decomposable if $\mathcal{A} = \bigoplus_{j=1}^{n} \mathfrak{L}_j$ for some non-trivial $\{\mathfrak{L}_j\}_{1\leq j\leq n} \subset \mathsf{LMod}(\mathcal{A})$ (left ideals).

(2) $\mathcal{A}$ is called right decomposable if $\mathcal{A} = \bigoplus_{j=1}^{n} \mathfrak{R}_j$ for some non-trivial $\{\mathfrak{R}_j\}_{1\leq j\leq n} \subset \mathsf{RMod}(\mathcal{A})$ (right ideals).

(3) $\mathcal{A}$ is called centrally decomposable if $\mathcal{A} = \bigoplus_{j=1}^{n} \mathfrak{A}_j$ for some non-trivial $\{\mathfrak{A}_j\}_{1\leq j\leq n} \subset \mathsf{BMod}(\mathcal{A})$ (two-sided ideals).

(4) $\mathcal{A}$ is called block-decomposable (or $\mathcal{A}$ admits a block decomposition) if $\mathcal{A}$ is **finitely** centrally decomposable. When that is the case, the pieces $\mathfrak{A}_j$, $1 \leq j \leq n$, will be called the blocks of $\mathcal{A}$.

Remarks:

(1) In particular, it follows from the definition that *a block* of $\mathcal{A}$ is itself indecomposable over $\mathcal{A}$ and that every centrally indecomposable $\mathcal{A}$ (equivalently, indecomposable as a bimodule over itself) is its (only) own block.

(2) One should be aware that the terminology of what constitutes a block varies in the literature. For our intents and purposes the above definition is completely sufficient.

Lemma 0.1.4 (Decomposability & Idempotents)**:** Let $\mathcal{A} \in \mathsf{Alg}_R$.

(1) Let $M \in \mathsf{LMod}(\mathcal{A})$. The following are equivalent:

  (i) ${}_{\mathcal{A}}M = \bigoplus_{k=1}^{m} {}_{\mathcal{A}}M_k$ with $0 \neq M_k \in \mathsf{LMod}(\mathcal{A})$, $1 \leq k \leq m$.

  (ii) $\mathrm{End}_{\mathcal{A}}({}_{\mathcal{A}}M)$ contains a complete system of mutually orthogonal non-trivial idempotents $\{\mathcal{E}_1, \ldots, \mathcal{E}_m\}$, in which case we also have $\forall\, 1 \leq k \leq m$: $M_k = \mathcal{E}_k(M)$.

  Analogously for $\mathsf{RMod}(\mathcal{A})$ and $\mathsf{BMod}(\mathcal{A})$.

(2) The following are equivalent:

  (i) $\mathcal{A}$ admits a complete system of mutually orthogonal non-trivial idempotents $\{e_1, \ldots, e_n\} \subset \mathcal{A}$.

  (ii) Left Peirce Decomposition: $\mathcal{A} = \bigoplus_{j=1}^{n} \mathfrak{L}_j$ in $\mathsf{LMod}(\mathcal{A})$, where $\mathfrak{L}_j = \mathcal{A}e_j \notin \{0, \mathcal{A}\}$, $1 \leq j \leq n$, are left ideals.

  (iii) Right Peirce Decomposition: $\mathcal{A} = \bigoplus_{j=1}^{n} \mathfrak{R}_j$ in $\mathsf{RMod}(\mathcal{A})$, where $\mathfrak{R}_j = e_j\mathcal{A} \notin \{0, \mathcal{A}\}$, $1 \leq j \leq n$, are right ideals.

(3) The following are equivalent:

  (i) The system of idempotents $\{e_1, \ldots, e_n\}$ in *(2)* is central.

  (ii) We have $\forall\, 1 \leq j \leq n$: $\mathfrak{L}_j = \mathcal{A}e_j = e_j\mathcal{A} = \mathfrak{R}_j =: \mathfrak{A}_j \lhd \mathcal{A}$ and $\mathcal{A} = \bigoplus_{j=1}^{n} \mathfrak{A}_j$ in $\mathsf{BMod}(\mathcal{A})$.

  (iii) $\mathcal{A} \cong \prod_{j=1}^{n} \mathfrak{A}_j$ in Alg_R, where $\mathfrak{A}_j \ncong 0, \mathcal{A}$, $1 \leq j \leq n$.

(4) The above decompositions are complete iff, additionally, the system $\{e_1, \ldots, e_n\}$ is primitive.

(5) If $\mathcal{A}$ admits a block decomposition $\mathcal{A} \cong \prod_{j=1}^{n} \mathcal{A}_j$, then the latter is unique up to permutation of the blocks $\mathcal{A}_j$, $1 \leq j \leq n$.

Proof: To (1): This is a standard generalization of, say, [Jac89][Prop. 3.1, p. 111]. Since we don't

[4] the submodule M_k is assumed to be strongly indecomposable, i.e. $\mathrm{End}_R(M_k)$ is a local ring, $1 \leq k \leq m$;

know a precise reference for the general statement, we include a proof. We only prove the $\mathsf{LMod}(\mathcal{A})$ case since the other cases are completely analogous.

"(i) $\Rightarrow$ (ii)": As $\prod \cong \coprod$ in $\mathsf{LMod}(\mathcal{A})$, we have retraction diagrams

$$\begin{tikzcd} M_k \arrow[r, hook, "\varepsilon_k"] \arrow[dr, "\mathrm{id}_{M_k}"'] & M \arrow[d, "\mathrm{pr}_k"] \\ & M_k \end{tikzcd}$$

for all $1 \leq k \leq m$. Define $\mathcal{E}_k \coloneqq \varepsilon_k \circ \mathrm{pr}_k \colon M \twoheadrightarrow M_k \hookrightarrow M$, $1 \leq k \leq m$. Then clearly $\forall\, 1 \leq k \leq m\colon \mathcal{E}_k \in \mathrm{End}_{\mathcal{A}}(M) \setminus \{0, \mathrm{id}_M\}$ and $\mathcal{E}_k^2 \overset{\text{def}}{=} \varepsilon_k \circ \mathrm{pr}_k \circ \varepsilon_k \circ \mathrm{pr}_k = \varepsilon_k \circ \mathrm{id}_{M_k} \circ \mathrm{pr}_k = \varepsilon_k \circ \mathrm{pr}_k \overset{\text{def}}{=} \mathcal{E}_k$. Writing $x = x_1 \oplus \cdots \oplus x_m$ for $x \in M$, we have $(\sum_{k=1}^m \mathcal{E}_k)(x) \overset{\text{def}}{=} \sum_{k=1}^m \sum_{\ell=1}^m (\varepsilon_k \circ \mathrm{pr}_k)(x_\ell) = \sum_{k=1}^m \varepsilon_k(\sum_{\ell=1}^m \delta_{k\ell} x_\ell) = \sum_{k=1}^m \varepsilon_k(x_k) = \sum_{k=1}^m x_k = x$, that is, $\sum_{k=1}^m \mathcal{E}_k = \mathrm{id}_M$. Finally, if $1 \leq k \neq \ell \leq m$, then $\mathrm{pr}_k \circ \varepsilon_\ell = 0$ in $\mathrm{Hom}_{\mathcal{A}}(M_\ell, M_k)$ since $M_k \cap M_\ell = 0$, hence $\mathcal{E}_k \mathcal{E}_\ell \overset{\text{def}}{=} \varepsilon_k \circ \mathrm{pr}_k \circ \varepsilon_\ell \circ \mathrm{pr}_\ell = \varepsilon_k \circ 0 \circ \mathrm{pr}_\ell = 0$. Thus, $\{\mathcal{E}_1, \ldots, \mathcal{E}_m\}$ is indeed a complete system of mutually orthogonal non-trivial idempotents in $\mathrm{End}_{\mathcal{A}}(M)$.

"(ii) $\Rightarrow$ (i)": Define $M_k \coloneqq \mathcal{E}_k(M) \subseteq M$, $1 \leq k \leq m$. Since $\mathcal{E}_k \neq 0$, we have $M_k \neq 0$, $1 \leq k \leq m$. Since $\sum_{k=1}^m \mathcal{E}_k = \mathrm{id}_M$, we clearly have $M = \sum_{k=1}^m M_k$. It remains to show that this is a direct sum. Observe that $\forall\, x_k \in M_k\colon \mathcal{E}_k(x_k) = x_k$ as $\mathcal{E}_k^2 = \mathcal{E}_k$, $1 \leq k \leq m$. Let $K \subsetneqq [m]$, $\ell \in [m] \setminus K$, and $x \in M_\ell \cap \sum_{k \in K} M_k$ with $x = \sum_{k \in K} \mathcal{E}_k(y_k)$ for some $y_k \in M$, $k \in K$. Then, using idempotence and orthogonality, we get $x = \mathcal{E}_\ell(x) = \mathcal{E}_\ell(\sum_{k \in K} \mathcal{E}_k(y_k)) = \sum_{k \in K} \mathcal{E}_\ell(\mathcal{E}_k(y_k)) = 0$. This completes the proof.

To (2): "(i) $\Rightarrow$ (ii)": Define $\mathfrak{L}_j \coloneqq \mathcal{A} e_j$, which is clearly a left ideal of $\mathcal{A}$, and for $a \in \mathcal{A}$ put $a_j \coloneqq a e_j \in \mathfrak{L}_j$, $1 \leq j \leq n$. Since $1_{\mathcal{A}} = \sum_{j=1}^n e_j$, we have $a = \sum_{j=1}^n a_j$, hence $\mathcal{A} = \sum_{j=1}^n \mathfrak{L}_j$. Observe that if $a \in \mathfrak{L}_j$, writing $a = \tilde{a} e_j$ for some $\tilde{a} \in \mathcal{A}$, we get $a_j \overset{\text{def}}{=} a e_j = \tilde{a} e_j^2 = \tilde{a} e_j = a$, $1 \leq j \leq n$. Suppose now that $a \in \mathfrak{L}_i \cap \mathfrak{L}_j$ for some $i \neq j$, hence $a = a_i = a_i e_i = a_j = a_j e_j$. Then $a = a_i = a_i e_i = a_j e_i = a_j e_j e_i = 0$ by orthogonality. It follows that $\forall\, i \neq j\colon \mathfrak{L}_i \cap \mathfrak{L}_j = 0$, hence $\mathcal{A} = \bigoplus_{j=1}^n \mathfrak{L}_j$.

"(ii) $\Rightarrow$ (i)": If $a \in \mathcal{A}$, then by assumption there exist unique $a_j \in \mathfrak{L}_j$, $1 \leq j \leq n$, such that $a = \oplus_{j=1}^n a_j$. Define accordingly $1_{\mathcal{A}} \eqqcolon \oplus_{j=1}^n e_j$. We have $\forall\, a \in \mathcal{A}\ \forall\, 1 \leq j \leq n\colon \mathfrak{L}_j \ni a_j = a_j 1_{\mathcal{A}} = \oplus_{i=1}^n a_j e_i \in \bigoplus_{i=1}^n \mathfrak{L}_i$ (since $\mathfrak{L}_i$, $1 \leq i \leq n$, are *left* ideals), therefore $a_j = a_j e_j$, hence $\mathfrak{L}_j \subseteq \mathcal{A} e_j$, and $\forall\, i \neq j\colon a_j e_i = 0$. In particular, taking $a_j \coloneqq e_j$, $1 \leq j \leq n$, it follows that $\forall\, 1 \leq j \leq n\colon e_j^2 = e_j$ and $\forall\, i \neq j\colon e_j e_i = 0$, that is, $\{e_1, \ldots, e_n\} \subset \mathcal{A}$ is a complete system of mutually orthogonal idempotents. Similarly, $\forall\, a \in \mathcal{A}\colon a = \oplus_{j=1}^n a_j = a 1_{\mathcal{A}} = a(\oplus_{j=1}^n e_j) = \oplus_{j=1}^n a e_j \in \bigoplus_{j=1}^n \mathfrak{L}_j$, therefore $\forall\, a \in \mathcal{A}\ \forall\, 1 \leq j \leq n\colon a_j = a e_j$, hence $\mathcal{A} e_j \subseteq \mathfrak{L}_j$. Thus, $\forall\, 1 \leq j \leq n\colon \mathfrak{L}_j = \mathcal{A} e_j$.

"(i) $\Leftrightarrow$ (iii)": Analogously, but with the corresponding accomodations for *right* ideals $\mathfrak{R}_j$, $1 \leq j \leq n$.

To (3): "(i) $\Rightarrow$ (ii)": If, additionally, $\{e_1, \ldots, e_n\} \subset \mathcal{Z}(\mathcal{A})$, then clearly $\forall\, 1 \leq j \leq n\colon \mathfrak{A}_j \coloneqq \mathfrak{L}_j = \mathcal{A} e_j = e_j \mathcal{A} = \mathfrak{R}_j$ are two-sided ideals of $\mathcal{A}$. The decomposition follows from (1).

"(ii) $\Rightarrow$ (i)": We only need to show that $e_1, \ldots, e_n \in \mathcal{Z}(\mathcal{A})$. Since $\mathfrak{A}_j \lhd \mathcal{A}$, the identity $a 1_{\mathcal{A}} = 1_{\mathcal{A}} a$, $a \in \mathcal{A}$, implies that $a_j e_j = e_j a_j$, and moreover $\forall\, i \neq j\colon a_j e_i = 0 = e_i a_j$, $1 \leq j \leq n$, which proves the claim since $a = \sum_{j=1}^n a_j$.

"(i) & (ii) $\Rightarrow$ (iii)": By hypothesis, we have $\mathcal{A} = \bigoplus_{j=1}^n \mathfrak{A}_j$, where $\forall\, 1 \leq j \leq n\colon \mathfrak{A}_j = \langle e_j \rangle \lhd \mathcal{A}$. Since $e_j \in \mathcal{Z}(\mathcal{A})$, we have that $\mathfrak{A}_j$ is itself a unital R-algebra $1_{\mathfrak{A}_j} = e_j$, $1 \leq j \leq n$. Furthermore, in

$\mathsf{BMod}(\mathcal{A})$ finite coproducts coincide with finite products:

$$\phi\colon \mathcal{A} = \bigoplus_{j=1}^{n} \mathfrak{A}_j \xrightarrow{\cong} \prod_{j=1}^{n} \mathfrak{A}_j =: \mathfrak{A}$$
$$a_1 \oplus \cdots \oplus a_n \mapsto (a_1, \ldots, a_n)$$

We are going to show that ϕ is in fact also an isomorphism in Alg_R. We only need to show that ϕ is multiplicative, but this follows from

$$ab = (\oplus_{i=1}^n a_i)(\oplus_{j=1}^n b_j) = \sum_{1\le i,j\le n} a_i b_j = \sum_{1\le i,j\le n} e_i e_j ab = \sum_{j=1}^{n} e_j a e_j b = \sum_{j=1}^{n} a_j b_j = \oplus_{j=1}^n a_j b_j,$$

using orthogonality and centrality of system $\{e_1, \ldots, e_n\}$.

"(iii) $\Rightarrow$ (ii) & (i)": To avoid notational overload, we write instead $\mathcal{A} \cong \prod_{j=1}^n \mathcal{A}_j =: \mathfrak{A}$ in Alg_R for the right hand side and denote by $\phi\colon \mathcal{A} \xrightarrow{\cong} \mathfrak{A}$ the given isomorphism. Now set $\mathfrak{A}_j' := \{0_{\mathcal{A}_1}\} \times \cdots \times \{0_{\mathcal{A}_{j-1}}\} \times \mathcal{A}_j \times \{0_{\mathcal{A}_{j+1}}\} \times \cdots \times \{0_{\mathcal{A}_n}\} \subseteq \mathfrak{A}$ and $e_j' := (0_{\mathcal{A}_1}, \ldots, 0_{\mathcal{A}_{j-1}}, 1_{\mathcal{A}_j}, 0_{\mathcal{A}_{j+1}}, \ldots, 0_{\mathcal{A}_n}) \in \mathfrak{A}$, $1 \le j \le n$. Clearly, $\forall\, 1 \le j \le n\colon \mathfrak{A}_j' = \langle e_j' \rangle \lhd \mathfrak{A}$ such that $\mathfrak{A} = \bigoplus_{j=1}^n \mathfrak{A}_j'$ in $\mathsf{BMod}(\mathfrak{A})$, and $\{e_1', \ldots, e_n'\} \subset \mathcal{Z}(\mathfrak{A})$ is a complete system of mutually orthogonal idempotents. Putting

$$\mathfrak{A}_j := \phi^{-1}(\mathfrak{A}_j') \text{ and } e_j := \phi^{-1}(e_j'),\ 1 \le j \le n, \tag{0.1.6}$$

it is easy to check that they do the job.

To (4): "$\Rightarrow$": Suppose that the system $\{e_1, \ldots, e_n\}$ is not primitive, i.e. suppose without loss of generality for concreteness that $e_1 = e_1' + e_2''$ for some non-trivial orthogonal idempotents $e_1', e_1'' \in \mathcal{A}$. Then by the same arguments as in "(i) $\Rightarrow$ (ii)" in (1) we have $\mathfrak{L}_1 = \mathfrak{L}_1' \oplus \mathfrak{L}_1''$, where $\mathfrak{L}_1' := \mathcal{A}e_1'$ and $\mathfrak{L}_1'' := \mathcal{A}e_1''$ are non-trivial left ideals. Alternatively, it suffices to observe that $\mathfrak{L}_1 = \mathfrak{L}_1' + \mathfrak{L}_1''$ and that $\{e_1', e_1'', e_2, \ldots, e_n\}$ is again a complete system of mutually orthogonal non-trivial idempotents: for $2 \le j \le n$ we have $e_1'e_j + e_1''e_j = e_1e_j = 0$, hence $0 = e_1'(e_1'e_j + e_1''e_j) = e_1'e_j + e_1'e_1''e_j = e_1'e_j$ and similarly $e_je_1' = e_1''e_j = e_je_1'' = 0$. (This would imply that $\mathfrak{L}_1' \oplus \mathfrak{L}_1'' \oplus \mathfrak{L}_2 \oplus \cdots \oplus \mathfrak{L}_n$ is indeed a refinement of $\mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus \cdots \oplus \mathfrak{L}_n$.)

"$\Leftarrow$": Suppose without loss of generality for concreteness that $\mathfrak{L}_1$ is further decomposable as $\mathfrak{L}_1 = \mathfrak{L}_1' \oplus \mathfrak{L}_1''$ and for the corresponding $e_1 \in \mathfrak{L}_1$ write $e_1 = e_1' \oplus e_2''$ accordingly. Since $e_1', e_1'' \in \mathfrak{L}_1$, we have $e_1' = e_1'e_1$ and $e_1'' = e_1''e_1$, hence $\mathfrak{L}_1' \ni e_1' = e_1'e_1 = e_1'^2 \oplus e_1'e_1''$ in $\mathfrak{L}_1' \oplus \mathfrak{L}_1''$, hence $e_1' = e_1'^2$ and $e_1'e_1'' = 0$. Analogously, $e_1'' = e_1''^2$ and $e_1''e_1' = 0$. Finally, since $\mathfrak{L}_1', \mathfrak{L}_1'' \notin \{0, \mathcal{A}\}$, it is clear that e_1', e_1'' are non-trivial.

To (5): This is completely standard, see e.g. [Lam01, Prop. 22.1 on p. 327]. □

Remarks:

(1) In other words, $\mathcal{A} \in \mathsf{Alg}_R$ is *left* decomposable iff it is *right* decomposable, or equivalently, $\mathcal{A}$ is *left indecomposable* iff it is *right indecomposable* (over itself). Thus, we can refer to $\mathcal{A} \in \mathsf{Alg}_R$ as just **decomposable** or **indecomposable** (over itself) without having to choose sides. In the same vein, $\mathcal{A}$ is *finitely decomposable* (over itself) iff it contains a complete (in particular, finite) system of mutually orthogonal non-trivial *primitive* idempotents. Finally, Lemma 0.1.4 (3) justifies calling $\mathcal{A} \in \mathsf{Alg}_R$ *indecomposable in* Alg_R if $\mathcal{A}$ cannot be written as a (finite) *product* $\prod_{j=1}^n \mathcal{A}_j$ in Alg_R, even though $\prod$ is *not the coproduct* in Alg_R. Likewise for calling $\mathcal{A}$ *finitely decomposable* and $\prod_{j=1}^n \mathcal{A}_j$ a *complete decomposition* in Alg_R.

(2) The results in Lemma 0.1.4 are only equivalence-of-existence. In general, a complete decomposition need not even exist to begin with despite an abundance of idempotents (e.g. infinite direct products of rings).

(3) A byproduct of the proof of Lemma 0.1.4 (2) is that $\mathcal{A} \in \mathsf{Alg}_R$ can be decomposed into at most finitely many (non-trivial) ideals.
(4) An important relevant for us example of blocks comes from local rings. First recall that if $\mathcal{B} = (\mathcal{B}, \mathfrak{n})$, then $\mathcal{B}$ does not admit non-trivial idempotents: if $e \in \mathcal{B}$ is an idempotent, then so is $1 - e$, and $e(1 - e) = 0$, but we have either $e \in \mathfrak{n}$, $1 - e \in \mathcal{B}^\times$, or vice versa, hence $e \in \{0_{\mathcal{B}}, 1_{\mathcal{B}}\}$ regardless. It follows that $(\mathcal{B}, \mathfrak{n})$ is indecomposable over itself by Lemma 0.1.4. Thus any finite product of local R-algebras is automatically a *block decomposition*.
(5) If $\mathcal{A} \in \mathsf{CAlg}_R$, then Lemma 0.1.4 (3) automatic as every idempotent will be central.
(6) In many standard references on Commutative Algebra, versions of Lemma 0.1.4 (3) are proved by passing to quotients and, say, invoking the Chinese Remainder Theorem. We remark that the included proof, among other things, does not utilize quotients.
(7) **Internalization:** but Lemma 0.1.4 (3) is first and foremost about *internalizing* the inherently *external* notion of (categorical) products in Alg_R so as to not have to constantly drag around or worry about some isomorphism $\phi\colon \mathcal{A} \xrightarrow{\cong} \prod_{j=1}^n \mathcal{A}_j$ that *may or may not* (but usually can be arranged to) stay an isomorphism in additional settings like Banach algebras, metric spaces, submanifolds etc. Indeed, product decompositions (like for instance that of commutative Artinian rings into local Artinian rings) generally hold only up to not even unique permutation isomorphisms, *unless* we have explicitly postulated $\mathcal{A} := \prod_{j=1}^n \mathcal{A}_j$ to begin with. On the other hand, the above $\bigoplus$-decompositions are *internal* in the sense of decompositions into subobjects, and the isomorphism $\phi\colon \bigoplus \cong \prod$ (finite) in $\mathsf{Mod}(\mathcal{A})$ *is unique* (as a result of the universal property), however $\bigoplus$ is a coproduct only in $\mathsf{Mod}(\mathcal{A})$, but not in Alg_R. So, here is how to eat the cake and have it too without having to break basic category theory conventions, spelled out: if we denote by $p_i\colon \mathfrak{A} := \prod_{j=1}^n \mathfrak{A}_j \twoheadrightarrow \mathfrak{A}_i$ the canonical i-th projection in Alg_R, then defining $\mathrm{pr}_i := \mathrm{pr}_i^{\mathcal{A}} := \phi^* p_i \overset{\text{def}}{=} p_i \circ \phi$ gives the projection $\mathrm{pr}_i^{\mathcal{A}}\colon \mathcal{A} = \bigoplus_{j=1}^n \mathfrak{A}_j \twoheadrightarrow \mathfrak{A}_i$ in $\mathsf{Mod}(\mathcal{A})$, that also *happens* to be a projection in Alg_R, $1 \leq i \leq n$. In other words, *the cake* is the *frankensteined* pair $(\mathcal{A} = \bigoplus_{j=1}^n \mathfrak{A}_j, \{\mathrm{pr}_j^{\mathcal{A}}\}_{1 \leq j \leq n})$, and there is only one way to bake it since ϕ is unique. Essentially, this is Wedderburn's outdated notion of a *direct sum of algebras. The overarching issue being that later on it will be much more convenient to do our analysis set-theoretically than to try so categorically and generally avoiding checking things up to some (no top of all)* ***algebraic*** *isomorphism.* Equivalently and perhaps more cleanly, one can instead simply and unambiguously postulate $\mathcal{A} := \prod_{j=1}^n \mathcal{A}_j$ from the outset. After all, $\mathcal{A}$ is isomorphic to only finitely many different sets, so one of them ought to be the right one, right? Anyhow, we will revisit the internalization issue for modules in Section 0.1.3.

Corollary 0.1.5 (Translation between M and $\mathrm{End}(M)$ (in)decomposability)**:** By Lemma 0.1.4 (1) and (2) we have in particular the following *translation* between module language and algebra language:

$$M \text{ indecomposable } \Leftrightarrow \mathrm{End}(M) \text{ indecomposable.} \tag{0.1.7}$$

In particular, if M is indecomposable, then $\mathrm{End}(M)$ is a block (i.e. centrally indecomposable) since the latter is just a special form of decomposability. We thus have the following "commutative diagram" of implications:

$$\begin{array}{ccccc} {}_R M \text{ indec.} & \Longrightarrow & {}_{\mathcal{A}} M \text{ indec.} & \Longrightarrow & {}_{\mathrm{End}_R(M)} M \text{ indec.} \\ \Updownarrow & & \Updownarrow & & \Updownarrow \\ \mathrm{End}_R(M) \text{ indec.} & \overset{\supseteq}{\Longrightarrow} & \mathrm{End}_{\mathcal{A}}(M) \text{ indec.} & \overset{\supseteq}{\Longrightarrow} & \mathrm{End}_{\mathrm{End}_R(M)}(M) = \mathcal{Z}(\mathrm{End}_R(M)) \text{ indec.} \end{array} \tag{0.1.8}$$

where the first row is induced by restriction of scalars and the second row is induced by inclusion of the corresponding endomorphism algebras. □

Let now $\{\mathcal{A}_j\}_{1\le j\le n} \subset \mathsf{Alg}_R$ and $\mathcal{A} \coloneqq \prod_{j=1}^n \mathcal{A}_j$. We thus have natural epimorphisms

$$\begin{aligned}\mathrm{pr}_j \coloneqq \mathrm{pr}_j^{\mathcal{A}} \colon \mathcal{A} &\twoheadrightarrow \mathcal{A}_j\\ (a_1,\dots,a_n) &\mapsto a_j,\end{aligned}$$

for all $1 \le j \le n$.

Definition 0.1.6: Let $\varnothing \neq I \subseteq [n]$ and $X \subseteq \mathcal{A}$. We put:
(1) $\mathcal{A}_I \coloneqq \prod_{j\in I} \mathcal{A}_j$.
(2) $X_j \coloneqq \mathrm{pr}_j(X) \coloneqq \mathrm{pr}_j^{\mathcal{A}}(X) \subseteq \mathcal{A}_j$, $1 \le j \le n$.
(3) $X_I \coloneqq \prod_{j\in I} X_j$ and

$$X_{(I)} \coloneqq \begin{cases} X_j, \text{ if } j \in I\\ \mathcal{A}_j, \text{ otherwise.}\end{cases}$$

Furthermore, for any $\varnothing \neq I \subseteq [n]$ we have an obvious *multi-projection* morphism

$$\mathrm{pr}_I \coloneqq \mathrm{pr}_I^{\mathcal{A}} \colon \mathcal{A} \twoheadrightarrow \mathcal{A}_I.$$

In the other direction we also have:

Definition 0.1.7:
(1) Canonical non-unital inclusions (embeddings[5]) of R-algebras: $\forall\, 1 \le j \le n$:

$$\begin{aligned}\varepsilon_j \coloneqq \varepsilon_j^{\mathcal{A}} \colon \mathcal{A}_j &\hookrightarrow \mathcal{A}\\ a_j &\mapsto (0,\dots,0,a_j,0,\dots,0)\end{aligned}$$

(2) Multi-Inclusions: if $\varnothing \neq I \subseteq [n]$, then

$$\begin{aligned}\varepsilon_I \coloneqq \varepsilon_I^{\mathcal{A}} \colon \mathcal{A}_I &\hookrightarrow \mathcal{A}\\ (a_j)_{j\in I} &\mapsto (a_1,\dots,a_n),\end{aligned}$$

where $a_j = 0$ whenever $j \notin I$.

Following the notations of Lemma 0.1.4 and its proof, we clearly have $e_j = \varepsilon_j(1_{\mathcal{A}_j})$ and $\varepsilon_j(\mathcal{A}_j) = \mathfrak{A}_j \lhd \mathcal{A}$, $1 \le j \le n$, as well as $\varepsilon_I(\mathcal{A}_I) = \bigoplus_{j\in I} \mathfrak{A}_j \lhd \mathcal{A}$. Moreover, $\mathrm{pr}_j \circ \varepsilon_j = \mathrm{id}_{\mathcal{A}_j}$, $\varepsilon_j \circ \mathrm{pr}_j = \varepsilon_j$, $1 \le j \le n$, $\sum_{j=1}^n \varepsilon_j \circ \mathrm{pr}_j = \mathrm{id}_{\mathcal{A}}$, $\mathrm{pr}_I \circ \varepsilon_I = \mathrm{id}_{\mathcal{A}_I}$, and $\varepsilon_I \circ \mathrm{pr}_I = \varepsilon_I$.

To finish off the section, we recall the following relaxed form of the notion of direct products, that oftentimes turns out to be completely sufficient:

Definition 0.1.8 (Subdirect Products)**:** An object $\mathcal{A} \in \mathsf{Alg}_R$ is called a subdirect product of a family $\{\mathcal{B}_i\}_{i\in I} \subset \mathsf{Alg}_R$, if there exists an inclusion[6] $\iota\colon \mathcal{A} \hookrightarrow \mathcal{B} \coloneqq \prod_{i\in I} \mathcal{B}_i$ such that the induced map

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook, "\iota"] \arrow[dr, dashed, two heads, "\pi_i"'] & \mathcal{B} \arrow[d, two heads, "\mathrm{pr}_i^{\mathcal{B}}"] \\ & \mathcal{B}_i \end{tikzcd}$$

is surjective for all $i \in I$.

0.1.3. Internal vs. External Direct Products

Even though we will be only interested in finite *direct products*, we should temporarily distinguish between two different kinds – *internal* direct products (or just direct products) and *external* direct

[5] we take here some poetic freedom to call non-unital inclusions of possibly unital R-algebras embeddings.
[6] the monomorphisms in Ring are always injective;

products (for the lack of a better term). As it will turn out, the distinction is somewhat superficial, but nonetheless conceptually convenient as it will reappear in Section 1.1.

We first briefly discuss the *internal* direct product of modules and its *internalization*[7]. To this end we fix a single $\mathcal{A} \in \mathsf{Alg}_R$ and consider a family $\{M_i\}_{i\in I} \subset \mathsf{LMod}(\mathcal{A})$ with $\lambda_i \colon \mathcal{A} \to \operatorname{End}_R(M_i)$, $1 \in I$, being the corresponding representations. Then the *internal* direct product is just the familiar categorical direct product $\prod_{i\in I} M_i \in \mathsf{LMod}(\mathcal{A})$ with the obvious $\mathcal{A}$-action $a \cdot (x_i)_{i\in I} := (a \cdot x_i)_{i\in I}$. As already mentioned in Section 0.1.2, even for $|I| < \infty$ a pet peeve of the direct product is the categorical artefact that such product representations generically hold only up to (unique permutation) isomorphisms unless one actually specifies the object explicitly as in $M := \prod_{k=1}^m M_k$. For instance, if $M_1, M_2 \in \mathsf{LMod}(\mathcal{A})$, then both $(M_1 \times M_2, \mathrm{pr}_1, \mathrm{pr}_2)$ and $(M_2 \times M_1, \mathrm{pr}_2, \mathrm{pr}_1)$ are products for the pair (M_1, M_2), yet in general $M_1 \times M_2 \neq M_2 \times M_1$ already in Set, but only $M_1 \times M_2 \cong M_2 \times M_1$ in $\mathsf{LMod}(\mathcal{A})$ (as it should). In the finite case ($|I| < \infty$) this can be rectified since the finite *internal* direct product in $\mathsf{LMod}(\mathcal{A})$ can be itself *internalized* as it coincides with the finite coproduct in $\mathsf{LMod}(\mathcal{A})$ (=direct sum), whose corresponding diagram consists of actual set-theoretical inclusions. More precisely, if we put $M := M_1 \oplus_{\mathcal{A}} M_2$ for the *external direct sum*, we have indeed $M_1 \times M_2 \cong M_1 \oplus_{\mathcal{A}} M_2$ (equipped with projections $M \twoheadrightarrow M_1, M_2$, or equivalently, $M_1 \times M_2$ is a categorical coproduct via the inclusions $M_1, M_2 \hookrightarrow M_1 \times M_2$, $x_1, x_2 \mapsto (x_1, 0), (0, x_2)$), but then we also have $M_1, M_2 \hookrightarrow M$ set-theoretically and the external direct sum becomes an *internal direct sum* with M itself being the *ambient* $\mathcal{A}$-module with equalities $M_2 \oplus_{\mathcal{A}} M_1 = M = M_1 \oplus_{\mathcal{A}} M_2$ rather than just isomorphisms. Such *internalization* is very convenient as it allows us to stop dragging around up-to-isomorphisms and pass to (set-theoretical) equalities instead.

Lemma-Definition 0.1.9 (External Products & The Category of All Modules)**:**

(1) Compatibility of morphisms: given $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ in Alg_R^* and ${}_RM \xrightarrow{\Phi} {}_RN$ in $\mathsf{Mod}(R)$, the pair (φ, Φ) is called compatible, if $M \in \mathsf{LMod}(\mathcal{A})$, $N \in \mathsf{LMod}(\mathcal{B})$, and
$$\forall\, A \in \mathcal{A} \colon \varphi(A) \circ \Phi = \Phi \circ A \tag{0.1.9}$$
with the understanding that $A \in \operatorname{End}_R(M)$ and $\varphi(A) \in \operatorname{End}_R(N)$. In that case we also call φ to be over Φ and write diagramatically
$$\begin{tikzcd} \mathcal{A} \arrow[r, "\varphi"] \arrow[d, no head, "\curvearrowright"] & \mathcal{B} \arrow[d, no head, "\curvearrowright"] \\ {}_RM \arrow[r, "\Phi"] & {}_RN \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} {}_RM \arrow[r, "\Phi"] \arrow[d, no head, "\curvearrowleft"] & {}_RN \arrow[d, no head, "\curvearrowleft"] \\ \mathcal{A} \arrow[r, "\varphi"] & \mathcal{B} \end{tikzcd} \tag{0.1.10}$$
In other words, we simply have $\Phi \in \operatorname{Hom}_{\mathcal{A}}({}_{\mathcal{A}}M, \varphi^*({}_{\mathcal{B}}N))$.

(2) The category $\mathsf{LMod}(\mathsf{Alg}_R^*)$ of **all** left modules over non-unital R-algebras: it is given by the data

 (i) Objects:
$$\mathrm{Obj}(\mathsf{LMod}(\mathsf{Alg}_R^*)) := \{{}_{\mathcal{A}}M := (\mathcal{A}, M) : \mathcal{A} \in \mathsf{Alg}_R^*, M \in \mathsf{LMod}(\mathcal{A})\}$$

 (ii) Morphisms:
$$\operatorname{Hom}_{\mathcal{A},\mathcal{B}}(M, N) := \operatorname{Mor}(\mathcal{A}, M; \mathcal{B}, N) :=$$
$$:= \operatorname{Mor}((\mathcal{A}, {}_{\mathcal{A}}M), (\mathcal{B}, {}_{\mathcal{B}}N)) := \{(\varphi, \Phi) : M \xrightarrow{\Phi} N \text{ is over } \mathcal{A} \xrightarrow{\varphi} \mathcal{B}\}.$$

 One defines $\mathsf{RMod}(\mathsf{Alg}_R^*)$ and $\mathsf{BMod}(\mathsf{Alg}_R^*)$ analogously.

(3) External product: if $\{{}_{\mathcal{A}_i}M_i\}_{i\in I} \subset \mathsf{LMod}(\mathsf{Alg}_R^*)$ is a family, then the eXternal direct product

[7] There is an unfortunate overload of terminology here, but both notions of *internal* are unrelated: the *internal direct product* refers to a product that is internal to some fixed category, e.g. $\mathsf{LMod}(\mathcal{A})$, while *internalization* refers here to replacing category-theoretical isomorphisms by set-theoretical equalities.

$\bigtimes_{i\in I}({}_{\mathcal{A}_i}M_i)$ is given by $\prod_{i\in I} M_i$ equipped with $\prod_{i\in I}\mathcal{A}_i \curvearrowright \prod_{i\in I} M_i$, $(a_i)_{i\in I}\cdot(x_i)_{i\in I} \coloneqq (a_i\cdot x_i)_{i\in I}$. Thus, putting $\mathcal{A} \coloneqq \prod_{i\in I}\mathcal{A}_i$, $M \coloneqq \prod_{i\in I} M_i$, we have in particular $M \in \mathsf{LMod}(\mathcal{A})$, and $\bigtimes_{i\in I}({}_{\mathcal{A}_i}M_i)$ coincides with the (internal) categorical product in $\mathsf{LMod}(\mathsf{Alg}^*_R)$, which is given by $(\mathcal{A}, M)$ equipped with the pairs of canonical projections $\{(\mathcal{A} \xrightarrow{\pi_i}\!\!\!\!\rightarrow \mathcal{A}_i, M \xrightarrow{\Pi_i}\!\!\!\!\rightarrow M_i)\}_{i\in I}$.

Proof: To (3): Firstly, given $M_j \in \mathsf{LMod}(\mathcal{A}_j)$, the canonical projection $\pi_j\colon \mathcal{A} \twoheadrightarrow \mathcal{A}_j$ in Alg^*_R induces ${}_{\mathcal{A}}M_j \coloneqq \pi_j^*({}_{\mathcal{A}_j}M_j)$, and we have $\forall\, a = (a_i)_{i\in I} \in \mathcal{A}\ \forall\, x = (x_i)_{i\in I} \in M\colon \Pi_j(a\cdot x) \overset{\text{def}}{=} \Pi_j((a_i\cdot x_i)_{i\in I}) = a_j\cdot x_j \overset{\text{def}}{=} a\cdot x_j$, hence (π_j, Π_j) is indeed a compatible pair, $j \in I$. It remains to show that $(\mathcal{A}, M)$ together with the given projections satisfies the universal property of a categorical product in $\mathsf{LMod}(\mathsf{Alg}^*_R)$:

$$\begin{tikzcd} (\mathcal{B}, N) \arrow[r, "(\varphi_j,\Phi_j)"] \arrow[d, dashed, "\exists_1(\varphi,\Phi)"'] & (\mathcal{A}_j, M_j) \\ (\mathcal{A}, M) \arrow[ur, two heads, "(\pi_j,\Pi_j)"'] & \end{tikzcd} \tag{0.1.11}$$

Clearly, $\varphi\colon \mathcal{B} \to \mathcal{A}$, $b \mapsto (\varphi_i(b))_{i\in I}$, and $\Phi\colon N \to M$, $x \mapsto (\Phi_i(x))_{i\in I}$, is the unique pair of maps that satisfies the above diagram for all $j \in I$, and it is a compatible pair by element-wise compatability (of the pairs (φ_i, Φ_i), $i \in I$). $\square$

Remark: The reader will immediately recognize that the external product $\bigtimes_{i\in I}{}_{\mathcal{A}_i}M_i$ really is an object in $\prod_{i\in I}\mathsf{LMod}(\mathcal{A}_i)$, the (internal) categorical product (of these categories) within CAT, the category of all categories. One should keep in mind that module categories are not small. $\square$

Lemma 0.1.10 (Translation between *internal* and *external* products)**:**

(1) Internalization: if $\{(\mathcal{A}_i, M_i)\}_{i\in I} \subset \mathsf{LMod}(\mathsf{Alg}^*_R)$ is an arbitrary family, then

$${}_{\mathcal{A}}\Big(\bigtimes_{i\in I}{}_{\mathcal{A}_i}M_i\Big) = \prod_{i\in I}{}_{\mathcal{A}}M_i \overset{\text{def}}{=} \prod_{i\in I}\pi_i^*({}_{\mathcal{A}_i}M_i),$$

where $\mathcal{A} \coloneqq \prod_{i\in I}\mathcal{A}_i$ in Alg^*_R. That is, the external product is internal with respect to $\mathcal{A}$.

(2) Externalization: if $\mathcal{A} \in \mathsf{Alg}^*_R$ and $\{M_i\}_{i\in I} \subset \mathsf{LMod}(\mathcal{A})$ is an arbitrary family, then

$$\prod_{i\in I}{}_{\mathcal{A}}M_i = \Delta^*\Big(\bigtimes_{i\in I}{}_{\mathcal{A}}M_i\Big),$$

where $\Delta\colon \mathcal{A} \hookrightarrow \mathcal{A}^I$, $a \mapsto (a)_{i\in I}$, is the canonical diagonal embedding. That is, the internal product is a diagonal restriction of scalars of the external one.

Proof: To (1): this is just Lemma-Definition 0.1.9 (3).

To (2): For every $i \in I$ we have a commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook, "\Delta"] \arrow[dr, "\mathrm{id}_{\mathcal{A}}"'] & \mathcal{A}^I \arrow[d, "\pi_i"] \\ & \mathcal{A} \end{tikzcd} \tag{0.1.12}$$

Thus:

$$\begin{aligned} \Delta^*\Big(\bigtimes_{i\in I}{}_{\mathcal{A}}M_i\Big) &\overset{\text{def}}{=} \Delta^*\Big(\prod_{i\in I}\pi_i^*({}_{\mathcal{A}}M_i)\Big) = \prod_{i\in I}\Delta^*\pi_i^*({}_{\mathcal{A}}M_i) = \prod_{i\in I}(\pi_i\circ\Delta)^*({}_{\mathcal{A}}M_i) = \\ &= \prod_{i\in I}\mathrm{id}^*_{\mathcal{A}}({}_{\mathcal{A}}M_i) = \prod_{i\in I}{}_{\mathcal{A}}M_i \end{aligned} \tag{0.1.13}$$

as desired. $\square$

One can combine Lemma 0.1.4 and Lemma 0.1.10 to show the following well-known fact (really, rather a restatement of these):

Proposition 0.1.11: Let $\{\mathcal{A}_j\}_{1\leq j\leq n} \subset \mathsf{Alg}_R$ and $\mathcal{A} := \prod_{j=1}^n \mathcal{A}_j$. Then there is an isomorphism of categories

$$\mathsf{LMod}(\prod_{j=1}^n \mathcal{A}_j) \cong \prod_{j=1}^n \mathsf{LMod}(\mathcal{A}_j)$$

given by the functors

$$F\colon \mathsf{LMod}(\prod_{j=1}^n \mathcal{A}_j) \to \prod_{j=1}^n \mathsf{LMod}(\mathcal{A}_j)$$
$$M \mapsto (e_1 M, \ldots, e_n M)$$

and

$$G\colon \prod_{j=1}^n \mathsf{LMod}(\mathcal{A}_j) \to \mathsf{LMod}(\prod_{j=1}^n \mathcal{A}_j)$$
$$(M_1, \ldots, M_n) \mapsto {}_{\mathcal{A}}\Big(\bigtimes_{j=1}^n {}_{\mathcal{A}_j}M_j\Big),$$

where $\{e_1, \ldots, e_n\}$ denotes the corresponding complete system of mutually orthogonal non-trivial central idempotents for $\mathcal{A}$. Analogously for RMod. □

Remark: Proposition 0.1.11 fails for infinite products or for Alg_R^* in place of Alg_R, though.

We end the subsection with fixing *matching* product terminology[8] for morphisms:

Definition 0.1.12: Let $\mathsf{C} \in \mathsf{CAT}$.

(1) Internal product of morphisms: if $X \in \mathsf{C}$ and $X \xrightarrow{\varphi_i} Y_i$, $i \in I$, is a family of morphisms in C such that $Y := \prod_{i\in I} Y_i$ exists, then their **internal** product is defined to be the **universal** morphism

$$(\varphi_i)_{i\in I}\colon X \to Y.$$

For example, if $\mathsf{C} = \mathsf{Set}$, we simply have

$$(\varphi_i)_{i\in I}\colon X \to \prod_{i\in I} Y_i$$
$$x \mapsto (\varphi_i(x))_{i\in I}.$$

(2) External product[9] of morphisms: if $X_i \xrightarrow{\varphi_i} Y_i$, $i \in I$, is a family of morphisms in C such that both $X := (X, \{\mathrm{pr}_i^X\}_{i\in I}) := \prod_{i\in I} X_i$ and $Y := \prod_{i\in I} Y_i$ exist, then their **external** product is defined to be the **internal** product

$$\times_{i\in I} \varphi_i := (\varphi_i \circ \mathrm{pr}_i^X)_{i\in I}\colon X \to Y.$$

For example, if $\mathsf{C} = \mathsf{Set}$, we simply have

$$\times_{i\in I} \varphi_i\colon \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$$
$$(x_i)_{i\in I} \mapsto (\varphi_i(x_i))_{i\in I}.$$

Given $\{X \xrightarrow{\varphi_i} Y_i\}_{i\in I}$ in Set, we have of course

$$(\varphi_i)_{i\in I} = (\times_{i\in I} \varphi_i) \circ \Delta,$$

where $\Delta\colon X \hookrightarrow X^I$, $x \mapsto (x)_{i\in I}$, is the obvious *diagonal* inclusion.

[8] as far as we know, there is none universally accepted despite the ubiquity of both constructions;
[9] also called Cartesian product of morphisms by some authors;

0.1.4. Convenient Corestrictions

Generically, given an arbitrary family $\{M_i\}_{i\in I} \subset \mathsf{Mod}(R)$, we have $M_i \in \mathsf{LMod}(\mathrm{End}_R(M_i))$, $i \in I$, hence $\bigtimes_{i\in I}({}_{\mathrm{End}_R(M_i)}M_i) \in \mathsf{LMod}(\prod_{i\in I}\mathrm{End}_R(M_i))$. Accordingly, putting $M := \prod_{i\in I} M_i$ in $\mathsf{Mod}(R)$, we have a natural faithful *"diagonal" product representation*

$$\iota_{\mathrm{prod}}\colon \prod_{i\in I}\mathrm{End}_R(M_i) \hookrightarrow \mathrm{End}_R(M) \tag{0.1.14}$$

via factor-wise action. *With regard to direct products*, this homomorphism plays a key role for the translation between the *representation*[10] point of view and the *module* point of view as follows. If $\lambda_i\colon \mathcal{A}_i \to \mathrm{End}_R(M_i)$, $i \in I$, denotes a family of algebra representations and $\mathcal{A} := \prod_{i\in I}\mathcal{A}_i$, then their *external* product (as morphisms) in Alg_R is given by $\times_{i\in I}\lambda_i\colon \prod_{i\in I}\mathcal{A}_i \to \prod_{i\in I}\mathrm{End}_R(M_i)$. In other words, the representation corresponding to the external product of $({}_{\mathcal{A}_i}M_i)_{i\in I}$ (as an $\mathcal{A}$-module) is *almost, but not quite* the external product of the individual representations, since it is actually given by the composition

$$\iota_{\mathrm{prod}} \circ (\mathop{\times}_{i\in I}\lambda_i)\colon \mathcal{A} \overset{\mathrm{def}}{=} \prod_{i\in I}\mathcal{A}_i \to \prod_{i\in I}\mathrm{End}_R(M_i) \hookrightarrow \mathrm{End}_R(M).$$

Since ι_{prod} is an Alg_R-inclusion, it thus makes sense to treat it as an **identification** of sorts (onto its image), a perspective that we shall adopt henceforth. Then one can get away with the following:

Meta-Lemma 1: The *external* product of representations as morphisms corresponds to the *external* product of the respective modules. □

On the other hand, if $\forall\, i \in I\colon \mathcal{A}_i = \mathcal{A}$, then the representation corresponding to the *internal* product $\prod_{i\in I\ \mathcal{A}}M_i$ is given by the diagonal pull-back of the product of the individual representations:

$$\iota_{\mathrm{prod}} \circ (\mathop{\times}_{i\in I}\lambda_i) \circ \Delta\colon \mathcal{A} \hookrightarrow \mathcal{A}^I \to \prod_{i\in I}\mathrm{End}_R(M_i) \hookrightarrow \mathrm{End}_R(M). \tag{0.1.15}$$

(cf. Lemma 0.1.10). In particular, the *Cartesian* product of $\mathcal{A}$-modules does **not** directly match the *Cartesian* product of the corresponding representations as morphisms. This would grow into a technical annoyance later on that therefore needs to be nipped in the bud (as one should). We come conceptually prepared:

Meta-Lemma 2: The (internal) product of modules corresponds to the *internal* product of the corresponding representations.

Proof: By Section 0.1.3 we have $(\lambda_i)_{i\in I} = (\times_{i\in I}\lambda_i) \circ \Delta$, hence

$$\iota_{\mathrm{prod}} \circ (\lambda_i)_{i\in I}\colon \mathcal{A} \to \prod_{i\in I}\mathrm{End}_R(M_i) \hookrightarrow \mathrm{End}_R(M)$$

by Eq. (0.1.15). □

When the product $M \overset{\mathrm{def}}{=} \prod_{i\in I} M_i$ is finite (which is what we are mainly interested in), there is a rather explicit description of the "identification" ι_{prod} as follows. Let $\mathsf{C} \in \mathsf{Cat}$, let $X \in \mathsf{C}$, and recall the functor $h_X(-) := \mathrm{Mor}_{\mathsf{C}}(X, -)\colon \mathsf{C} \to \mathsf{Set}$ and the cofunctor $h^X(-) := \mathrm{Mor}_{\mathsf{C}}(-, X)\colon \mathsf{C} \to \mathsf{Set}$, or equivalently, $h^X(-)\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ as a *functor*. As *functors*, both $h_X(-)$ and $h^X(-)$ are *continuous*, i.e. $h_X(\lim_{i\in I} Y_i) \cong \lim_{i\in I} h_X(Y_i)$ and $h^X(\lim_{i\in I} Y_i) \cong \lim_{i\in I} h^X(Y_i)$ for any diagram $\{Y_i\}_{i\in I}$ in C or C^{op}, respectively. Since limits in C^{op} correspond to colimits in C, this means in particular that $h_X(-)$ preserves products and $h^X(-)$ preserves coproducts in C. Taking $\mathsf{C} := \mathsf{Mod}(R)$, we have that $\forall\, M, N \in \mathsf{Mod}(R)\colon \mathrm{Hom}_R(M, N) \in \mathsf{Mod}(R)$ via $\forall\, r \in R\ \forall\, f \in \mathrm{Hom}_R(M, N)\ \forall\, x \in M\colon (r \cdot f)(x) := rf(x) =$

[10] as a morphism;

$f(rx)$, hence $\mathrm{Hom}_R(-,-)$ is in fact an *internal*[11] Hom, i.e. $\mathrm{Hom}_R(M,-), \mathrm{Hom}_R(-,N)\colon \mathsf{Mod}(R) \to \mathsf{Mod}(R)$ rather than just Set, where $\mathrm{Hom}_R(M,-)$ preserves arbitrary direct products in $\mathsf{Mod}(R)$ and $\mathrm{Hom}_R(-,N)$ preserves arbitrary direct sums in $\mathsf{Mod}(R)$ as just observed. In particular, since $\bigoplus_{\mathrm{fin}} \cong \prod_{\mathrm{fin}}$ in $\mathsf{Mod}(R)$, we have that $\mathrm{Hom}_R(M,-)$ preserves *finite direct sums* and $\mathrm{Hom}_R(-,N)$ preserves *finite direct products*.[12] More specifically, following our *internalization philosophy* from Section 0.1.3, we thus obtain the *equalities*

$$\mathrm{Hom}_R(M, \bigoplus_{k=1}^m M_k) = \bigoplus_{k=1}^m \mathrm{Hom}_R(M, M_k)$$
$$\mathrm{Hom}_R(\bigoplus_{k=1}^m M_k, M) = \bigoplus_{k=1}^m \mathrm{Hom}_R(M_k, M)$$

rather than mere isomorphisms, where explicitly

$$\begin{aligned} (i_\ell)_*\colon \mathrm{Hom}_R(M, M_\ell) &\hookrightarrow \mathrm{Hom}_R(M, \bigoplus_{k=1}^m M_k) \\ f &\mapsto i_\ell \circ f \end{aligned}$$

for $i_\ell\colon M_\ell \hookrightarrow \bigoplus_{k=1}^m M_k$ denoting the canonical inclusion of R-modules, $1 \leq \ell \leq m$, and

$$\begin{aligned} \mathrm{Hom}_R(M_\ell, M) &\hookrightarrow \mathrm{Hom}_R(\bigoplus_{k=1}^m M_k, M) \\ f &\mapsto (\oplus_{k=1}^m x_k \mapsto f(x_\ell)). \end{aligned}$$

Now let $M := \bigoplus_{k=1}^m M_k$ in $\mathsf{LMod}(\mathcal{A})$, hence also in $\mathsf{Mod}(R)$, and let $\lambda\colon \mathcal{A} \to \mathrm{End}_R(M)$ denote the corresponding representation. It follows that we have the *equality*

$$\mathrm{End}_R(M) \overset{\mathrm{def}}{=} \mathrm{End}_R\left(\bigoplus_{k=1}^m M_k\right) \overset{\mathrm{def}}{=} \mathrm{Hom}_R\left(\bigoplus_{k=1}^m M_k, \bigoplus_{\ell=1}^m M_\ell\right) = \bigoplus_{k=1}^m \bigoplus_{\ell=1}^m \mathrm{Hom}_R(M_k, M_\ell), \tag{0.1.16}$$

where the last object can be neatly organized as the matrix $\left(\bigoplus_{k=1}^m \bigoplus_{\ell=1}^m \mathrm{Hom}_R(M_k, M_\ell)\right)_{1\leq \ell,k\leq m}$ and is indeed canonically an R-algebra in the R-linear category $\mathsf{Mod}(R)$ via the composition $\circ$ of morphisms. It comes equipped with the obvious $\mathsf{Mod}(R)$-projections

$$\mathrm{pr}_{k,\ell}^{\mathrm{End}(M)}\colon \mathrm{End}_R(M) \twoheadrightarrow \mathrm{Hom}_R(M_k, M_\ell),\ 1 \leq k, \ell \leq m,$$

which, however, are in general **not** Alg_R-morphisms in the *corner* cases $k = \ell$, unless the decomposition is actually *orthogonal* in the sense that $\forall\, 1 \leq k \neq \ell \leq m\colon \mathrm{Hom}_R(M_k, M_\ell) = 0$. On the other hand, following Lemma 0.1.4 (3) and Definition 0.1.7 we do get a natural (diagonal) inclusion of the *corner* R-algebras

$$\begin{tikzcd} \mathrm{End}_R(M_\ell) \arrow[r, hook, "\varepsilon_\ell"] & \prod_{k=1}^m \mathrm{End}_R(M_k) \arrow[r, hook, dashed, "\iota_{\mathrm{cor}}"] \arrow[dr, "\cong"'] & \mathrm{End}_R(M) \\ & & \bigoplus_{k=1}^m \mathrm{Hom}_R(M_k, M_k) \arrow[u, hook, "\subseteq"'] \end{tikzcd} \tag{0.1.17}$$

But while ε_ℓ, $1 \leq \ell \leq m$, are only Alg_R^*-inclusions, ι_{cor} is in fact an honest Alg_R-inclusion, as one can easily check. Consequently, even though usually products act on products, we have in the finite case Alg_R *products acting on* $\mathsf{Mod}(R)$ *direct sums*:

$$\prod_{k=1}^m \mathrm{End}_R(M_k) \curvearrowright M = \bigoplus_{k=1}^m M_k \cong \prod_{k=1}^m M_k$$

[11] yet another usage of *internal*;

[12] Without further assumptions on $R \in \mathsf{CRing}$ and/or $M, N \in \mathsf{Mod}(R)$ this is about the best one can expect as there are counter-examples otherwise.

(factor-wise). Moreover, we clearly have a commutative diagram

$$\begin{tikzcd} \prod_{k=1}^m \mathrm{End}_R(M_k) \arrow[r, hook, "\iota_{\mathrm{cor}}"] \arrow[dr, "\mathrm{pr}_\ell"'] & \mathrm{End}_R(M) \arrow[d, "\mathrm{pr}_{\ell,\ell}^{\mathrm{End}(M)}"] \\ & \mathrm{End}_R(M_\ell) \end{tikzcd} \tag{0.1.18}$$

for all $1 \leq \ell \leq m$, but only in $\mathsf{Mod}(R)$. All in all, this gives a complete description of ι_{prod} in the finite product case: $\iota_{\mathrm{prod}} = \iota_{\mathrm{cor}}$.

Let us now apply our starting point of view of this section to our finite case $M = \bigoplus_{k=1}^m M_k$ a little further. Tautologically, we have $M \in \mathsf{LMod}(\mathrm{End}_R(M))$ with the original ${}_RM$ being of course the restriction of scalars of ${}_{\mathrm{End}_R(M)}M$ along the canonical inclusion $R \hookrightarrow \mathrm{End}_R(M)$, $r \mapsto r\,\mathrm{id}_M$. Writing $x = \oplus_{k=1}^m x_k \in M$, the restriction of scalars $\iota_{\mathrm{cor}}^* M$ is simply given by

$$\prod_{k=1}^m \mathrm{End}_R(M_k) \curvearrowright M\colon (\Phi_1, \dots, \Phi_m) \cdot x \overset{\mathrm{def}}{=} (\Phi_1, \dots, \Phi_m) \cdot \oplus_{k=1}^m x_k = \oplus_{k=1}^m \Phi_k(x_k). \tag{0.1.19}$$

Accordingly, the subsequent restriction of scalars $\varepsilon_\ell^* \iota_{\mathrm{cor}}^* M$ is in turn given explicitly by

$$\begin{aligned} \mathrm{End}_R(M_\ell) \curvearrowright M\colon \Phi_\ell \cdot x &\overset{\mathrm{def}}{=} (\iota_{\mathrm{cor}} \circ \varepsilon_\ell)(\Phi_\ell) \cdot x \overset{\mathrm{def}}{=} (0, \dots, 0, \Phi_\ell, 0, \dots, 0) \cdot \oplus_{k=1}^m x_k = \\ &= 0 \oplus \dots \oplus 0 \oplus \Phi_\ell(x_\ell) \oplus 0 \oplus \dots \oplus 0 \overset{\mathrm{def}}{=} i_\ell(\Phi_\ell(x_\ell)), \end{aligned} \tag{0.1.20}$$

$1 \leq \ell \leq m$. It follows that

$$i_\ell\colon {}_{\mathrm{End}_R(M_\ell)} M_\ell \hookrightarrow \varepsilon_\ell^* \iota_{\mathrm{cor}}^* M \tag{0.1.21}$$

is in fact well-defined in $\mathsf{LMod}(\mathrm{End}_R(M_\ell))$, $1 \leq \ell \leq m$. In other words, Eq. (0.1.21) simply says that the inclusion $\iota_{\mathrm{cor}} \circ \varepsilon_\ell\colon \mathrm{End}_R(M_\ell) \hookrightarrow \mathrm{End}_R(M)$ in Alg_R^* is compatible with the inclusion $i_\ell\colon {}_RM_\ell \hookrightarrow {}_RM$ in $\mathsf{Mod}(R)$ in the sense of Lemma-Definition 0.1.9, that is,

$$\forall\, \Phi_\ell \in \mathrm{End}_R(M_\ell)\colon (\iota_{\mathrm{cor}} \circ \varepsilon_\ell)(\Phi_\ell) \circ i_\ell = i_\ell \circ \Phi_\ell, \tag{0.1.22}$$

or put differently, that we have a morphism $(\iota_{\mathrm{cor}} \circ \varepsilon_\ell, i_\ell)\colon (\mathrm{End}_R(M_\ell), M_\ell) \hookrightarrow (\mathrm{End}_R(M), M)$ in $\mathsf{LMod}(\mathsf{Alg}_R^*)$, $1 \leq \ell \leq m$. Using this, we can now, among other things, address the inverse problem of characterizing the necessary and sufficient conditions under which the containment of a product of endomorphism algebras gives rise to a *matching* decomposition of the corresponding module:

Proposition-Definition 0.1.13 (Corner Corestriction)**:** Let $M \in \mathsf{LMod}(\mathcal{A})$ via $\lambda\colon \mathcal{A} \to \mathrm{End}_R(M)$ and let $i_k\colon {}_RM_k \hookrightarrow {}_RM$, $1 \leq k \leq m$, be a finite family of R-submodules of M that are also $\mathcal{A}$-modules via corresponding representations $\lambda_k\colon \mathcal{A} \to \mathrm{End}_R(M_k)$, $1 \leq k \leq m$. Then ${}_{\mathcal{A}}M = \bigoplus_{k=1}^m {}_{\mathcal{A}}M_k$ in $\mathsf{LMod}(\mathcal{A})$ if and only if the following two conditions are fulfilled:

(i) Factorization (or compatibility of the $\mathcal{A}$-action): $\lambda = \iota_{\mathrm{cor}} \circ \lambda^{\mathrm{cor}}$ in Alg_R for **some** inclusion $\iota_{\mathrm{cor}}\colon \prod_{k=1}^m \mathrm{End}_R(M_k) \hookrightarrow \mathrm{End}_R(M)$ and $\lambda^{\mathrm{cor}} := (\lambda_1, \dots, \lambda_m) \overset{\mathrm{def}}{=} \sum_{k=1}^m \varepsilon_k \circ \lambda_k$.

(ii) Compatibility (between the module inclusion and its endomorphism algebra inclusion): $i_k\colon M_k \hookrightarrow M$ is $\mathrm{End}_R(M_k)$-linear, where $\mathrm{End}_R(M_k) \curvearrowright M$ via $\iota_{\mathrm{cor}} \circ \varepsilon_k\colon \mathrm{End}_R(M_k) \hookrightarrow \mathrm{End}_R(M)$, for all $1 \leq k \leq m$.

Moreover, when this is the case, the inclusion ι_{cor} is precisely the one induced by the decomposition ${}_RM = \bigoplus_{k=1}^m {}_RM_k$ in $\mathsf{Mod}(R)$ as described in Eq. (0.1.17), and hence the action $\mathrm{End}_R(M_k) \curvearrowright M$ is as given by Eq. (0.1.20). Equivalently, we have the following succinct diagram

$$
\begin{tikzcd}
 & \mathcal{A} \arrow[dl, "\lambda_\ell"'] \arrow[d, dashed, "\lambda^{\mathrm{cor}}"'] \arrow[dr, "\lambda"] & \\
\mathrm{End}_R(M_\ell) \arrow[r, shift right, "\varepsilon_\ell"'] \arrow[d, dash, "\curvearrowright"] & \prod_{k=1}^m \mathrm{End}_R(M_k) \arrow[l, two heads, shift right, "\mathrm{pr}_\ell"'] \arrow[r, hook, dashed, "\iota_{\mathrm{cor}}"] & \mathrm{End}_R(M) \arrow[d, dash, "\curvearrowright"] \\
M_\ell \arrow[rr, hook, "i_\ell"] & & M
\end{tikzcd}
\tag{0.1.23}
$$

where the **upper** left triangle (above ε_ℓ) and the right triangle commute, $1 \leq \ell \leq m$. The morphism λ^{cor} will be called the corner corestriction of λ. Analogously for $\mathsf{RMod}(\mathcal{A})$, **anti**-representations $\rho\colon \mathcal{A} \to \mathrm{End}_R(M)$ and $\rho_k\colon \mathcal{A} \to \mathrm{End}_R(M_k)$, $1 \leq k \leq m$, and an **anti-**homomorphism ρ^{cor}.

Proof: "$\Rightarrow$": This basically follows from the above discussion. Note that $i_k\colon M_k \hookrightarrow M$, $1 \leq k \leq m$, also become inclusions in $\mathsf{LMod}(\mathcal{A})$ since by assumption ${}_{\mathcal{A}}M = \bigoplus_{k=1}^m {}_{\mathcal{A}}M_k$. It only remains to show that the right triangle commutes. Denote by $\phi\colon M = \bigoplus_{k=1}^m M_k \xrightarrow{\cong} \prod_{k=1}^m M_k$, $x = \oplus_{k=1}^m x_k \mapsto \times_{k=1}^m x_k$, the canonical isomorphism (in $\mathsf{LMod}(\mathcal{A})$) and observe that $\forall\, a \in \mathcal{A}$: $(\iota_{\mathrm{cor}} \circ \lambda^{\mathrm{cor}})(a) \overset{\mathrm{def}}{=} \phi^{-1} \circ \lambda^{\mathrm{cor}}(a) \circ \phi$. Thus $\forall\, a \in \mathcal{A}\; \forall\, x \in M$: $(\iota_{\mathrm{cor}}(\lambda^{\mathrm{cor}}(a)))(x) \overset{\mathrm{def}}{=} \phi^{-1}(\lambda^{\mathrm{cor}}(a)(\phi(x))) \overset{\mathrm{def}}{=} \phi^{-1}\big((\times_{k=1}^m \lambda_k(a))(\times_{k=1}^m x_k)\big) = \phi^{-1}(\times_{k=1}^m(\lambda_k(a)(x_k))) \overset{\mathrm{def}}{=} \phi^{-1}(\times_{k=1}^m(a \cdot x_k)) \overset{\mathrm{def}}{=} \phi^{-1}(a \cdot (\times_{k=1}^m x_k)) \overset{\mathrm{def}}{=} \phi^{-1}(a \cdot \phi(x)) = \phi^{-1}(\phi(a \cdot x)) = a \cdot x \overset{\mathrm{def}}{=} \lambda(a)(x)$, i.e. $\forall\, a \in \mathcal{A}$: $(\iota_{\mathrm{cor}} \circ \lambda^{\mathrm{cor}})(a) = \lambda(a)$ as desired.

"$\Leftarrow$": Conversely, put $\mathcal{B} \coloneqq \prod_{k=1}^m \mathrm{End}_R(M_k)$ and suppose that there exists *some* inclusion $\iota_{\mathrm{cor}}\colon \mathcal{B} \hookrightarrow \mathrm{End}_R(M)$ in Alg_R (technically not a *corner* inclusion yet, either) such that $\iota_{\mathrm{cor}} \circ \varepsilon_\ell$ is compatible with i_ℓ, $1 \leq \ell \leq m$, and $\lambda = \iota_{\mathrm{cor}} \circ \lambda^{\mathrm{cor}}$, where $\lambda^{\mathrm{cor}}\colon \mathcal{A} \to \mathcal{B}$, $a \mapsto (\lambda_1(a), \ldots, \lambda_m(a))$, is as given (technically not a *corner* restriction yet). We first claim that very generally $\forall\, \Phi \in \mathrm{C}_{\mathcal{B}}(\lambda^{\mathrm{cor}}(\mathcal{A})) = \prod_{k=1}^m \mathrm{C}_{\mathrm{End}_R(M_k)}(\lambda_k(\mathcal{A}))$: $\iota_{\mathrm{cor}}(\Phi) \in \mathrm{End}_{\mathcal{A}}(M) \subseteq \mathrm{End}_R(M)$. Indeed, setting $\mathcal{E} \coloneqq \iota_{\mathrm{cor}}(\Phi)$, we have $\forall\, a \in \mathcal{A}\; \forall\, x \in M$:

$$
\begin{aligned}
a \cdot \mathcal{E}(x) &\overset{\mathrm{def}}{=} (\lambda(a)\mathcal{E})(x) = \big(\iota_{\mathrm{cor}}(\lambda^{\mathrm{cor}}(a))\iota_{\mathrm{cor}}(\Phi)\big)(x) = \big(\iota_{\mathrm{cor}}(\lambda^{\mathrm{cor}}(a)\Phi)\big)(x) = \big(\iota_{\mathrm{cor}}(\Phi\lambda^{\mathrm{cor}}(a))\big)(x) = \\
&= \big(\iota_{\mathrm{cor}}(\Phi)\iota_{\mathrm{cor}}(\lambda^{\mathrm{cor}}(a))\big)(x) = (\mathcal{E}\lambda(a))(x) \overset{\mathrm{def}}{=} \mathcal{E}(a \cdot x),
\end{aligned}
$$

Now, if $\{e_1, \ldots, e_m\}$ denotes the corresponding canonical complete central system of mutually orthogonal non-trivial idempotents for $\mathcal{B}$, explicitly $e_\ell \overset{\mathrm{def}}{=} \varepsilon_\ell(\mathrm{id}_{M_\ell}) \overset{\mathrm{def}}{=} \varepsilon_\ell(\lambda_\ell(1_{\mathcal{A}})) \overset{\mathrm{def}}{=} (0, \ldots, 0, \lambda_\ell(1_{\mathcal{A}}), 0, \ldots, 0)$, then clearly $e_\ell \in \mathrm{C}_{\mathcal{B}}(\lambda^{\mathrm{cor}}(\mathcal{A}))$, $1 \leq \ell \leq m$, hence, setting $\mathcal{E}_\ell \coloneqq \iota_{\mathrm{cor}}(e_\ell) \overset{\mathrm{def}}{=} (\iota_{\mathrm{cor}} \circ \varepsilon_\ell \circ \lambda_\ell)(1_{\mathcal{A}}) \in \mathrm{End}_R(M)$, $1 \leq \ell \leq m$, we actually get that $\{\mathcal{E}_1, \ldots, \mathcal{E}_m\} \subset \mathrm{End}_{\mathcal{A}}(M) \subseteq \mathrm{End}_R(M)$ is itself a complete system of mutually orthogonal non-trivial idempotents (not necessarily central anymore though). Taking $M'_\ell \coloneqq \mathcal{E}_\ell(M) \in \mathsf{LMod}(\mathcal{A})$, $1 \leq \ell \leq m$, it follows from Lemma 0.1.4 that ${}_{\mathcal{A}}M = \bigoplus_{k=1}^m {}_{\mathcal{A}}M'_k$. Next we claim that $\mathcal{E}_\ell \circ i_\ell\colon {}_{\mathcal{A}}M_\ell \xrightarrow{=} {}_{\mathcal{A}}M'_\ell$ is in fact an equality of $\mathcal{A}$-submodules of M, $1 \leq \ell \leq m$. To this end, we have $\forall\, x_\ell \in M_\ell$: $(\mathcal{E}_\ell \circ i_\ell)(x_\ell) \overset{\mathrm{def}}{=} \big((\iota_{\mathrm{cor}} \circ \varepsilon_\ell)(\mathrm{id}_{M_\ell})\big)(i_\ell(x_\ell)) = (\mathrm{id}_{M_\ell} \circ i_\ell)(x_\ell) = i_\ell(x_\ell)$ by *compatibility*, i.e. $\mathcal{E}_\ell \circ i_\ell = i_\ell$ in $\mathsf{Mod}(R)$, hence ${}_RM_\ell = {}_RM'_\ell$ and $\forall\, k \neq \ell$: $\mathcal{E}_k \circ i_\ell = \mathcal{E}_k \circ \mathcal{E}_\ell \circ i_\ell = 0$ by orthogonality. It remains to show that i_ℓ is also $\mathcal{A}$-linear. We have $\forall\, a \in \mathcal{A}$:

$$
\begin{aligned}
\iota_{\mathrm{cor}}(\lambda_{\mathrm{cor}}(a)) &\overset{\mathrm{def}}{=} \iota_{\mathrm{cor}}\left(\sum_{k=1}^m (\varepsilon_k \circ \lambda_k)(a)\right) = \sum_{k=1}^m (\iota_{\mathrm{cor}} \circ \varepsilon_k \circ \lambda_k)(a1_{\mathcal{A}}) = \\
&= \sum_{k=1}^m (\iota_{\mathrm{cor}} \circ \varepsilon_k \circ \lambda_k)(a) \cdot (\iota_{\mathrm{cor}} \circ \varepsilon_k \circ \lambda_k)(1_{\mathcal{A}}) \overset{\mathrm{def}}{=} \sum_{k=1}^m (\iota_{\mathrm{cor}} \circ \varepsilon_k \circ \lambda_k)(a)\mathcal{E}_k.
\end{aligned}
$$

Hence, using $\forall\, 1 \leq k, \ell \leq m$: $\mathcal{E}_k \circ i_\ell = \delta_{k\ell} i_\ell$ from above, we have $\forall\, a \in \mathcal{A}\; \forall\, x_\ell \in M_\ell$:

$$
\begin{aligned}
a \cdot i_\ell(x_\ell) &\overset{\mathrm{def}}{=} \lambda(a)(i_\ell(x_\ell)) = \big(\iota_{\mathrm{cor}}(\lambda_{\mathrm{cor}}(a))\big)(i_\ell(x_\ell)) = \left(\sum_{k=1}^m (\iota_{\mathrm{cor}} \circ \varepsilon_k \circ \lambda_k)(a)\mathcal{E}_k\right)(i_\ell(x_\ell)) = \\
&= \big((\iota_{\mathrm{cor}} \circ \varepsilon_\ell \circ \lambda_\ell)(a)\big)(i_\ell(x_\ell)) = \big((\iota_{\mathrm{cor}} \circ \varepsilon_\ell)(\lambda_\ell(a))\big)(i_\ell(x_\ell)) = i_\ell(\lambda_\ell(a)(x_\ell)) \overset{\mathrm{def}}{=} i_\ell(a \cdot x_\ell)
\end{aligned}
$$

by *compatibility*. This completes the proof. □

Remark: Since $i_k\colon M_k \to M$ is $\mathrm{End}_R(M_k)$-linear, it follows that $\mathrm{Ker}(i_k) \in \mathsf{LMod}(\mathrm{End}_R(M_k))$, so with some additional assumptions on M_k, e.g. M_k simple, the injectivity hypothesis on i_k could be even dropped, $1 \leq k \leq m$. □

In other words, we have just given a characterization of when a module is a direct sum (=*internal* direct product) of finitely many submodules *in terms of inclusions of the corresponding endomorphism algebras*. Next we give a characterization of the parallel question of when a representation is (essentially) an *internal* direct product of other representations *as morphisms*:

Lemma 0.1.14: Let $\{M_k\}_{1\leq k\leq m} \subset \mathsf{LMod}(\mathcal{A})$ be a finite family with corresponding representations $\lambda_k\colon \mathcal{A} \to \mathrm{End}_R(M_k)$, $1 \leq k \leq m$, let ${}_RM = \bigoplus_{k=1}^m {}_RM_k$ in $\mathsf{Mod}(R)$, and let $\lambda\colon \mathcal{A} \to \mathrm{End}_R(M)$ be some representation. Then $\lambda = \iota_{\mathrm{cor}} \circ (\lambda_1, \ldots, \lambda_m)$, i.e. λ is the internal direct product λ^{cor} up to ι_{cor}, if and only if the following two conditions are satisfied:

(i) $\lambda(\mathcal{A}) \subseteq \prod_{\ell=1}^m \mathrm{End}_R(M_\ell)$;

(ii) $\forall\, 1 \leq k \leq m\colon\ \mathrm{pr}_{k,k}^{\mathrm{End}(M)} \circ \lambda = \lambda_k$ (in $\mathsf{Mod}(R)$).

Schematically, we have the diagram

$$\begin{tikzcd}
 & \lambda(\mathcal{A}) \arrow[d, dashed, hook, "\iota_{\lambda(\mathcal{A})}"] \arrow[dr, hook, "\subseteq"] & \\
\mathcal{A} \arrow[ur, "\lambda^{\mathrm{im}}"] \arrow[r, "(\lambda_1,\ldots,\lambda_m)"] \arrow[dr, "\lambda_k"'] & \prod_{\ell=1}^m \mathrm{End}_R(M_\ell) \arrow[r, hook, "\iota_{\mathrm{cor}}"] \arrow[d, "\mathrm{pr}_k"] & \mathrm{End}_R(M) \arrow[dl, two heads, "\mathrm{pr}_{k,k}^{\mathrm{End}(M)}"] \\
 & \mathrm{End}_R(M_k) &
\end{tikzcd} \tag{0.1.24}$$

for all $1 \leq k \leq m$, where $\lambda = \subseteq \circ \lambda^{\mathrm{im}}$ and $\iota_{\lambda(\mathcal{A})}$ denotes the inclusion from (i). Analogously for $\mathsf{RMod}(\mathcal{A})$, **anti**-representations $\rho\colon \mathcal{A} \to \mathrm{End}_R(M)$ and $\rho_k\colon \mathcal{A} \to \mathrm{End}_R(M_k)$, $1 \leq k \leq m$, and **anti**-homomorphisms ρ^{cor} and ρ^{im}.

Proof: We need to show that the upper triangle commutes if and only if there exists a set-theoretical containment $\iota_{\lambda(\mathcal{A})}$ and the outer parallelogram commutes for all $1 \leq k \leq m$.

"⇒": obvious from the discussion so far.

"⇐": By commutativity of the upper right triangle it suffices to show that the upper left triangle commutes. But

$$\lambda_k = \mathrm{pr}_{k,k}^{\mathrm{End}(M)} \circ \lambda = \mathrm{pr}_{k,k}^{\mathrm{End}(M)} \circ \subseteq \circ \lambda^{\mathrm{im}} = \mathrm{pr}_k \circ \iota_{\lambda(\mathcal{A})} \circ \lambda^{\mathrm{im}}$$

by commutativity of the *whole* right triangle, hence the issue reduces to the commutative "double" diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, shift left, "(\lambda_1,\ldots,\lambda_m)"] \arrow[r, shift right, "\iota_{\lambda(\mathcal{A})}\circ\lambda^{\mathrm{im}}"'] \arrow[dr, "\lambda_k"'] & \prod_{\ell=1}^m \mathrm{End}_R(M_\ell) \arrow[d, "\mathrm{pr}_k"] \\
 & \mathrm{End}_R(M_k)
\end{tikzcd}$$

where we need to show that the two parallel arrows agree, $1 \leq k \leq m$. But this is just the universal property of the product. □

Lemma-Definition 0.1.15 (Convenient Corestriction)**:** Let ${}_{\mathcal{A}}M = \bigoplus_{k=1}^m {}_{\mathcal{A}}M_k$ in $\mathsf{LMod}(\mathcal{A})$ with corresponding representations $\lambda^{\mathrm{cor}} = (\lambda_1, \dots, \lambda_m)$. Put $\mathcal{B}_k \coloneqq \lambda_k(\mathcal{A})$, $1 \le k \le m$, and $\mathcal{B} \coloneqq \prod_{k=1}^m \mathcal{B}_k$. Then the morphism λ^{mor} completing the commutative diagram

$$\begin{tikzcd}
 & & \mathcal{A} \arrow[dll, two heads, "\lambda^{\mathrm{im}}"'] \arrow[dl, dashed, "\lambda^{\mathrm{mor}}"] \arrow[d, "\lambda^{\mathrm{cor}}"] \arrow[dr, "\lambda"] & \\
\lambda(\mathcal{A}) \arrow[r, hook] & \mathcal{B} \arrow[r, hook, "\times_{k=1}^m \iota_{\mathcal{B}_k}"] & \prod_{k=1}^m \mathrm{End}_R(M_k) \arrow[r, hook, "\iota_{\mathrm{cor}}"] & \mathrm{End}_R(M)
\end{tikzcd} \tag{0.1.25}$$

will be called the convenient corestriction of λ and $\mathcal{B}$ will be called the convenient target (or convenient codomain) of λ, where $\iota_{\mathcal{B}_k} \colon \mathcal{B}_k \hookrightarrow \mathrm{End}_R(M_k)$, $1 \le k \le m$, denote the image inclusions. In other words, we have

$$\lambda^{\mathrm{mor}} \overset{\mathrm{def}}{=} (\lambda_1^{\mathrm{im}}, \dots, \lambda_m^{\mathrm{im}}). \tag{0.1.26}$$

Analogously for $\mathsf{RMod}(\mathcal{A})$, **anti**-representations $\rho\colon \mathcal{A} \to \mathrm{End}_R(M)$ and $\rho_k \colon \mathcal{A} \to \mathrm{End}_R(M_k)$, $1 \le k \le m$, and **anti**-homomorphisms ρ^{cor}, ρ^{im}, and ρ^{mor}.

Proof: λ^{cor} exists "commutatively" by Proposition-Definition 0.1.13, from where the "commutative" existence of λ^{mor} is clear by construction. $\square$

Remarks:

(1) If $m = 1$, then indeed $\lambda^{\mathrm{mor}} = \lambda^{\mathrm{im}}$.

(2) Certainly, both λ^{cor} and λ^{mor} depend on the choice of a decomposition for M. However, as already pointed out in Section 0.1.2, under favourable circumstances M admits a unique (up to permutations) complete decomposition, in which case λ^{cor} and λ^{mor} only depend on the choice of ordering / labelling of the direct summands.

(3) If $\mathcal{A} \in \mathsf{CAlg}_R$, then $(-)^{\mathrm{mor}}$ allows us to abandon the inherently non-commutative world of the representation λ and pass to a morphism in CAlg_R. This is especially useful when $\mathcal{B}_k$, $1 \le k \le m$, turn out to be (commutative) *blocks*, e.g. commutative local rings. Using λ^{im} instead is less convenient since the block decomposition of the image *need not match the (say, complete) decomposition of M* (e.g. think diagonal inclusions like $\Delta \colon \mathcal{A} \hookrightarrow \mathcal{A}^2$).

Recalling Definition 0.1.8, we now show that even though it may not be possible to match a decomposition of the image to the decomposition of the target endomorphism algebra factor-wise, the image still happens to be a *subdirect product* of an object matching said decomposition, namely of the *convenient target*:

Lemma 0.1.16: Let ${}_{\mathcal{A}}M = \bigoplus_{k=1}^m {}_{\mathcal{A}}M_k$ in $\mathsf{LMod}(\mathcal{A})$ with corresponding representations $\lambda \colon \mathcal{A} \to \mathrm{End}_R(M)$, $\lambda_k \colon \mathcal{A} \to \mathrm{End}_R(M_k)$, $\mathcal{B}_k \coloneqq \lambda_k(\mathcal{A})$, $1 \le k \le m$. Then the image of λ, $\lambda(\mathcal{A})$, is a subdirect product of the convenient target of λ, $\mathcal{B} \coloneqq \prod_{k=1}^m \mathcal{B}_k$. That is, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, two heads, "\lambda^{\mathrm{im}}"] \arrow[drr, two heads, "\lambda_k^{\mathrm{im}}"'] & \lambda(\mathcal{A}) \arrow[r, hook, "\subseteq"] \arrow[dr, dashed, two heads, "\pi_k"] & \mathcal{B} \arrow[r, hook, "\iota_{\mathrm{cor}}|_{\mathcal{B}}"] \arrow[d, two heads, "\mathrm{pr}_k^{\mathcal{B}}"] & \mathrm{End}_R(M) \arrow[dl, "\mathrm{pr}_{k,k}^{\mathrm{End}(M)}"] \\
 & & \mathcal{B}_k &
\end{tikzcd} \tag{0.1.27}$$

where the rightmost triangle is only in $\mathsf{Mod}(R)$, for all $1 \le k \le m$. Analogously for $\mathsf{RMod}(\mathcal{A})$, **anti**-representations $\rho \colon \mathcal{A} \to \mathrm{End}_R(M)$ and $\rho_k \colon \mathcal{A} \to \mathrm{End}_R(M_k)$, $1 \le k \le m$, and an **anti**-homomorphism ρ^{im}.

Proof: We only need to show that the maps π_k, $1 \le k \le m$, are surjective. To this end we show that the leftmost triangle commutes. But this is clear from the commutativity of the bigger left triangle in which it is contained: $\pi_k \circ \lambda^{\mathrm{im}} \overset{\mathrm{def}}{=} \mathrm{pr}_k^{\mathcal{B}} \circ \subseteq \circ \lambda^{\mathrm{im}} = \lambda_k^{\mathrm{im}}$, $1 \le k \le m$. Since λ_k^{im} is itself surjective by definition, it now follows that so is π_k, $1 \le k \le m$. $\square$

Corollary 0.1.17: Let $\mathcal{B} = \prod_{k=1}^m \mathcal{B}_k$ in Alg_R with corresponding regular representations $\lambda_{\mathcal{B}} \colon \mathcal{B} \hookrightarrow \operatorname{End}_R(\mathcal{B})$, $\lambda_{\mathcal{B}_k} \colon \mathcal{B}_k \hookrightarrow \operatorname{End}_R(\mathcal{B}_k)$, $1 \le k \le m$. We then have a commutative diagram

$$\begin{tikzcd}
\mathcal{B}_\ell \arrow[r, hook, shift left, "\varepsilon_\ell^{\mathcal{B}}"] \arrow[d, hook, "\lambda_{\mathcal{B}_\ell}"'] & \mathcal{B} = \prod_{k=1}^m \mathcal{B}_k \arrow[l, two heads, shift left, "\mathrm{pr}_\ell^{\mathcal{B}}"] \arrow[r, hook, "\lambda_{\mathcal{B}}"] \arrow[d, hook, dashed, "\times_{k=1}^m \lambda_{\mathcal{B}_k}"'] & \operatorname{End}_R(\mathcal{B}) \\
\operatorname{End}_R(\mathcal{B}_\ell) \arrow[r, hook, shift left, "\varepsilon_\ell"] & \prod_{k=1}^m \operatorname{End}_R(\mathcal{B}_k) \arrow[l, two heads, shift left, "\mathrm{pr}_\ell"] \arrow[ur, hook, "\iota_{\mathrm{cor}}"'] &
\end{tikzcd} \tag{0.1.28}$$

for all $1 \le \ell \le m$. It follows that $\lambda_{\mathcal{B}}^{\mathrm{cor}} = \times_{k=1}^m \lambda_{\mathcal{B}_k}$ and $\lambda_{\mathcal{B}}^{\mathrm{im}} = \lambda_{\mathcal{B}}^{\mathrm{mor}} = \mathrm{id}_{\mathcal{B}}$. Analogously for the right regular **anti**-representations $\rho_{\mathcal{B}} \colon \mathcal{B} \to \operatorname{End}_R(\mathcal{B})$ and $\rho_{\mathcal{B}_k} \colon \mathcal{B}_k \to \operatorname{End}_R(\mathcal{B}_k)$, $1 \le k \le m$.

Proof: The square on the left is self-evident (when interpreted appropriately), so nothing to show there. By Lemma 0.1.4 and the subsequent internalization discussion we can write ${}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}} = \bigoplus_{k=1}^m {}_{\mathcal{B}}(\mathcal{B}_k)_{\mathcal{B}}$ in $\mathsf{BMod}(\mathcal{B})$, hence also in $\mathsf{Mod}(R)$, which in turn induces $\iota_{\mathrm{cor}} \colon \prod_{k=1}^m \operatorname{End}_R(\mathcal{B}_k) \hookrightarrow \operatorname{End}_R(\mathcal{B})$ in Alg_R. Next we show that $\lambda_{\mathcal{B}}^{\mathrm{cor}} = \times_{k=1}^m \lambda_{\mathcal{B}_k}$. Writing $b = \times_{k=1}^m b_k \in \mathcal{B} \in \mathsf{Alg}_R$ and $y = \oplus_{k=1}^m y_k \in \mathcal{B} \in \mathsf{Mod}(R)$, we get

$$\begin{aligned}
\forall\, b, y \in \mathcal{B} \colon \lambda_{\mathcal{B}}(b)(y) &\overset{\mathrm{def}}{=} by \overset{\mathrm{def}}{=} \times_{k=1}^m b_k \cdot \times_{k=1}^m y_k = \times_{k=1}^m b_k y_k \overset{\mathrm{def}}{=} \times_{k=1}^m \lambda_{\mathcal{B}_k}(b_k)(y_k) \overset{\mathrm{def}}{=} \\
&= \iota_{\mathrm{cor}}(\times_{k=1}^m \lambda_{\mathcal{B}_k}(b_k))(\oplus_{k=1}^m y_k) \overset{\mathrm{def}}{=} \iota_{\mathrm{cor}}(\times_{k=1}^m \lambda_{\mathcal{B}_k}(b_k))(y) \overset{\mathrm{def}}{=} \\
&\overset{\mathrm{def}}{=} \iota_{\mathrm{cor}}\big((\times_{k=1}^m \lambda_{\mathcal{B}_k})(\times_{k=1}^m b_k)\big)(y) \overset{\mathrm{def}}{=} \Big(\big(\iota_{\mathrm{cor}} \circ (\times_{k=1}^m \lambda_{\mathcal{B}_k})\big)(b)\Big)(y),
\end{aligned}$$

hence $\forall\, b \in \mathcal{B} \colon \lambda_{\mathcal{B}}(b) = \big(\iota_{\mathrm{cor}} \circ (\times_{k=1}^m \lambda_{\mathcal{B}_k})\big)(b)$, i.e. $\lambda_{\mathcal{B}} = \iota_{\mathrm{cor}} \circ (\times_{k=1}^m \lambda_{\mathcal{B}_k})$ as desired. Finally, $\lambda_{\mathcal{B}}^{\mathrm{im}} = \lambda_{\mathcal{B}}^{\mathrm{mor}} = \mathrm{id}_{\mathcal{B}}$ follows from Lemma-Definition 0.1.15. □

Corollary 0.1.18: Upon identification via ι_{cor}, we simply have:

$$\lambda_{\prod_{k=1}^m \mathcal{B}_k} = \times_{k=1}^m \lambda_{\mathcal{B}_k}. \tag{0.1.29}$$

That is, the regular representation "commutes" with finite direct products. □

In CAlg_R we have one additional type of useful natural corestriction that we examine next:

Lemma-Definition 0.1.19 (Tautological Corestriction)**:** Let $\mathcal{A} \in \mathsf{CAlg}_R$ and let ${}_{\mathcal{A}}M = \bigoplus_{k=1}^m {}_{\mathcal{A}}M_k$ in $\mathsf{Mod}(\mathcal{A})$. Then we have a commutative diagram

$$\begin{tikzcd}
 & & \mathcal{A} \arrow[dll, two heads, "\lambda^{\mathrm{im}}"'] \arrow[dl, "\lambda^{\mathrm{cor}}_{\operatorname{End}_{\mathcal{A}}}"] \arrow[d, dashed, shift right, "\lambda^{\mathrm{taut}}"'] \arrow[d, shift left, "\lambda^{\mathrm{cor}}_{\operatorname{End}_R}"] \arrow[dr, "\lambda"] & \\
\lambda(\mathcal{A}) \arrow[r, hook] & \prod_{k=1}^m \operatorname{End}_{\mathcal{A}}(M_k) \arrow[r, hook, shift left, "\iota_{\mathrm{cor}}^{\operatorname{End}_{\mathcal{A}}}"] \arrow[r, hook, shift right, "\times_{k=1}^m \iota_k^{\mathrm{taut}}"'] & \left(\begin{matrix}\operatorname{End}_{\mathcal{A}}(M), \\ \prod_{k=1}^m \operatorname{End}_R(M_k)\end{matrix}\right) \arrow[r, hook, shift left, "\iota_{\mathrm{taut}}"] \arrow[r, hook, shift right, "\iota_{\mathrm{cor}}^{\operatorname{End}_R}"'] & \operatorname{End}_R(M)
\end{tikzcd} \tag{0.1.30}$$

with **no a priori inclusion relations between** $\operatorname{End}_{\mathcal{A}}(M)$ and $\prod_{k=1}^m \operatorname{End}_R(M_k)$. The map λ^{taut} will be called *the tautological corestriction* of λ.

Proof: Since $\mathcal{A}$ is commutative, the representation $\lambda \colon \mathcal{A} \to \operatorname{End}_R(M)$ factors through the natural, almost *tautological* inclusion $\iota_{\mathrm{taut}} \colon \operatorname{End}_{\mathcal{A}}(M) \hookrightarrow \operatorname{End}_R(M)$ via a map λ^{taut}. Analogously, we have maps $\iota_k^{\mathrm{taut}} \colon \operatorname{End}_{\mathcal{A}}(M_k) \hookrightarrow \operatorname{End}_R(M_k)$, $1 \le k \le m$. The other notations are self-explanatory. □

Lemma 0.1.20 ($(-)^{\mathrm{rep}}$ vs. $(-)^{\mathrm{mor}}$)**:**

(1) Let $\varphi_1, \dots, \varphi_m$ be morphisms in Alg_R. Then, after an ι_{cor}-identification, we have:
- Internal Products: $(\varphi_1, \dots, \varphi_m)^{\mathrm{rep}} = (\varphi_1^{\mathrm{rep}}, \dots, \varphi_m^{\mathrm{rep}})$.

- External Products: $(\times_{k=1}^m \varphi_k)^{\mathrm{rep}} = \times_{k=1}^m \varphi_k^{\mathrm{rep}}$.

(2) Let $\varphi = (\varphi_1, \dots, \varphi_m)\colon \mathcal{A} \to \mathcal{B} = \prod_{k=1}^m \mathcal{B}_k$ be an internal morphism product in Alg_R. Then we have a commutative diagram

$$\begin{tikzcd}
 & & \mathcal{A} \arrow[dll, "\varphi"'] \arrow[dl, "\lambda_\varphi^{\mathrm{im}}"'] \arrow[d, "\lambda_\varphi^{\mathrm{mor}}"] \arrow[dr, "\lambda_\varphi^{\mathrm{cor}}"] \arrow[drr, "\lambda_\varphi"] & & \\
\mathcal{B} \arrow[r, "\cong", "\lambda_{\mathcal{B}}^{\mathrm{im}}"'] \arrow[rrr, bend right, "\times_{k=1}^m \lambda_{\mathcal{B}_k}"'] \arrow[rrrr, bend right=40, "\lambda_{\mathcal{B}}"'] & \lambda_\varphi(\mathcal{A}) \arrow[r, hook] & \prod_{k=1}^m \lambda_{\varphi_k}(\mathcal{A}) \arrow[r, hook] & \prod_{k=1}^m \mathrm{End}_R(\mathcal{B}_k) \arrow[r, hook, "\iota_{\mathrm{cor}}"] & \mathrm{End}_R(\mathcal{B})
\end{tikzcd} \tag{0.1.31}$$

in Alg_R, where:

$$(\varphi^{\mathrm{rep}})^{\mathrm{mor}} \overset{\mathrm{def}}{=} \lambda_\varphi^{\mathrm{mor}} = ((\varphi_1^{\mathrm{rep}})^{\mathrm{im}}, \dots, (\varphi_m^{\mathrm{rep}})^{\mathrm{im}}) \neq \varphi. \tag{0.1.32}$$

Proof: To (1): We follow here Eq. (0.1.29). First let $\mathcal{A} \xrightarrow{\varphi_k} \mathcal{B}$, $1 \le k \le m$. Then:

$$\begin{aligned}(\varphi_1, \dots, \varphi_m)^{\mathrm{rep}} &\overset{\mathrm{def}}{=} (\lambda_{\prod_{k=1}^m \mathcal{B}_k}) \circ (\varphi_1, \dots, \varphi_m) = (\times_{k=1}^m \lambda_{\mathcal{B}_k}) \circ (\varphi_1, \dots, \varphi_m) = \\ &= (\lambda_{\mathcal{B}_1} \circ \varphi_1, \dots, \lambda_{\mathcal{B}_m} \circ \varphi_m) \overset{\mathrm{def}}{=} (\varphi_1^{\mathrm{rep}}, \dots, \varphi_m^{\mathrm{rep}}).\end{aligned}$$

Next let $\mathcal{A}_k \xrightarrow{\varphi_k} \mathcal{B}$, $1 \le k \le m$. Then:

$$(\times_{k=1}^m \varphi_k)^{\mathrm{rep}} \overset{\mathrm{def}}{=} (\lambda_{\prod_{k=1}^m \mathcal{B}_k}) \circ (\times_{k=1}^m \varphi_k) = (\times_{k=1}^m \lambda_{\mathcal{B}_k}) \circ (\times_{k=1}^m \varphi_k) = \times_{k=1}^m (\lambda_{\mathcal{B}_k} \circ \varphi_k) \overset{\mathrm{def}}{=} \times_{k=1}^m \varphi_k^{\mathrm{rep}}.$$

To (2): This follows from Lemma-Definition 0.1.15. In particular:

$$\begin{aligned}(\varphi^{\mathrm{rep}})^{\mathrm{mor}} \overset{\mathrm{def}}{=} \lambda_\varphi^{\mathrm{mor}} \overset{\mathrm{def}}{=} ((\lambda_\varphi)_1^{\mathrm{im}}, \dots, (\lambda_\varphi)_m^{\mathrm{im}}) &\overset{\mathrm{def}}{=} ((\mathrm{pr}_1^{\mathrm{End}} \circ \lambda_\varphi)^{\mathrm{im}}, \cdots, (\mathrm{pr}_m^{\mathrm{End}} \circ \lambda_\varphi)^{\mathrm{im}}) = \\ &= ((\lambda_{\mathcal{B}_1} \circ \varphi_1)^{\mathrm{im}}, \dots, (\lambda_{\mathcal{B}_m} \circ \varphi_m)^{\mathrm{im}}) \overset{\mathrm{def}}{=} (\lambda_{\varphi_1}^{\mathrm{im}}, \dots, \lambda_{\varphi_m}^{\mathrm{im}})\end{aligned}$$

by Eq. (0.1.28), where we have denoted $\mathrm{pr}_k^{\mathrm{End}}\colon \prod_{\ell=1}^m \mathrm{End}_R(\mathcal{B}_\ell) \twoheadrightarrow \mathrm{End}_R(\mathcal{B}_k)$, $1 \le k \le m$. Hence $\mathrm{Codom}(\lambda_\varphi^{\mathrm{mor}}) = \mathrm{Codom}(\lambda_{\varphi_1}^{\mathrm{im}}, \dots, \lambda_{\varphi_m}^{\mathrm{im}}) = \prod_{k=1}^m \lambda_{\varphi_k}(\mathcal{A})$. The inclusion $\lambda_\varphi(\mathcal{A}) \subseteq \prod_{k=1}^m \lambda_{\varphi_k}(\mathcal{A})$ follows *automatically* from Lemma-Definition 0.1.15. Alternatively, we have

$$\begin{aligned}\lambda_\varphi(\mathcal{A}) \overset{\mathrm{def}}{=} \varphi^{\mathrm{rep}}(\mathcal{A}) \overset{\mathrm{def}}{=} (\varphi_1, \dots, \varphi_m)^{\mathrm{rep}}(\mathcal{A}) &\overset{(2)}{=} (\varphi_1^{\mathrm{rep}}, \dots, \varphi_m^{\mathrm{rep}})(\mathcal{A}) \overset{\mathrm{def}}{=} \\ &\overset{\mathrm{def}}{=} (\lambda_{\varphi_1}, \dots, \lambda_{\varphi_m})(\mathcal{A}) \subseteq \prod_{k=1}^m \lambda_{\varphi_k}(\mathcal{A}).\end{aligned}$$

□

0.1.5. Standard Factorizations

Lemma 0.1.21 (Morphisms from Decomposable Algebras into Indecomposable Algebras)**:** Let $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ be a morphism in Alg_R. If $\mathcal{A}$ is decomposable over itself with ${}_{\mathcal{A}}\mathcal{A} = \bigoplus_{j=1}^n \mathfrak{L}_j$ and $\mathcal{A}_{\mathcal{A}} = \bigoplus_{j=1}^n \mathfrak{R}_j$ and $\mathcal{B}$ is indecomposable over itself, then there exists a unique $i \in [n]$ such that

$$\begin{tikzcd}
{}_{\mathcal{A}}\mathcal{A} \arrow[r, "{}_{\mathcal{A}}\varphi"] \arrow[d, "\mathrm{pr}_i"'] & {}_{\mathcal{A}}\mathcal{B} \\
\mathfrak{L}_i \arrow[ur, dashed, "\exists_1\, {}_{\mathcal{A}}\bar{\varphi}"']
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\mathcal{A}_{\mathcal{A}} \arrow[r, "\varphi_{\mathcal{A}}"] \arrow[d, "\mathrm{pr}_i"'] & \mathcal{B}_{\mathcal{A}} \\
\mathfrak{R}_i \arrow[ur, dashed, "\exists_1\, \bar{\varphi}_{\mathcal{A}}"']
\end{tikzcd} \tag{0.1.33}$$

are commutative diagrams in $\mathsf{LMod}(\mathcal{A})$ and $\mathsf{RMod}(\mathcal{A})$, respectively. In particular, if $\mathcal{A} = \prod_{j=1}^{n} \mathcal{A}_j$ in Alg_R, then there exists a unique $i \in [n]$ such that

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\varphi"] \arrow[d, "\mathrm{pr}_i"'] & \mathcal{B} \\ \mathcal{A}_i \arrow[ur, dashed, "\exists_1\, \bar{\varphi}"'] & \end{tikzcd} \tag{0.1.34}$$

is a commutative diagram in Alg_R.

Proof: We prove only the $\mathsf{LMod}(\mathcal{A})$ case since the $\mathsf{RMod}(\mathcal{A})$ case follows analogously. Let $\{e_1, \dots, e_n\} \subset \mathcal{A}$ denote the corresponding complete system of mutually orthogonal non-trivial idempotents. Putting $f_j := \varphi(e_j)$, $1 \leq j \leq n$, we have that $\{f_j\}_{1 \leq j \leq n}$ is itself a complete system of mutually orthogonal idempotents in $\mathcal{B}$. Since $\mathcal{B}$ is indecomposable over itself, it follows that $\forall\, 1 \leq j \leq n\colon f_j \in \{0_{\mathcal{B}}, 1_{\mathcal{B}}\}$. As φ is non-trivial and unital, we cannot have $\forall\, 1 \leq j \leq n\colon f_j = 0$, since otherwise we get $0 = \sum_{j=1}^{n} f_j = \varphi(\sum_{j=1}^{n} e_j) = \varphi(1_{\mathcal{A}})$. Suppose now that there exist $i, j \in [n]$, $i \neq j$, such that $f_i = f_j = 1_{\mathcal{B}}$. Then by orthogonality $0_{\mathcal{B}} = f_i f_j = 1_{\mathcal{B}}$, a contradiction to φ being non-trivial. Thus $\exists_1\, 1 \leq i \leq n\colon f_i = 1_{\mathcal{B}}$, and $\forall\, j \neq i\colon f_j = 0_{\mathcal{B}}$, that is, $\forall\, j \neq i\colon e_j \in \mathrm{Ker}(\varphi)$. Putting $\mathfrak{l}_i := \sum_{j \neq i} \mathcal{A} e_j \overset{\text{def}}{=} \bigoplus_{j \neq i} \mathfrak{L}_j \in \mathsf{LMod}(\mathcal{A})$, we have $\mathfrak{l}_i \subset \mathrm{Ker}(\varphi) = \mathrm{Ker}({}_{\mathcal{A}}\varphi)$, hence ${}_{\mathcal{A}}\varphi$ factors through ${}_{\mathcal{A}}\mathcal{A}/{}_{\mathcal{A}}\mathfrak{l}_i \cong {}_{\mathcal{A}}\mathfrak{L}_i$. Finally, if the decomposition of $\mathcal{A}$ is central, then in the notations of Lemma 0.1.4 without loss of generality $\mathfrak{L}_j = \mathfrak{R}_j =: \mathfrak{A}_j$, $1 \leq j \leq n$, are two-sided ideals, hence the quotient is an object in Alg_R isomorphic to $\mathcal{A}_i$. □

Lemma 0.1.22 (Indecomposable Modules over Centrally Decomposable Algebras)**:** Let $\mathcal{A} \in \mathsf{Alg}_R$ and $M \in \mathsf{LMod}(\mathcal{A})$ via a representation $\lambda\colon \mathcal{A} \to \mathrm{End}_R(M)$ or $M \in \mathsf{RMod}(\mathcal{A})$ via an anti-representation $\rho\colon \mathcal{A} \to \mathrm{End}_R(M)$. If $\mathcal{A} = \prod_{j=1}^{n} \mathcal{A}_j$ in Alg_R and M is indecomposable in $\mathsf{LMod}(\mathcal{A})$ or $\mathsf{RMod}(\mathcal{A})$, respectively, then there exists a unique $i \in [n]$ such that

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\lambda{,}\rho"] \arrow[d, "\mathrm{pr}_i^{\mathcal{A}}"'] & \mathrm{End}_R(M) \\ \mathcal{A}_i \arrow[ur, dashed, "\exists_1\, \bar{\lambda}{,}\bar{\rho}"'] & \end{tikzcd} \tag{0.1.35}$$

is a commutative diagram in Alg_R, where in the second case $\bar{\rho}$ is an anti-homomorphism.

Proof: Let $\{e_1, \dots, e_n\} \subset \mathcal{A}$ denote the corresponding complete system of mutually orthogonal non-trivial *central* idempotents as in Lemma 0.1.4. Put $f_j := \lambda(e_j)$, $1 \leq j \leq n$. Since $e_j \in \mathcal{Z}(\mathcal{A})$, we have in fact $f_j \in \mathrm{End}_{\mathcal{A}}(M) \subseteq \mathrm{End}_R(M)$, $1 \leq j \leq n$. Thus $\{f_j\}_{1 \leq j \leq n}$ itself is a complete system of mutually orthogonal idempotents in $\mathrm{End}_{\mathcal{A}}(M)$. Since M is indecomposable in $\mathsf{LMod}(\mathcal{A})$, it follows from Lemma 0.1.4 (1) that $\forall\, 1 \leq j \leq n\colon f_j \in \{0, \mathrm{id}_M\}$. It is not possible that $\forall\, 1 \leq j \leq n\colon f_j = 0$, since then we would have $\lambda(1_{\mathcal{A}}) = 0$, a contradiction as $M \neq 0$ by definition. Therefore, at least one f_j is id_M. Suppose now that $f_i = f_j = \mathrm{id}_M$ for some $i \neq j$. Then $\mathrm{id}_M = f_i f_j = 0$ by orthogonality, a contradiction again to $M \neq 0$. Thus there exists exactly one $1 \leq i \leq n$ such that $f_i = \mathrm{id}_M$ and $\forall\, j \neq i\colon f_j = 0$, in particular $\forall\, j \neq i\colon e_j \in \mathrm{Ker}(\lambda)$. Setting $\mathfrak{a}_i := \langle e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n \rangle \lhd \mathcal{A}$, we have $\mathfrak{a}_i \subseteq \mathrm{Ker}(\lambda)$, hence λ factors through $\mathcal{A}/\mathfrak{a}_i \cong \mathcal{A}_i$ as a morphism of R-algebras. The proof for $\mathsf{RMod}(\mathcal{A})$ is analogous. □

Remark: Despite the apparent similitude between the proofs of Lemma 0.1.21 and Lemma 0.1.22, a moment's thought shows that neither of the lemmas implies the other one *naively*. We refer the reader here to the diagram in Eq. (0.1.8). On the one hand, taking $\mathcal{B} := \mathrm{End}_R(M)$ and $\varphi := \lambda$ as a special case of Lemma 0.1.21 implies that we are expecting M to be indecomposable in $\mathsf{Mod}(R)$, which is certainly too strong a requirement (think $R = K$ a field). However, this can be salvaged by additionally requiring $\mathcal{A}$ to be *commutative* (see Corollary 0.1.23 for the basic case of an indecomposable ${}_{\mathcal{A}}M$ and Corollary 0.1.29 for the general setting of ${}_{\mathcal{A}}M$ being finitely decomposable). On the other hand, taking instead $M := \mathcal{B}$ as a special case of Lemma 0.1.22 means that we are expecting ${}_{\mathcal{A}}M \overset{\text{def}}{=} {}_{\mathcal{A}}\mathcal{B}$ to be indecomposable, which is once again a much stronger hypothesis than the indecomposability of ${}_{\mathcal{B}}M \overset{\text{def}}{=} {}_{\mathcal{B}}\mathcal{B}$ needed for Lemma 0.1.21. Nevertheless, we include this short approach in Corollary 0.1.24 (for the case of ${}_{\mathcal{A}}\mathcal{B}$ indecomposable) and in Corollary 0.1.31 (for the more general case of ${}_{\mathcal{A}}\mathcal{B}$ being finitely decomposable) below. Finally, observe that the proof of Lemma 0.1.22 does not work anyway with (the more general) indecomposability of ${}_{\mathcal{B}}M$ for $\mathcal{B} := \mathrm{End}_R(M)$ in place of ${}_{\mathcal{A}}M$, since generically $\lambda(\mathcal{Z}(\mathcal{A})) \nsubseteq \mathcal{Z}(\mathcal{B})$. Thus, there does not appear to be a clear-cut way to *unconditionally* deduce the *morphism case* from the *representation case*, or vice versa in such complete generality. $\square$

As just mentioned, in the special case when $\mathcal{A}$ is commutative, one can actually deduce the representation factorization from the morphism factorization as follows:

Corollary 0.1.23: Let $\mathcal{A} \in \mathsf{CAlg}_R$ and $M \in \mathsf{Mod}(\mathcal{A})$ via $\lambda\colon \mathcal{A} \to \mathrm{End}_R(M)$. If $\mathcal{A} = \prod_{j=1}^n \mathcal{A}_j$ in CAlg_R and ${}_{\mathcal{A}}M$ is indecomposable, then there exists a unique $i \in [n]$ such that

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\lambda"] \arrow[d, "\mathrm{pr}_i"'] & \mathrm{End}_R(M) \\ \mathcal{A}_i \arrow[ur, dashed, "\exists_1 \bar{\lambda}"'] & \end{tikzcd} \tag{0.1.36}$$

is a commutative diagram in Alg_R. Moreover, this is the same $i \in [n]$ implied by Lemma 0.1.22.

Proof: Put $\mathcal{B} := \lambda(\mathcal{A}) \subseteq \mathrm{End}_R(M)$ and consider $\varphi_\lambda := \lambda^{\mathrm{mor}}\colon \mathcal{A} \twoheadrightarrow \mathcal{B}$. Since $\mathcal{A}$ is commutative, we have $\mathcal{B} \subseteq \mathrm{End}_{\mathcal{A}}(M) \subseteq \mathrm{End}_R(M)$. As ${}_{\mathcal{A}}M$ is indecomposable, it follows that $\mathcal{B}$ does not contain any non-trivial idempotents, hence $\mathcal{B}$ is itself indecomposable by Lemma 0.1.4. It now follows from Lemma 0.1.21 that we have a commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\lambda"] \arrow[d, "\mathrm{pr}_i"'] \arrow[dr, two heads, "\varphi_\lambda"'] & \mathrm{End}_K(M) \\ \mathcal{A}_i \arrow[r, dashed, "\bar{\varphi}_\lambda"'] & \mathcal{B} \arrow[u, hook] \end{tikzcd}$$

for some $i \in [n]$. It is the same $i \in [n]$ as in Lemma 0.1.22 by its uniqueness property in the diagram of Eq. (0.1.35). $\square$

In the special case when the morphism $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ yields an indecomposable left or right $\mathcal{A}$-module, one can conversely deduce the morphism factorization from the representation factorization as follows:

Corollary 0.1.24: Let $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ be a morphism in Alg_R. If $\mathcal{A} = \prod_{j=1}^n \mathcal{A}_j$ in Alg_R and ${}_{\mathcal{A}}\mathcal{B}$ or $\mathcal{B}_{\mathcal{A}}$ is indecomposable in $\mathsf{LMod}(\mathcal{A})$ or $\mathsf{RMod}(\mathcal{A})$, respectively, then there exists a unique $i \in [n]$ such that

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\varphi"] \arrow[d, "\mathrm{pr}_i"'] & \mathcal{B} \\ \mathcal{A}_i \arrow[ur, dashed, "\exists_1 \bar{\varphi}"'] & \end{tikzcd}$$

is a commutative diagram in Alg_R. Moreover, this is the same $i \in [n]$ as the one implied by Lemma 0.1.21.

Proof: We prove the $\mathsf{LMod}(\mathcal{A})$ case. Consider the induced representation $\lambda_\varphi := \varphi^{\mathrm{rep}} \overset{\text{def}}{=} \lambda_{\mathcal{B}} \circ \varphi$. Since ${}_{\mathcal{A}}\mathcal{B}$ is indecomposable in $\mathsf{LMod}(\mathcal{A})$, it follows from Lemma 0.1.22 that λ_φ induces a unique representation $\bar{\lambda}_\varphi$ such that

$$\begin{tikzcd}
\mathcal{A} \arrow[dd, "\mathrm{pr}_i"'] \arrow[dr, "\varphi"] \arrow[drr, "\lambda_\varphi"] & & \\
& \mathcal{B} \arrow[r, hook, "\lambda_{\mathcal{B}}"] & \mathrm{End}_R(\mathcal{B}) \\
\mathcal{A}_i \arrow[ur, dashed, "\bar{\varphi}"] \arrow[urr, "\bar{\lambda}_\varphi"'] & &
\end{tikzcd}$$

commutes in Alg_R. Since $\lambda_{\mathcal{B}}$ is injective, it follows from the commutativity of the diagram that there exist a unique morphism $\bar{\varphi}$ that completes it. Furthermore, the index $i \in [n]$ is the same as the one in Lemma 0.1.21 by its uniqueness property in the diagram of Eq. (0.1.34). Finally, the $\mathsf{RMod}(\mathcal{A})$ case is analogous except one considers the induced *anti-representation* $\rho_\varphi := \rho_{\mathcal{B}} \circ \varphi$ instead. □

Definition 0.1.25 (Projection selection morphism)**:** Suppose that $\mathcal{A} = \prod_{j=1}^n \mathcal{A}_j$ in Alg_R, $m \in \mathbb{N}$, and let $\tau\colon [m] \to [n]$, $k \mapsto \tau(k)$, be an index parametrization. We put

$$\Pi := \Pi_{(\tau)} := (\mathrm{pr}_{\tau(1)}, \ldots, \mathrm{pr}_{\tau(m)}),$$

that is

$$\begin{aligned} \Pi_{(\tau)}\colon \mathcal{A} &\to \prod_{k=1}^m \mathcal{A}_{\tau(k)} \\ (a_1, \ldots, a_n) &\mapsto (a_{\tau(1)}, \ldots, a_{\tau(m)}). \end{aligned}$$

Lemma-Definition 0.1.26 (Standard Form of $\Pi_{(\tau)}$)**:** Let $\mathcal{A} = \prod_{j=1}^n \mathcal{A}_j$, $m \in \mathbb{N}$, and $\tau\colon [m] \to [n]$ as above.

(1) Up to some $\pi \in \mathfrak{S}_m$, the map $\Pi_{(\tau)}$ can be written as

$$\Pi_{(\tau)}\colon \mathcal{A} \xrightarrow{\mathrm{pr}_I} \mathcal{A}_I \xhookrightarrow{\Delta_I} \textstyle\prod_{j\in I} \mathcal{A}_j^{m_j}, \tag{0.1.37}$$

which we will call a *standard form* of $\Pi_{(\tau)}$, where:
- $I := I_{(\tau)} := \tau([m]) \subseteq [n]$ (implicitly with the induced order from $\mathbb{N}$);
- $m_j := |\tau^{-1}(j)| \geq 1$, $j \in I$, and $\sum_{j\in I} m_j = m$ is an $\mathbb{N}$-partition of m of size $|I|$;
- $\Delta_I := \times_{j\in I}\Delta_j$ with $\Delta_j\colon \mathcal{A}_j \hookrightarrow \mathcal{A}_j^{m_j}$, $a_j \mapsto (a_j, \ldots, a_j)$, $j \in I$, being the obvious diagonal monomorphisms.

Moreover, the choice of said π is unique up to post-composition with a permutation from $\prod_{j\in I} \mathfrak{S}_j \leq \mathfrak{S}_m$. If $\Pi_{(\tau)}$ is in a *standard form*, then the index (re)mapping $\tau\colon [m] \to [n]$ will be called *fiber-grouping* or *fiber-consolidating*.

(2) If $X_k \subseteq \mathcal{A}_{\tau(k)}$, $1 \leq k \leq m$, then

$$\Pi_{(\tau)}^{-1}\left(\prod_{k=1}^m X_k\right) = \begin{cases} \bigcap\limits_{k\in\tau^{-1}(j)} X_k, \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise.} \end{cases} \tag{0.1.38}$$

Proof: To (1): This is simply a regrouping of the $\mathcal{A}_{\tau(k)}$-factors in the codomain by τ-fibers: clearly, there exists $\pi \in \mathfrak{S}_m$ inducing a permutation isomorphism (abuse of notation)

$$\pi_{(\mathcal{A})}\colon \prod_{k=1}^m \mathcal{A}_{\tau(k)} \xrightarrow{\cong} \prod_{k=1}^m \mathcal{A}_{\tau(\pi(k))} \equiv \prod_{j\in I} \mathcal{A}_j^{m_j}.$$

Then replacing $\Pi_{(\tau)}$ by the post-composition $\pi_{\mathcal{A}} \circ \Pi_{(\tau)}$ gives the desired map, and the factorization $(\times_{j\in I}\Delta_j) \circ \mathrm{pr}_I$ is now immediate. Finally, the uniqueness up to post-composition with a permutation from $\prod_{j\in I} \mathfrak{S}_{m_j} \leq \mathfrak{S}_m$ is also clear.

To (2): Indeed we have

$$\Pi_{(\tau)}^{-1}\Big(\prod_{k=1}^{m} X_k\Big) \overset{\text{def}}{=} (\mathrm{pr}^{\mathcal{A}}_{j(k)})^{-1}_{1\le k\le m}\Big(\prod_{k=1}^{m} X_k\Big) = \bigcap_{k=1}^{m}(\mathrm{pr}^{\mathcal{A}}_{j(k)})^{-1}(X_k) = \bigcap_{j\in I}\ \bigcap_{k\in\tau^{-1}(j)}(\mathrm{pr}^{\mathcal{A}}_{j(k)})^{-1}(X_k) =$$
$$= \bigcap_{j\in I}\ \bigcap_{k\in\tau^{-1}(j)}(\mathrm{pr}^{\mathcal{A}}_{j})^{-1}(X_k) = \bigcap_{j\in I}(\mathrm{pr}^{\mathcal{A}}_{j})^{-1}\Big(\bigcap_{k\in\tau^{-1}(j)} X_k\Big) = \begin{cases}\bigcap_{k\in\tau^{-1}(j)} X_k, \text{ if } j\in I\\ \mathcal{A}_j, \text{ otherwise.}\end{cases}$$

as claimed. □

Remarks:

(1) Conversely, any $\varnothing \neq I \subseteq [n]$ and any $\mathbb{N}$-partition $\{m_j : j\in I\}$ of m give together rise to a projection selection homomorphism $\Pi := (\times_{j\in I}\Delta_j)\circ \mathrm{pr}_I\colon \prod_{j=1}^{n}\mathcal{A}_j \twoheadrightarrow \prod_{j\in I}\mathcal{A}_j \hookrightarrow \prod_{j\in I}\mathcal{A}_j^{m_j}$.

(2) If $n\ge m$ and τ is injective, then $\Pi_{(\tau)}$ is an honest projection (surjective). If τ is not injective, then $\Pi_{(\tau)}$ cannot be surjective, since it will contain non-trivial diagonal algebra inclusions $\Delta_j\colon \mathcal{A}_j \hookrightarrow \mathcal{A}_j^{m_j}$, $m_j\ge 2$, as its components for some $1\le j\le n$. If $n\le m$ and τ is surjective, then $\Pi_{(\tau)}$ is injective as it does not discard any of the factors $\mathcal{A}_j$, $1\le j\le n$. If τ is not surjective, then we have

$$\operatorname{Ker}\Pi_{(\tau)} = \prod_{j=1}^{n}\begin{cases}0, \text{ if } j\in I\\ \mathcal{A}_j, \text{otherwise.}\end{cases}$$

(3) A slight variation of the above description of Π is the following: realizing that $\mathcal{A}_j^0 := 0$ is the singleton object in Alg_R, we have $\mathcal{A}_j\times 0\cong \mathcal{A}_j$ in Alg_R naturally. Then Π can be given as $\times_{j=1}^{n}\Delta_j$, where again $\Delta_j\colon \mathcal{A}_j\to\mathcal{A}_j^{m_j}$, but this time $m_j = |\tau^{-1}(j)|\ge 0$, $1\le j\le n$, and $m_1+\cdots+m_n = m$ is an $\mathbb{N}_0$-partition of m of size n, followed by discarding of the 0-factors.

(4) The related notion of $\Sigma_{(\tau)}$ will be introduced later on in Definition 0.1.48. The previous observations about $\Pi_{(\tau)}$ apply to it almost verbatim.

The following lemma too falls into the category of "unique up to permutations" nonsense.

Lemma 0.1.27 (Factorization of Morphisms from Finite Products to Finite Indecomposable Products in Alg_R)**:** Let $\mathcal{A}\xrightarrow{\varphi}\mathcal{B}$ be a morphism in Alg_R. If $\mathcal{A} = \prod_{j=1}^{n}\mathcal{A}_j$ in Alg_R and $\mathcal{B}$ is block-decomposable, then there exists a block decomposition $\beta\colon\mathcal{B}\cong\prod_{k=1}^{m}\mathcal{B}_k$ and a fiber-grouping index mapping $\tau := \tau_\varphi\colon [m]\to[n]$, $k\mapsto\tau(k)$, depending on said decomposition and φ, such that φ can be written as

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\varphi"] \arrow[d, "\Pi_\varphi"'] & \mathcal{B} \arrow[r, "\beta", "\cong"'] & \prod_{k=1}^{m}\mathcal{B}_k \\ \prod_{k=1}^{m}\mathcal{A}_{\tau(k)} \arrow[urr, dashed, "\exists_1\,\bar\varphi"'] & & \end{tikzcd} \tag{0.1.39}$$

where:

- $\Pi_\varphi := \Pi_{(\tau_\varphi)}$ is a projection selection morphism, in a standard form, associated to the morphism $\mathcal{A}\xrightarrow{\varphi}\mathcal{B}$ in Alg_R;
- $\bar\varphi = \times_{k=1}^{m}\bar\varphi_k$ for some morphisms $\bar\varphi_k\colon\mathcal{A}_{\tau(k)}\to\mathcal{B}_k$ such that $\varphi_k = \bar\varphi_k\circ\mathrm{pr}^{\mathcal{A}}_{\tau(k)}$, $1\le k\le m$, with $(\varphi_1,\ldots,\varphi_m) := \beta\circ\varphi$ being the "components" of φ with respect to the above block decomposition of $\mathcal{B}$.

Moreover:

$$\operatorname{Ker}(\varphi) = \begin{cases}\bigcap_{k\in\tau^{-1}(j)}\operatorname{Ker}(\bar\varphi_k), \text{ if } j\in\tau([m])\\ \mathcal{A}_j, \text{ otherwise.}\end{cases} \tag{0.1.40}$$

Proof: Let $\beta'\colon \mathcal{B} \xrightarrow{\cong} \prod_{k=1}^m \mathcal{B}'_k$ be *some initial* block decomposition of $\mathcal{B}$. Write $\varphi' := \beta' \circ \varphi\colon \mathcal{A} \to \prod_{k=1}^m \mathcal{B}'_k$ and $\varphi' =: (\varphi'_1, \ldots, \varphi'_m)$ for its respective components. By Lemma 0.1.21 we have that $\forall\, 1 \le k \le m\ \exists_1\, 1 \le i \le n\ \exists_1\, \bar{\varphi}'_k \in \mathrm{Mor}_{\mathsf{Alg}_R}(\mathcal{A}_i, \mathcal{B}'_k)\colon \varphi'_k = \bar{\varphi}'_k \circ \mathrm{pr}_i^{\mathcal{A}}$, where different k-s may share the same i. Putting $\tau'(k) := i$, i.e. $\bar{\varphi}'_k\colon \mathcal{A}_{\tau'(k)} \to \mathcal{B}'_k$, and $\bar{\varphi}' := \times_{k=1}^m \bar{\varphi}'_k$, we get a projection selection homomorphism $\Pi' := \Pi_{(\tau')}\colon \mathcal{A} \to \prod_{k=1}^m \mathcal{A}_{\tau'(k)}$ such that $\varphi' = \bar{\varphi}' \circ \Pi'$. As in Lemma-Definition 0.1.26 there exists $\pi \in \mathfrak{S}_m$ such that taking $\tau := \tau' \circ \pi$ we have $\Pi := \Pi_{(\tau)} \overset{\text{def}}{=} \pi_{(\mathcal{A})} \circ \Pi'$ in a standard form. Furthermore, π also induces a permutation isomorphism $\pi_{(\mathcal{B})}\colon \prod_{k=1}^m \mathcal{B}'_k \xrightarrow{\cong} \prod_{k=1}^m \mathcal{B}'_{\pi(k)}$. Now put $\mathcal{B}_k := \mathcal{B}'_{\pi(k)}$, $\varphi_k := \varphi'_{\pi(k)}\colon \mathcal{A} \to \mathcal{B}_k$, $\bar{\varphi}_k := \bar{\varphi}'_{\pi(k)}\colon \mathcal{A}_{\tau(k)} \to \mathcal{B}_k$, $1 \le k \le m$, and $\bar{\varphi} := \times_{k=1}^m \bar{\varphi}_k$, that is, $\bar{\varphi} \overset{\text{def}}{=} \pi_{(\mathcal{B})} \circ \bar{\varphi}' \circ \pi_{(\mathcal{A})}^{-1}$. Setting $\beta := \pi_{(\mathcal{B})} \circ \beta'\colon \mathcal{B} \xrightarrow{\cong} \prod_{k=1}^m \mathcal{B}'_k \xrightarrow{\cong} \prod_{k=1}^m \mathcal{B}_k$, we get $\beta \circ \varphi \overset{\text{def}}{=} \pi_{(\mathcal{B})} \circ \beta' \circ \varphi = \pi_{(\mathcal{B})} \circ \varphi' = \pi_{(\mathcal{B})} \circ \bar{\varphi}' \circ \Pi' = \pi_{(\mathcal{B})} \circ \bar{\varphi}' \circ \pi_{(\mathcal{A})}^{-1} \circ \Pi = \bar{\varphi} \circ \Pi$ as desired. Finally, since β is an isomorphism, we have

$$\mathrm{Ker}(\varphi) = \mathrm{Ker}(\bar{\varphi} \circ \Pi_\varphi) = \Pi_\varphi^{-1}(\mathrm{Ker}(\bar{\varphi})) \overset{\text{def}}{=} \Pi_\varphi^{-1}(\mathrm{Ker}(\times_{k=1}^m \bar{\varphi}_k)) = \Pi_\varphi^{-1}\Big(\prod_{k=1}^m \mathrm{Ker}(\bar{\varphi}_k)\Big),$$

and the last claim follows from Eq. (0.1.38). $\square$

Definition 0.1.28: A choice of product decomposition of the codomain $\mathcal{B} \cong \prod_{k=1}^m \mathcal{B}_k$ satisfying Eq. (0.1.39) will be called **φ-adapted.**

When $\mathcal{A}$ is commutative, we can actually deduce from Lemma 0.1.27 (morphism setting) a generalization of Corollary 0.1.23 to finitely decomposable ${}_{\mathcal{A}}M$ (representation setting) as follows:

Corollary 0.1.29 (Factorization of Finitely Decomposable Representations of Finite "Commutative" Products)**:** Let $\mathcal{A} \in \mathsf{CAlg}_R$ and $M \in \mathsf{Mod}(\mathcal{A})$ via $\lambda\colon \mathcal{A} \to \mathrm{End}_R(M)$. If $\mathcal{A} = \prod_{j=1}^n \mathcal{A}_j$ in CAlg_R and M is finitely decomposable in $\mathsf{Mod}(\mathcal{A})$, then there exists a complete decomposition ${}_{\mathcal{A}}M = \bigoplus_{k=1}^m {}_{\mathcal{A}}M_k$ and a fiber-grouping index mapping $\tau := \tau_\lambda\colon [m] \to [n]$, $k \mapsto \tau(k)$, depending on said decomposition and λ, such that λ can be written as

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\lambda^{\mathrm{cor}}"] \arrow[d, "\Pi_\lambda"'] & \prod_{k=1}^m \mathrm{End}_R(M_k) \arrow[r, hook, "\iota_{\mathrm{cor}}"] & \mathrm{End}_R(M) \\ \prod_{k=1}^m \mathcal{A}_{\tau(k)} \arrow[ur, dashed, "\exists_1\, \overline{\lambda^{\mathrm{cor}}}"'] & & \end{tikzcd} \tag{0.1.41}$$

where:

- $\Pi_\lambda := \Pi_{(\tau_\lambda)}$ is a projection selection morphism, in a standard form, associated to the representation λ;
- $\overline{\lambda^{\mathrm{cor}}} = \times_{k=1}^m \bar{\lambda}_k$ for some representations $\bar{\lambda}_k\colon \mathcal{A}_{\tau(k)} \to \mathrm{End}_R(M_k)$ such that $\lambda_k = \bar{\lambda}_k \circ \mathrm{pr}_{\tau(k)}^{\mathcal{A}}$, $1 \le k \le m$, where $\lambda^{\mathrm{cor}} = (\lambda_1, \ldots, \lambda_m)$ as usual.

Moreover:

$$\mathrm{Ker}(\lambda) = \begin{cases} \bigcap_{k \in \tau^{-1}(j)} \mathrm{Ker}(\bar{\lambda}_k), \text{ if } j \in \tau([m]) \\ \mathcal{A}_j, \text{ otherwise.} \end{cases} \tag{0.1.42}$$

Proof: Let ${}_{\mathcal{A}}M = \bigoplus_{k=1}^m {}_{\mathcal{A}}M'_k$ be *some initial* complete decomposition of M in $\mathsf{LMod}(\mathcal{A})$ and write instead $\lambda' := \lambda$ to signify that choice. Then by Section 0.1.4 we have a commutative diagram

$$\begin{tikzcd} & & \mathcal{A} \arrow[dll, two heads, "(\lambda')^{\mathrm{im}}"'] \arrow[dl, "(\lambda')^{\mathrm{mor}} =: \varphi"'] \arrow[d, "(\lambda')^{\mathrm{cor}}"] \arrow[dr, "\lambda' = \lambda"] & \\ \lambda'(\mathcal{A}) \arrow[r, hook] & \mathcal{B}' = \prod_{k=1}^m \mathcal{B}'_k \arrow[r, hook, "\beta'"] & \prod_{k=1}^m \mathrm{End}_R(M'_k) \arrow[r, hook, "\iota'_{\mathrm{cor}}"] & \mathrm{End}_R(M) \end{tikzcd}$$

where $\beta'_k\colon \mathcal{B}'_k \hookrightarrow \mathrm{End}_R(M'_k)$, $1 \le k \le m$, denote the canonical image inclusions and $\beta' := \times_{k=1}^m \beta'_k$. We set for short $\varphi := (\lambda')^{\mathrm{mor}}$. It clear from the proof of Corollary 0.1.23 that $\prod_{k=1}^m \mathcal{B}'_k$ is a *block decomposition* for $\mathcal{B}'$. It follows from Lemma 0.1.27 that there exists a permutation[13] $\pi \in \mathfrak{S}_m$ inducing block permutation isomorphisms $\pi_{(\mathcal{B}')}\colon \mathcal{B}' \overset{\mathrm{def}}{=} \prod_{k=1}^m \mathcal{B}'_k \xrightarrow{\cong} \prod_{k=1}^m \mathcal{B}_k$ and $\pi_{\mathrm{End}}\colon \prod_{k=1}^m \mathrm{End}_R(M'_k) \xrightarrow{\cong} \prod_{k=1}^m \mathrm{End}_R(M_k)$ with $\mathcal{B}_k := \mathcal{B}'_{\pi(k)}$ and $M_k := M'_{\pi(k)}$, $1 \le k \le m$, such that we get a commutative diagram in Alg_R

$$\begin{tikzcd}
\mathcal{A} \arrow[d, "\varphi"] \arrow[dr, dashed, "\lambda^{\mathrm{mor}}"'] \arrow[drr, dashed, "\lambda^{\mathrm{cor}}"] \arrow[drrr, "\lambda"] \arrow[dd, bend right=40, "\Pi_\varphi"'] & & & \\
\mathcal{B}' \arrow[r, "\pi_{(\mathcal{B}')}", "\cong"'] & \prod_{k=1}^m \mathcal{B}_k \arrow[r, hook, "\beta"] & \prod_{k=1}^m \mathrm{End}_R(M_k) \arrow[r, hook, "\iota_{\mathrm{cor}}"] & \mathrm{End}_R(M) \\
\prod_{k=1}^m \mathcal{A}_{\tau(k)} \arrow[ur, "\exists_1\, \bar{\varphi}"] \arrow[urr, dashed, "\exists_1\, \overline{\lambda^{\mathrm{cor}}}"'] & & &
\end{tikzcd} \tag{0.1.43}$$

where $\bar{\varphi} = (\bar{\varphi}_1, \ldots, \bar{\varphi}_m)$ with $\bar{\varphi}_k \circ \mathrm{pr}^{\mathcal{A}}_{\tau(k)} = \varphi_k$, $1 \le k \le m$, for $\pi_{(\mathcal{B}')} \circ \varphi =: (\varphi_1, \ldots, \varphi_m)$, $\beta := \times_{k=1}^m \beta_k$ for $\beta_k := \beta'_{\pi(k)}$, $1 \le k \le m$, i.e. $\beta \circ \pi_{(\mathcal{B}')} \overset{\mathrm{def}}{=} \pi_{\mathrm{End}} \circ \beta'$, $\iota_{\mathrm{cor}} := \iota'_{\mathrm{cor}} \circ \pi_{\mathrm{End}}^{-1}$, and $\overline{\lambda^{\mathrm{cor}}} := \times_{k=1}^m \bar{\lambda}_k$ for $\bar{\lambda}_k := \beta_k \circ \bar{\varphi}_k$, $1 \le k \le m$. We can also take now $\lambda^{\mathrm{cor}} := \pi_{\mathrm{End}} \circ (\lambda')^{\mathrm{cor}} \overset{\mathrm{def}}{=} \pi_{\mathrm{End}} \circ \beta' \circ \varphi \overset{\mathrm{def}}{=} \beta \circ \pi_{(\mathcal{B}')} \circ \varphi \overset{\mathrm{def}}{=} \beta \circ (\varphi_1, \ldots, \varphi_m) = (\beta_1 \circ \varphi_1, \ldots, \beta_m \circ \varphi_m)$. Then indeed $\bar{\lambda}_k \circ \mathrm{pr}^{\mathcal{A}}_{\tau(k)} \overset{\mathrm{def}}{=} \beta_k \circ \bar{\varphi}_k \circ \mathrm{pr}^{\mathcal{A}}_{\tau(k)} = \beta_k \circ \varphi_k =: \lambda_k$ is precisely the k-th component of λ^{cor} as just defined, $1 \le k \le m$. It remains to justify the notation $\Pi_\lambda := \Pi_{\lambda^{\mathrm{mor}}}$, which will be done after the next lemma... Finally, since $\mathrm{Ker}(\lambda) = \mathrm{Ker}(\lambda^{\mathrm{mor}})$ (as $\iota_{\mathrm{cor}} \circ \beta$ is injective) and $\mathrm{Ker}(\bar{\lambda}_k) = \mathrm{Ker}(\bar{\varphi}_k)$ (since β_k is injective), $1 \le k \le m$, the last claim follows from Eq. (0.1.40) applied to the pair $(\lambda^{\mathrm{mor}}, \bar{\varphi})$. □

Lemma 0.1.30 (Factorization of Finitely Decomposable Representations of Finite Products)**:** Let $\mathcal{A} \in \mathsf{Alg}_R$ and $M \in \mathsf{LMod}(\mathcal{A})$ via $\lambda\colon \mathcal{A} \to \mathrm{End}_R(M)$. If $\mathcal{A} = \prod_{j=1}^n \mathcal{A}_j$ in Alg_R and M is finitely decomposable in $\mathsf{LMod}(\mathcal{A})$, then there exist a complete decomposition $M = \bigoplus_{k=1}^m M_k$ in $\mathsf{LMod}(\mathcal{A})$ and a fiber-grouping index mapping $\tau := \tau_\lambda\colon [m] \to [n]$, $k \mapsto \tau(k)$, depending on said decomposition and λ, such that λ can be written as

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\lambda^{\mathrm{cor}}"] \arrow[d, "\Pi_\lambda"'] & \prod_{k=1}^m \mathrm{End}_R(M_k) \arrow[r, hook, "\iota_{\mathrm{cor}}"] & \mathrm{End}_R(M) \\
\prod_{k=1}^m \mathcal{A}_{\tau(k)} \arrow[ur, dashed, "\exists_1\, \overline{\lambda^{\mathrm{cor}}}"'] & &
\end{tikzcd} \tag{0.1.44}$$

where:

- $\Pi_\lambda := \Pi_{(\tau_\lambda)}$ is a projection selection morphism, in a standard form, associated to the representation λ;
- $\overline{\lambda^{\mathrm{cor}}} = \times_{k=1}^m \bar{\lambda}_k$ for some representations $\bar{\lambda}_k\colon \mathcal{A}_{\tau(k)} \to \mathrm{End}_R(M_k)$ such that $\lambda_k = \bar{\lambda}_k \circ \mathrm{pr}^{\mathcal{A}}_{\tau(k)}$, $1 \le k \le m$, where $\lambda^{\mathrm{cor}} = (\lambda_1, \ldots, \lambda_m)$ as usual.

Moreover:

$$\mathrm{Ker}(\lambda) = \begin{cases} \bigcap\limits_{k \in \tau^{-1}(j)} \mathrm{Ker}(\bar{\lambda}_k), \text{ if } j \in \tau([m]) \\ \mathcal{A}_j, \text{ otherwise.} \end{cases} \tag{0.1.45}$$

Analogously for $M \in \mathsf{RMod}(\mathcal{A})$ via an anti-representation $\rho\colon \mathcal{A} \to \mathrm{End}_R(M)$.

[13] even if we don't invoke the uniqueness of block decompositions up to permutation, it follows from the proof itself of Lemma 0.1.27 that it is sufficient to permute the existing block decomposition;

Proof: Let ${}_{\mathcal{A}}M = \bigoplus_{k=1}^{m} {}_{\mathcal{A}}M'_k$ be *some initial* complete decomposition of M in $\mathsf{LMod}(\mathcal{A})$ and as in Corollary 0.1.29 write $\lambda' := \lambda$ to indicate that choice. In particular, we have a commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[d, "(\lambda')^{\mathrm{cor}}"'] \arrow[dr, "\lambda'"] & \\ \prod_{k=1}^{m} \mathrm{End}_R(M'_k) \arrow[r, hook, "\iota'_{\mathrm{cor}}"] & \mathrm{End}_R(M) \end{tikzcd}$$

in Alg_R with $(\lambda')^{\mathrm{cor}} =: (\lambda'_1, \dots, \lambda'_m)$. By Lemma 0.1.22 we have that $\forall\, 1 \le k \le m\ \exists_1\, 1 \le i \le n\ \exists_1\, \bar{\lambda}'_k \in \mathrm{Mor}_{\mathsf{Alg}_R}(\mathcal{A}_i, \mathrm{End}_R(M'_k))\colon \lambda'_k = \bar{\lambda}'_k \circ \mathrm{pr}_i^{\mathcal{A}}$, where different k-s may share the same i. Putting $\tau'(k) := i$, i.e. $\bar{\lambda}'_k\colon \mathcal{A}_{\tau'(k)} \to \mathrm{End}_R(M'_k)$, and $\overline{(\lambda')^{\mathrm{cor}}} := \times_{k=1}^{m} \bar{\lambda}'_k$, we get a projection selection homomorphism $\Pi' := \Pi_{(\tau')}\colon \mathcal{A} \to \prod_{k=1}^{m} \mathcal{A}_{\tau'(k)}$ such that $(\lambda')^{\mathrm{cor}} = \overline{(\lambda')^{\mathrm{cor}}} \circ \Pi'$. As in Lemma-Definition 0.1.26 there exists $\pi \in \mathfrak{S}_m$ such that taking $\tau := \tau' \circ \pi$ we have $\Pi := \Pi_{(\tau)} \overset{\mathrm{def}}{=} \pi_{(\mathcal{A})} \circ \Pi'$ in standard form. Furthermore, π also induces a permutation isomorphism $\pi_{\mathrm{End}}\colon \prod_{k=1}^{n} \mathrm{End}_R(M'_k) \xrightarrow{\cong} \prod_{k=1}^{m} \mathrm{End}_R(M'_{\pi(k)})$. Now put $M_k := M'_{\pi(k)}$, $\lambda_k := \lambda'_{\pi(k)}\colon \mathcal{A} \to \mathrm{End}_R(M_k)$, $\bar{\lambda}_k := \bar{\lambda}'_{\pi(k)}\colon \mathcal{A}_{\tau(k)} \to \mathrm{End}_R(M_k)$, $1 \le k \le m$, and $\overline{\lambda^{\mathrm{cor}}} := \times_{k=1}^{m} \bar{\lambda}_k$, that is, $\overline{\lambda^{\mathrm{cor}}} = \pi_{\mathrm{End}} \circ \overline{(\lambda')^{\mathrm{cor}}} \circ \pi_{(\mathcal{A})}^{-1}$. For $\bigoplus_{k=1}^{m} M_k = \bigoplus_{k=1}^{m} M'_k = M$ (rather than just an isomorphism) take $\iota_{\mathrm{cor}}\colon \prod_{k=1}^{m} \mathrm{End}_R(M_k) \hookrightarrow \mathrm{End}_R(M)$, that is, $\iota_{\mathrm{cor}} = \iota'_{\mathrm{cor}} \circ \pi_{\mathrm{End}}^{-1}$. Thus $\lambda = \iota'_{\mathrm{cor}} \circ (\lambda')^{\mathrm{cor}} = (\iota'_{\mathrm{cor}} \circ \pi_{\mathrm{End}}^{-1}) \circ (\pi_{\mathrm{End}} \circ (\lambda')^{\mathrm{cor}}) = \iota_{\mathrm{cor}} \circ \pi_{\mathrm{End}} \circ \overline{(\lambda')^{\mathrm{cor}}} \circ \Pi' = \iota_{\mathrm{cor}} \circ (\pi_{\mathrm{End}} \circ \overline{(\lambda')^{\mathrm{cor}}} \circ \pi_{(\mathcal{A})}^{-1}) \circ (\pi_{(\mathcal{A})} \circ \Pi') = \iota_{\mathrm{cor}} \circ \overline{\lambda^{\mathrm{cor}}} \circ \Pi$ as desired. Thinking of ι_{cor} and ι'_{cor} as **identifications** (with the corner corestrictions), we simply have $\lambda = \pi_{\mathrm{End}} \circ \lambda'$. The $\mathrm{Ker}(\lambda)$ part follows as in the proof of Lemma 0.1.27 from Eq. (0.1.38). Finally, the $\mathsf{RMod}(\mathcal{A})$ case is analogous by invoking the *anti-representation* part of Lemma 0.1.22. □

Remarks:

(1) Comparison between Corollary 0.1.29 and Lemma 0.1.30: inspecting the proofs of Lemma 0.1.27, on which Corollary 0.1.29 is based, and of Lemma 0.1.30, it is clear from the uniqueness part of Corollary 0.1.23 that $\tau_{\lambda^{\mathrm{mor}}} = \tau_\lambda$ and hence $\Pi_{\lambda^{\mathrm{mor}}} = \Pi_\lambda$.

(2) The similitude, yet subtle difference between the proofs of Lemma 0.1.27 and Lemma 0.1.30 is an artefact of the same tension between Lemma 0.1.21 and Lemma 0.1.22 mentioned previously.

We can now also deduce the *weaker* form (mentioned previously) of the *morphism factorization* from the *representation factorization*:

Corollary 0.1.31 (Factorization of Morphisms from Finite Products to Finite Indecomposable Products of Domain-Indecomposables)**:** Let $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ be a morphism in Alg_R. If $\mathcal{A} = \prod_{j=1}^{n} \mathcal{A}_j$ in Alg_R and $\mathcal{B}$ is decomposable into $\mathsf{LMod}(\mathcal{A})$- or $\mathsf{RMod}(\mathcal{A})$-indecomposable blocks, then there exists a block decomposition $\beta\colon \mathcal{B} \cong \prod_{k=1}^{m} \mathcal{B}_k$ in Alg_R and a fiber-grouping index mapping $\tau := \tau_\varphi\colon [m] \to [n]$, $k \mapsto \tau(k)$, depending on said decomposition and φ, such that φ can be written as

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\varphi"] \arrow[d, "\Pi_\varphi"'] & \mathcal{B} \arrow[r, "\beta", "\cong"'] & \prod_{k=1}^{m} \mathcal{B}_k \\ \prod_{k=1}^{m} \mathcal{A}_{\tau(k)} \arrow[urr, dashed, "\exists_1\, \bar{\varphi}"'] & & \end{tikzcd} \tag{0.1.46}$$

where:

- $\Pi_\varphi := \Pi_{(\tau_\varphi)}$ is the projection selection morphism, in a standard form, associated to the morphism $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ in Alg_R;
- $\bar{\varphi} = \times_{k=1}^{m} \bar{\varphi}_k$ for some morphisms $\bar{\varphi}_k\colon \mathcal{A}_{\tau(k)} \to \mathcal{B}_k$ such that $\varphi_k = \bar{\varphi}_k \circ \mathrm{pr}_{\tau(k)}^{\mathcal{A}}$, $1 \le k \le m$, with $(\varphi_1, \dots, \varphi_m) := \beta \circ \varphi$ being the "components" of φ with respect to the above block decomposition of $\mathcal{B}$.

Moreover:

$$\operatorname{Ker}(\varphi) = \begin{cases} \bigcap_{k \in \tau^{-1}(j)} \operatorname{Ker}(\bar{\varphi}_k), \text{ if } j \in \tau([m]) \\ \mathcal{A}_j, \text{ otherwise.} \end{cases} \tag{0.1.47}$$

Proof: This is basically a slight generalization of the proof of Corollary 0.1.24. Consider again $\lambda_\varphi := \varphi^{\mathrm{rep}} \overset{\mathrm{def}}{=} \lambda_{\mathcal{B}} \circ \varphi$. By hypothesis and Lemma 0.1.4, $\mathcal{B}$ admits a finite decomposition $\mathcal{B} = \bigoplus_{k=1}^m \mathcal{B}_k$ with $\mathcal{B}_k \lhd \mathcal{B}$, $1 \leq k \leq m$, that is *simultaneously indecomposable* in both $\mathsf{Mod}(\mathcal{B})$ and $\mathsf{LMod}(\mathcal{A})$ (or $\mathsf{RMod}(\mathcal{A})$, respectively). It follows from Lemma 0.1.30 that we have a commutative diagram

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, "\Pi_{\lambda_\varphi}"'] \arrow[d, "\lambda_\varphi^{\mathrm{cor}}"] \arrow[dr, "\lambda_\varphi"] & \\ \prod_{k=1}^m \mathcal{A}_{\tau(k)} \arrow[r, "\overline{\lambda_\varphi^{\mathrm{cor}}}"] & \prod_{k=1}^m \operatorname{End}_R(\mathcal{B}_k) \arrow[r, hook, "\iota_{\mathrm{cor}}"] & \operatorname{End}_R(\mathcal{B}) \end{tikzcd}$$

in Alg_R. By Corollary 0.1.17 and Lemma 0.1.20 this diagram extends to a commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[d, "\varphi=(\varphi_1,\ldots,\varphi_m)"] \arrow[dr, "\lambda_\varphi^{\mathrm{cor}}"] \arrow[drr, "\lambda_\varphi"] \arrow[dd, bend right=60, "\Pi_{\lambda_\varphi}"'] & & \\ \mathcal{B} \cong \prod_{k=1}^m \mathcal{B}_k \arrow[r, hook, "\times_{k=1}^m \lambda_{\mathcal{B}_k}"] & \prod_{k=1}^m \operatorname{End}_R(\mathcal{B}_k) \arrow[r, hook, "\iota_{\mathrm{cor}}"] & \operatorname{End}_R(\mathcal{B}) \\ \prod_{k=1}^m \mathcal{A}_{\tau(k)} \arrow[u, dashed, "\times_{k=1}^m \bar{\varphi}_k"] \arrow[ur, "\overline{\lambda_\varphi^{\mathrm{cor}}}=\times_{k=1}^m \bar{\lambda}_{\varphi_k}"'] & & \end{tikzcd} \tag{0.1.48}$$

in Alg_R with $\lambda_\varphi^{\mathrm{cor}} = (\varphi_1^{\mathrm{rep}}, \ldots, \varphi_m^{\mathrm{rep}}) =: (\lambda_{\varphi_1}, \ldots, \lambda_{\varphi_m})$ and $\overline{\lambda_\varphi^{\mathrm{cor}}} = \times_{k=1}^m \bar{\lambda}_{\varphi_k}$, where $\forall\, 1 \leq k \leq m$: $\lambda_{\varphi_k} = \bar{\lambda}_{\varphi_k} \circ \mathrm{pr}_{\tau(k)}^{\mathcal{A}}$. Since $\lambda_{\mathcal{B}_k}$ is injective, it follows from the commutativity of the diagram that $\bar{\lambda}_{\varphi_k}$ induces a unique $\bar{\varphi}_k \colon \mathcal{A}_{\tau(k)} \to \mathcal{B}_k$ with $\bar{\lambda}_{\varphi_k} = \lambda_{\mathcal{B}_k} \circ \bar{\varphi}_k$, i.e.

$$\bar{\lambda}_{\varphi_k} \overset{\mathrm{def}}{=} \overline{\varphi_k^{\mathrm{rep}}} = \bar{\varphi}_k^{\mathrm{rep}} \overset{\mathrm{def}}{=} \lambda_{\bar{\varphi}_k},$$

that are also easily seen to satisfy $\varphi_k = \bar{\varphi}_k \circ \mathrm{pr}_{\tau(k)}^{\mathcal{A}}$, $1 \leq k \leq m$. The diagram is completed by taking $\times_{k=1}^m$ of the corresponding arrows. Comparing the proofs of Lemma 0.1.27 and Lemma 0.1.30, it is clear from the uniqueness part of Corollary 0.1.24 that the construction of $\tau' \colon [m] \to [n]$ produces the same index parametrization in both cases, *provided* that the *initial* complete decompositions agree, i.e. $\forall\, 1 \leq k \leq m$: $M_k' = \mathcal{B}_k'$, whence... one can / to arrange an equality $\tau_{\lambda_\varphi} = \tau_\varphi$, i.e. $\Pi_{\lambda_\varphi} = \Pi_\varphi$. Moreover, we have $\operatorname{Ker}(\varphi) = \operatorname{Ker}(\lambda_\varphi)$ and $\operatorname{Ker}(\bar{\varphi}_k) = \operatorname{Ker}(\bar{\lambda}_{\varphi_k})$, $1 \leq k \leq m$, by faithfulness of the regular representations, hence the $\operatorname{Ker}(\varphi)$ part follows from the corresponding claim in Lemma 0.1.30. Finally, the case $\mathsf{RMod}(\mathcal{A})$ is argued analogously by considering the induced *anti-representation* $\rho_\varphi := \rho_{\mathcal{B}} \circ \varphi \colon \mathcal{A} \to \mathcal{B} \hookrightarrow \operatorname{End}_R(\mathcal{B})$ instead. □

Notation: Whenever there is no ambiguity about the index (re)mapping $\tau \colon [m] \to [n]$, by abuse of notation we will simply write for convenience $j \colon [m] \to [n]$ and $j(k) := \tau(k)$ instead, e.g. $\Pi_\lambda \colon \prod_{j=1}^n \mathcal{A}_j \to \prod_{k=1}^m \mathcal{A}_{j(k)}$. □

With the notations of Lemma 0.1.27 and Lemma 0.1.30 we now finally put:

Definition 0.1.32 (Induced Index Subset)**:** Let $\mathcal{A} := \prod_{j=1}^n \mathcal{A}_j$ in Alg_R.

(1) If $\mathcal{B}$ is finitely decomposable in Alg_R and $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ is a morphism, then we set

$$I_\varphi := I_{(\tau_\varphi)} \overset{\mathrm{def}}{=} \tau_\varphi([m]) \subseteq [n].$$

(2) If $\lambda\colon \mathcal{A} \to \mathrm{End}_R(M)$ is a representation such that M is finitely decomposable in $\mathsf{LMod}(\mathcal{A})$, then we set
$$I_\lambda := I_{(\tau_\lambda)} \overset{\text{def}}{=} \tau_\lambda([m]) \subseteq [n].$$
Analogously for anti-representations $\rho\colon \mathcal{A} \to \mathrm{End}_R(M)$.

Express $\mathcal{B}$ being finitely decomposable through some property of φ... Then also: M is finitely decomposable in $\mathsf{LMod}(\mathcal{A})$ if and only if λ is how decomposable?...

Remarks:

(1) If $J \subseteq [n]$, then $I_{\mathrm{pr}_J} \overset{\text{def}}{=} J$ since $\mathrm{pr}_J = \mathrm{id}_{\prod_{j\in J}\mathcal{A}_j} \circ \mathrm{id}_{\prod_{j\in J}\mathcal{A}_j} \circ \mathrm{pr}_J$. More succinctly, $I_{\mathrm{pr}_I} = I$.
(2) If φ is injective or ρ faithful, then *necessarily* $I_\varphi = I_\rho = [n]$.
(3) If $\mathcal{A} \xrightarrow{\varphi} \mathcal{B} \xrightarrow{\psi} \mathcal{C}$, then $I_{\psi\circ\varphi} \subseteq I_\varphi \subseteq [n]$.
(4) Internal Products: if $\varphi := (\varphi_1, \ldots, \varphi_m)\colon \mathcal{A} \to \prod_{k=1}^m \mathcal{B}_k$ (i.e. $I_{\varphi_k} \subseteq [n]$, $1 \le k \le m$), then $I_\varphi = \bigcup_{1\le k\le m} I_{\varphi_k}$. This boils down to the fact that pr_I is itself an internal product $(\mathrm{pr}_j)_{j\in I}$.
(5) External Products: if $\varphi := \times_{j=1}^n \varphi_j$, then *necessarily* $I_{\times_{j=1}^n\varphi_j} = \bigcup_{j=1}^n I_{\varphi_j}$. □

0.1.6. Relative Notions

Definition 0.1.33 (Relative closures, units, radicals, classes)**:** Let $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ be a morphism in Alg_R.

(1) Closure[14] with respect to φ: if $X \subseteq \mathcal{A}$, then
$$X^\varphi := \varphi^{-1}(\varphi(X)).$$
(2) Multiplicative submonoid with respect to φ:
$$\mathcal{A}_\varphi^\times := \mathrm{U}_\varphi(\mathcal{A}) := \varphi^{-1}(\mathrm{U}(\mathcal{B})) \overset{\text{def}}{=} \varphi^{-1}(\mathcal{B}^\times) \supseteq \mathcal{A}^\times.$$
(3) Singular cone with respect to φ:
$$\mathrm{Sing}_\varphi(\mathcal{A}) := \mathcal{A} \setminus \mathcal{A}_\varphi^\times = \varphi^{-1}(\mathrm{Sing}(\mathcal{B})) \subseteq \mathrm{Sing}(\mathcal{A}).$$
(4) Jacobson radical with respect to φ:
$$\mathfrak{J}_\varphi(\mathcal{A}) := \varphi^{-1}(\mathfrak{J}(\mathcal{B})).$$
(5) Nilcone with respect to φ:
$$\mathrm{nil}_\varphi(\mathcal{A}) := \varphi^{-1}(\mathrm{nil}(\mathcal{B})) \supseteq \mathrm{nil}(\mathcal{A}).$$
If additionally $\mathcal{A}, \mathcal{B} \in \mathsf{CAlg}_R$, we call this the nilradical with respect to φ.
(6) Singular "class" of $a \in \mathcal{A}$ with respect to φ:
$$[a]_{\mathrm{sing}} := a + \mathrm{Sing}(\mathcal{A})$$
and
$$[a]_\varphi := a + \mathrm{Sing}_\varphi(\mathcal{A}) \subseteq [a]_{\mathrm{sing}}.$$
(7) Multiplicative "class" of $a \in \mathcal{A}$ with respect to φ:
$$[a]^\times := a + \mathcal{A}^\times$$
and
$$[a]_\varphi^\times := a + \mathcal{A}_\varphi^\times \overset{\text{def}}{=} \mathcal{A} \setminus [a]_\varphi \supseteq [a]^\times.$$

Remarks:

(1) As usual, the φ-saturation $(-)^\varphi$ is a closure operator.

[14] also called *φ-saturation* of X;

(2) Observe that Definition 0.1.33 also includes the representation case as a morphism $\mathcal{A} \xrightarrow{\lambda} \mathrm{End}_R(M)$ in Alg_R.

(3) Note that in general $\mathfrak{J}_\varphi(\mathcal{A}) \not\supseteq \mathfrak{J}(\mathcal{A})$ (it would be true if for example φ is surjective since $\varphi(\mathfrak{J}(\mathcal{A})) \subseteq \mathfrak{J}(\mathcal{B})$ then).

(4) The *nilcone* (or nilpotent cone) is indeed a cone in the following *algebraic* sense: $\forall\, r \in R\ \forall\, x \in \mathrm{nil}_\varphi(\mathcal{A})\colon rx \in \mathrm{nil}_\varphi(\mathcal{A})$ as $\varphi(rx) = r\varphi(x) \in \mathrm{nil}(\mathcal{B})$.

(5) Clearly, $[0]_\varphi \overset{\mathrm{def}}{=} \mathrm{Sing}_\varphi(\mathcal{A})$, $[0]_\varphi^\times \overset{\mathrm{def}}{=} \mathcal{A}_\varphi^\times$, $[a]_{\mathrm{sing}} \overset{\mathrm{def}}{=} [a]_{\mathrm{id}_\mathcal{A}}$, and $[a]^\times \overset{\mathrm{def}}{=} [a]_{\mathrm{id}_\mathcal{A}}^\times$.

(6) In general, $[a]_{\mathrm{sing}}$ (and $[a]^\times$) are not equivalence classes in quotient R-algebras.

(7) Of course, since $\mathfrak{J}_\varphi(\mathcal{A}) \lhd \mathcal{A}$ and $\mathrm{nil}_\varphi(\mathcal{A}) \lhd \mathcal{A}$ too in the commutative case, we also have the more "traditional" cosets $[a]_{\mathfrak{J}_\varphi} := a + \mathfrak{J}_\varphi(\mathcal{A}) \in \mathcal{A}/\mathfrak{J}_\varphi(\mathcal{A})$ and $[a]_{\mathrm{nil}_\varphi} := a + \mathrm{nil}_\varphi(\mathcal{A}) \in \mathcal{A}/\mathrm{nil}_\varphi(\mathcal{A})$, but these won't suffice for our purposes later on.

Lemma 0.1.34: Let $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ be a morphism in Alg_R. Then:

(1) $\mathcal{A}_\varphi^\times$ is indeed a multiplicative submonoid of $(\mathcal{A}, \times)$. Moreover, if $\mathcal{B}$ is Dedekind-finite, then $\forall\, a_1, a_2 \in \mathcal{A}\colon a_1 a_2 \in \mathcal{A}_\varphi^\times \Leftrightarrow a_1, a_2 \in \mathcal{A}_\varphi^\times$.

(2) If $\mathcal{B}$ is Dedekind-finite, then $\mathrm{Sing}_\varphi(\mathcal{A})$ is indeed a (two-sided algebraic) cone, i.e. $\forall\, a \in \mathcal{A}\ \forall\, x \in \mathrm{Sing}_\varphi(\mathcal{A})\colon ax, xa \in \mathrm{Sing}_\varphi(\mathcal{A})$.

Proof: To (1): We have $a_1, a_2 \in \mathcal{A}_\varphi^\times \overset{\mathrm{def}}{\Leftrightarrow} \varphi(a_1), \varphi(a_2) \in \mathcal{B}^\times \Rightarrow \varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2) \in \mathcal{B}^\times \Rightarrow a_1 a_2 \in \mathcal{A}_\varphi^\times$. If $\mathcal{B}$ is Dedekind-finite, then $a_1 a_2 \in \mathcal{A}_\varphi^\times \Leftrightarrow \varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2) \in \mathcal{B}^\times \Leftrightarrow \varphi(a_1), \varphi(a_2) \in \mathcal{B}^\times$, i.e. $a_1, a_2 \in \mathcal{A}_\varphi^\times$.

To (2): If there exist $a \in \mathcal{A}$ and $x \in \mathrm{Sing}_\varphi(\mathcal{A})$ such that $ax \notin \mathrm{Sing}_\varphi(\mathcal{A})$, i.e. $ax \in \mathcal{A}_\varphi^\times$, then $a, x \in \mathcal{A}_\varphi^\times$ by (1), a contradiction to $x \in \mathrm{Sing}_\varphi(\mathcal{A})$. Analogously for xa. □

0.1.7. Product Notions

Definition 0.1.35 (Index-wise Projection Closure)**:** Let $\mathcal{A} := \prod_{j=1}^n \mathcal{A}_j$ in Alg_R, $I \subseteq [n]$, and $X \subseteq \mathcal{A}$. Then, as a continuation of Definition 0.1.6, we put

$$\widetilde{X} := \widetilde{X}_{(I)} := \bigcap_{j \in I} \mathrm{pr}_j^{-1}(X_j) \overset{\mathrm{def}}{=} \bigcap_{j \in I} X^{\mathrm{pr}_j} = \mathrm{pr}_I^{-1}(X_I) = \prod_{j=1}^n \begin{cases} X_j, \text{if } j \in I \\ \mathcal{A}_j, \text{otherwise} \end{cases} \overset{\mathrm{def}}{=} X_{(I)}$$

(See also Definition 0.1.6.)

Remarks:

(1) $\widetilde{X}$ manifestly depends on the choice of a product decomposition for $\mathcal{A}$, i.e. it is in general not invariant under permutation of the factors in any reasonable sense, even though the latter would produce an algebra of the same isomorphism type.

(2) $\widetilde{(-)}$ is indeed a closure operator, i.e. $X \subseteq \widetilde{X}$, $X \subseteq Y \Rightarrow \widetilde{X} \subseteq \widetilde{Y}$, and $\widetilde{\widetilde{X}} = \widetilde{X}$.

(3) There is an additional notion of a *uniform projection closure* defined by the saturation $X^{\mathrm{pr}_I} \overset{\mathrm{def}}{=} \mathrm{pr}_I^{-1}(\mathrm{pr}_I(X)) \overset{\mathrm{def}}{=} X + \mathrm{Ker}(\mathrm{pr}_I)$. Clearly, we only have $X^{\mathrm{pr}_I} \subseteq \widetilde{X}$ since $\mathrm{pr}_I(X) \subseteq X_I$ only. In the case of the simple projections we won't have any need for it, but we remark that nevertheless an analogue of it will appear in Section 0.1.9 along with an analogue of the just defined index-wise[15] projection closure.

(4) Complements: since $X \subseteq \widetilde{X}$, we have $\widetilde{X}^c \subseteq X^c \subseteq \widetilde{X^c}$. Moreover, continuing the previous point, there are in principle also corresponding notions of an *uniform projection complement* $(X^{\mathrm{pr}_I})^c$ and an *index-wise projection complement* $\bigcap_{j \in I} \mathrm{pr}_j^{-1}(X_j)^c$ in parallel with the upcoming Definition 0.1.42. However, in the present work we won't have any use for them either,

[15] for the lack of a better term; suggestions are most welcome, though;

so we will refrain here from formally introducing them since we are already boldly riding the limits of reasonable notation.

(5) If $I_1 \subseteq I_2$, then clearly $\widetilde{X}_{(I_1)} \supseteq \widetilde{X}_{(I_2)}$.

(6) If X is open (closed) in some *product* topology on $\mathcal{A}$, then so is $\widetilde{X}$ since the projections pr_j, $j \in J$, would be open and continuous.

(7) Projection closure of a point: if $a = (a_j)_{1 \le j \le n} \in \mathcal{A}$, then:

$$\widetilde{\{a\}} \overset{\text{def}}{=} \prod_{j=1}^{n} \begin{cases} \{a_j\}, \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise.} \end{cases}$$

In view of Section 0.1.5 and Definition 0.1.32 we also introduce the following short notations:

Definition 0.1.36 (Projection closure with respect to a morphism or a representation)**:** Given $\mathcal{A} := \prod_{j=1}^{n} \mathcal{A}_j$ in Alg_R, a morphism φ with corresponding index subset $I_\varphi \subseteq [n]$ or an (anti-)representation λ with corresponding index subset $I_\lambda \subseteq [n]$, and $X \subseteq \mathcal{A}$, we put

$$\widetilde{X} := \widetilde{X}_\varphi := \widetilde{X}_{(I_\varphi)} \text{ or } \widetilde{X} := \widetilde{X}_\lambda := \widetilde{X}_{(I_\lambda)},$$

respectively.

Lemma-Definition 0.1.37 (Product-relative[16] units, radicals, classes)**:** Let $\mathcal{A} := \prod_{j=1}^{n} \mathcal{A}_j$ in Alg_R and $I \subseteq [n]$.

(1) Multiplicative submonoid with respect to I:

$$\mathcal{A}^\times_{(I)} := \mathrm{U}_{(I)}(\mathcal{A}) := \mathrm{U}_{\mathrm{pr}_I}(\mathcal{A}) \overset{\text{def}}{=} \mathrm{pr}_I^{-1}\left(\mathrm{U}\Big(\prod_{j \in I} \mathcal{A}_j\Big)\right) = \mathrm{pr}_I^{-1}\left(\prod_{j \in I} \mathrm{U}(\mathcal{A}_j)\right) =$$

$$= \prod_{j=1}^{n} \begin{cases} \mathcal{A}_j^\times, \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise} \end{cases} = \widetilde{\mathcal{A}^\times} \supseteq \mathcal{A}^\times.$$

(2) Singular cone with respect to I:

$$\mathrm{Sing}_{(I)}(\mathcal{A}) := \mathrm{Sing}_{\mathrm{pr}_I}(\mathcal{A}) \overset{\text{def}}{=} \mathcal{A} \setminus \mathcal{A}_I^\times = \bigcup_{i \in I} \prod_{j=1}^{n} \begin{cases} \mathrm{Sing}(\mathcal{A}_j), \text{ if } j = i \\ \mathcal{A}_j, \text{ otherwise} \end{cases} \subseteq \mathrm{Sing}(\mathcal{A}) \subseteq \widetilde{\mathrm{Sing}(\mathcal{A})}.$$

(3) Jacobson radical with respect to I:

$$\mathfrak{J}_{(I)}(\mathcal{A}) := \mathfrak{J}_{\mathrm{pr}_I}(\mathcal{A}) \overset{\text{def}}{=} \mathrm{pr}_I^{-1}\left(\mathfrak{J}\Big(\prod_{j \in I} \mathcal{A}_j\Big)\right) = \mathrm{pr}_I^{-1}\left(\prod_{j \in I} \mathfrak{J}(\mathcal{A}_j)\right) =$$

$$= \prod_{j=1}^{n} \begin{cases} \mathfrak{J}(\mathcal{A}_j), \text{if } j \in I \\ \mathcal{A}_j, \text{otherwise} \end{cases} = \widetilde{\mathfrak{J}(\mathcal{A})} \supseteq \mathfrak{J}(\mathcal{A}).$$

(4) Nilcone with respect to I:

$$\mathrm{nil}_{(I)}(\mathcal{A}) := \mathrm{nil}_{\mathrm{pr}_I}(\mathcal{A}) \overset{\text{def}}{=} \mathrm{pr}_I^{-1}\left(\mathrm{nil}\left(\prod_{j \in I} \mathcal{A}_j\right)\right) = \mathrm{pr}_I^{-1}\left(\prod_{j \in I} \mathrm{nil}(\mathcal{A}_j)\right) =$$

$$= \prod_{j=1}^{n} \begin{cases} \mathrm{nil}(\mathcal{A}_j), \text{if } j \in I \\ \mathcal{A}_j, \text{otherwise} \end{cases} = \widetilde{\mathrm{nil}(\mathcal{A})} \supseteq \mathrm{nil}(\mathcal{A}).$$

If additionally $\mathcal{A} \in \mathsf{CAlg}_R$, we also call this the nilradical with respect to I.

(5) Singular "class" of $a \in \mathcal{A}$ with respect to I:

$$[a]_{(I)} := a + \mathrm{Sing}_{(I)}(\mathcal{A}) \subseteq [a]_{\mathrm{sing}}.$$

[16] here is a mnemonic: pr_I stands both for projection and ***product-relative***;

(6) Multiplicative "class" of $a = (a_1, \dots, a_n) \in \mathcal{A}$ with respect to I:

$$[a]^\times_{(I)} \coloneqq a + \mathcal{A}^\times_{(I)} \overset{\text{def}}{=} \prod_{j=1}^n \begin{cases} a_j + \mathcal{A}_j^\times, \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise} \end{cases} = \widetilde{[a]^\times} \supseteq [a]^\times.$$

Proof: To (1), (3), (4): Let $\mathrm{F} \in \{\mathrm{U}, \mathfrak{J}, \mathrm{nil}\}$. Since $\mathrm{F}(-)$ respects products, we have $\forall\, 1 \leq j \leq n$: $\mathrm{pr}_j(\mathrm{F}(\mathcal{A})) = \mathrm{F}(\mathcal{A}_j)$, hence $\mathrm{F}_{(I)}(\mathcal{A}) = \widetilde{\mathrm{F}(\mathcal{A})}$.

To (6): We have $\forall\, 1 \leq j \leq n$: $\mathrm{pr}_j([a]^\times) = \mathrm{pr}_j(a) + \mathrm{pr}_j(\mathcal{A}^\times) = a_j + \mathcal{A}_j^\times$, hence $[a]^\times_{(I)} = \widetilde{[a]^\times}$ as well.

The rest is clear from Definition 0.1.33. $\square$

Remarks:

(1) The two key properties in Lemma-Definition 0.1.37 are that pr_I is surjective and that $\mathrm{U}(-)$, $\mathrm{nil}(-)$, and $\mathfrak{J}(-)$ all respect (in fact arbitrary) direct products (see e.g. [Lam01, Ex. 4.12B on p. 65] for the latter).

(2) Since $\mathrm{pr}_{[n]} \overset{\text{def}}{=} \mathrm{id}_\mathcal{A}$, we have $\mathcal{A}^\times \overset{\text{def}}{=} \mathcal{A}^\times_{([n])}$, $\mathrm{Sing}(\mathcal{A}) \overset{\text{def}}{=} \mathrm{Sing}_{([n])}(\mathcal{A})$, $\mathfrak{J}(\mathcal{A}) \overset{\text{def}}{=} \mathfrak{J}_{([n])}(\mathcal{A})$, $\mathrm{nil}(\mathcal{A}) \overset{\text{def}}{=} \mathrm{nil}_{([n])}(\mathcal{A})$, $[a]_{([n])} \overset{\text{def}}{=} [a]_{\mathrm{sing}}$, and $[a]^\times_{([n])} \overset{\text{def}}{=} [a]^\times$.

(3) Note that at this point we establish no connection between $\mathcal{A}^\times_\varphi$ vs. $\mathcal{A}^\times_{(I_\varphi)}$, or $\mathrm{Sing}_\varphi(\mathcal{A})$ vs. $\mathrm{Sing}_{(I_\varphi)}(\mathcal{A})$, or $\mathfrak{J}_\varphi(\mathcal{A})$ vs. $\mathfrak{J}_{(I_\varphi)}(\mathcal{A})$, or $\mathrm{nil}_\varphi(\mathcal{A})$ vs. $\mathrm{nil}_{(I_\varphi)}(\mathcal{A})$, or $[a]_\varphi$ vs. $[a]_{(I_\varphi)}$, or $[a]^\times_\varphi$ vs. $[a]^\times_{(I_\varphi)}$, respectively, with I_φ as in Definition 0.1.32. Likewise for representations (λ, I_λ) or anti-representations (ρ, I_ρ). This will happen in Section 0.1.10 and Section 0.1.12.

(4) Furthermore, $\mathrm{Sing}_{(I)}(\mathcal{A})$ should not be confused with the expression

$$\prod_{j=1}^n \begin{cases} \mathrm{Sing}(\mathcal{A}_j), \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise} \end{cases} = \bigcap_{j \in I} \mathrm{pr}_j^{-1}(\mathrm{pr}_j(\mathcal{A}^\times)^c),$$

which might be called a *projection complement* of $\mathcal{A}^\times$ with respect to $I \subseteq [n]$ (see *spectral complement* in Definition 0.1.42 below).

(5) Since in general we only have $\mathrm{nil}(\mathcal{A}_j) \subseteq \mathrm{Sing}(\mathcal{A}_j)$, $1 \leq j \leq n$, this also extends to $\mathrm{nil}_{(I)}(\mathcal{A}) \subseteq \mathrm{Sing}_{(I)}(\mathcal{A})$ only, or equivalently, $\mathcal{A}^\times_{(I)} \subseteq \mathrm{nil}_{(I)}(\mathcal{A})^c$ only. This obviously disappears for products of local rings with nilpotent maximal ideals.

Finally, we need a small additional observation about $\mathfrak{J}(-)$ and *subdirect products*.

Lemma 0.1.38 ($\mathfrak{J}(-)$ for subdirect products)**:** Let $\varnothing \neq I$ be an arbitrary index set and let $\mathcal{A} \overset{\iota}{\hookrightarrow} \mathcal{B} \coloneqq \prod_{i \in I} \mathcal{B}_i$ be a subdirect product in Alg_R. Then $\mathfrak{J}(\mathcal{A}) \subseteq \mathfrak{J}(\mathcal{B}) = \prod_{i \in I} \mathfrak{J}(\mathcal{B}_i)$.

Proof: Since π_i is surjective, we have $\pi_i(\mathfrak{J}(\mathcal{A})) \subseteq \mathfrak{J}(\mathcal{B}_i)$, i.e. $\mathfrak{J}(\mathcal{A}) \subseteq \pi_i^{-1}(\mathfrak{J}(\mathcal{B}_i))$, $i \in I$. Thus, $\mathfrak{J}(\mathcal{A}) \subseteq \bigcap_{i \in I} \pi_i^{-1}(\mathfrak{J}(\mathcal{B}_i)) = \bigcap_{i \in I} (\mathrm{pr}_i^\mathcal{B} \circ \iota)^{-1}(\mathfrak{J}(\mathcal{B}_i)) = \bigcap_{i \in I} \iota^{-1}\big((\mathrm{pr}_i^\mathcal{B})^{-1}(\mathfrak{J}(\mathcal{B}_i))\big) = \iota^{-1}\big(\bigcap_{i \in I} (\mathrm{pr}_i^\mathcal{B})^{-1}(\mathfrak{J}(\mathcal{B}_i))\big) = \iota^{-1}(\mathfrak{J}(\mathcal{B}))$ since $\mathfrak{J}(\mathcal{B}) = \mathfrak{J}(\prod_{i \in I} \mathcal{B}_i) = \prod_{i \in I} \mathfrak{J}(\mathcal{B}_i)$, that is, $\iota(\mathfrak{J}(\mathcal{A})) \subseteq \mathfrak{J}(\mathcal{B})$ as desired. $\square$

Remark: Thus for *subdirect products* $\mathfrak{J}(-)$ behaves in the same way as $\mathrm{U}(-)$ and $\mathrm{nil}(-)$ do with respect to inclusions into products.

0.1.8. Unital Saturation

Definition 0.1.39: Let $R \in \mathsf{CRing}$, $\mathcal{A} \in \mathsf{Alg}_R$, and $X \subseteq \mathcal{A}$ with $X \cap \mathcal{A}^\times \neq \varnothing$.

(1) Unital closure of X in $\mathcal{A}$:

$$\bar{\mathrm{U}}_\mathcal{A}(X) \coloneqq \langle X \cap \mathcal{A}^\times \rangle \leq \mathcal{A}^\times \tag{0.1.49}$$

will denote smallest subgroup of $\mathcal{A}^\times$ containing all elements of X invertible in $\mathcal{A}^\times$.

(2) X is called unitally closed (or unitally saturated) in $\mathcal{A}$ if $X \supseteq \bar{\mathrm{U}}_\mathcal{A}(X)$.

If $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ is a morphism in Alg_R, then we only have $\varphi(\mathcal{A}^\times) \subseteq \mathcal{B}^\times$ and $\varphi^{-1}(\mathcal{B}^\times) \supseteq \mathcal{A}^\times$. Moreover, $\mathrm{U}(-)$ is a product-preserving functor $\mathsf{Alg}_R \to \mathsf{Grp}$, inducing $\varphi^\times := \mathrm{U}(\varphi)\colon \mathcal{A}^\times \to \mathcal{B}^\times$, but in general $\varphi^{-1}(\mathcal{B}^\times) \neq (\varphi^\times)^{-1}(\mathcal{B}^\times)$. In determining $\varphi^{-1}(\mathcal{B}^\times)$, it is sometimes useful to work with $\bar{\mathrm{U}}_{\mathcal{B}}(\varphi(\mathcal{A})) \leq \mathcal{B}^\times$ instead since $\varphi^{-1}(\mathcal{B}^\times) = \varphi^{-1}(\varphi(\mathcal{A}) \cap \mathcal{B}^\times) = \varphi^{-1}(\bar{\mathrm{U}}_{\mathcal{B}}(\varphi(\mathcal{A})))$. We remark that *unital saturatedness* is a transitive property under appropriate inclusions. We list now some examples central to our applications later:

Lemma 0.1.40: Let $\mathcal{A} \in \mathsf{Alg}_R$ and ${}_{\mathcal{A}}M = \bigoplus_{k=1}^m {}_{\mathcal{A}}M_k$ in $\mathsf{LMod}(\mathcal{A})$ via $\lambda\colon \mathcal{A} \to \mathrm{End}_R(M)$ and $\lambda_k\colon \mathcal{A} \to \mathrm{End}_R(M_k)$, $1 \leq k \leq m$. Then:

(1) The R-subalgebra $\prod_{k=1}^m \mathrm{End}_R(M_k)$ is unitally saturated in $\mathrm{End}_R(M)$, that is:
$$\mathrm{GL}_R(M) \cap \prod_{k=1}^m \mathrm{End}_R(M_k) = \prod_{k=1}^m \mathrm{GL}_R(M_k).$$

(2) The R-subalgebra $\mathrm{End}_{\mathcal{A}}(M)$ is unitally saturated in $\mathrm{End}_R(M)$, that is:
$$\mathrm{End}_{\mathcal{A}}(M) \cap \mathrm{GL}_R(M) = \mathrm{GL}_{\mathcal{A}}(M).$$
It follows that the R-subalgebra $\prod_{k=1}^m \mathrm{End}_{\mathcal{A}}(M_k)$ is unitally saturated in $\mathrm{End}_R(M)$.

(3) If $\mathcal{A} \in \mathsf{CAlg}_R$, then
$$\mathcal{A}_\lambda^\times = \lambda^{-1}\left(\prod_{k=1}^m \mathrm{GL}_{\mathcal{A}}(M_k)\right) = \bigcap_{k=1}^m \mathcal{A}_{\lambda_k}^\times \tag{0.1.50}$$

Analogously for $\mathsf{RMod}(\mathcal{A})$ and $\mathsf{BMod}(\mathcal{A})$.

Proof: To (1): By induction it suffices to consider only the case $M = M_1 \oplus_{\mathcal{A}} M_2 = M_1 \oplus_R M_2$. Writing explicitly
$$\mathrm{End}_R(M) = \mathrm{End}_R(M_1 \oplus M_2) = \begin{pmatrix} \mathrm{End}_R(M_1) & \mathrm{Hom}_R(M_2, M_1) \\ \mathrm{Hom}_R(M_1, M_2) & \mathrm{End}_R(M_2) \end{pmatrix}$$
one checks that any block-diagonal element invertible in $\mathrm{GL}_R(M)$ must already be an element of $\mathrm{GL}_R(M_1) \times \mathrm{GL}_R(M_2)$.

To (2): Let $\Phi \in \mathrm{End}_{\mathcal{A}}(M) \cap \mathrm{GL}_R(M)$ with $\Psi := \Phi^{-1} \in \mathrm{GL}_R(M)$. We only need to show that Ψ is $\mathcal{A}$-linear. But since $\Phi\Psi = \mathrm{id}_M$, we have $\forall\, a \in \mathcal{A}\ \forall\, x \in M\colon \Phi\Psi a x = a x = a\Phi\Psi x = \Phi a \Psi x$, hence $\forall\, a \in \mathcal{A}\ \forall\, x \in M\colon \Psi a x = a\Psi x$ by taking Φ^{-1} on the left.

To (3): follows from (2) and Eq. (0.1.30). For example,
$$\begin{aligned}\mathcal{A}_\lambda^\times &\overset{\text{def}}{=} \lambda^{-1}(\mathrm{GL}_{\mathbb{K}}(M)) = \lambda^{-1}(\lambda(\mathcal{A}) \cap \mathrm{GL}_{\mathbb{K}}(M)) = \lambda^{-1}\left(\lambda(\mathcal{A}) \cap \prod_{k=1}^m \mathrm{GL}_{\mathbb{K}}(M_k)\right) = \\ &= \{a \in \mathcal{A} \mid \forall\, 1 \leq k \leq m\colon \lambda_k(a) \in \mathrm{GL}_{\mathbb{K}}(M_k)\} = \bigcap_{k=1}^m \lambda_k^{-1}(\mathrm{GL}_{\mathbb{K}}(M_k)) \overset{\text{def}}{=} \bigcap_{k=1}^m \mathcal{A}_{\lambda_k}^\times\end{aligned}$$
as desired. □

Lemma 0.1.41: Let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a morphism in Alg_R. If $(\mathcal{A}, \mathfrak{m}) \in \mathsf{Alg}_R$ is local with $\mathfrak{m}$ nil, then $\varphi(\mathcal{A})$ is unitally saturated in $\mathcal{B}$.

Proof: It follows that $\mathcal{A}' := \varphi(\mathcal{A})$ too is a local ring with *nil* maximal ideal, say $\mathcal{A}' = (\mathcal{A}', \mathfrak{m}')$, hence $(\mathcal{A} \setminus \mathcal{A}'^\times) \cap \mathcal{B}^\times = \mathfrak{m}' \cap \mathcal{B}^\times = \varnothing$. □

0.1.9. Formal Characters & Spectral Closures

Fix $R \in \mathsf{CRing}$, $\mathcal{A} \in \mathsf{CAlg}_R$, and consider $\mathrm{Spm}(\mathcal{A}) \overset{\text{def}}{=} \{\mathfrak{M} : \mathfrak{M} \lhd \mathcal{A} \text{ maximal}\}$. Let $\mathcal{I} \subseteq \mathrm{Spm}(\mathcal{A})$ be a collection of maximal ideals also thought of as an *indexing subset*, that is, we will use $j \in \mathcal{I}$ and $\mathfrak{M}_j \in \mathcal{I}$ interchangeably to mean the same thing. We put $\kappa_j \coloneqq \mathcal{A}/\mathfrak{M}_j$ for the corresponding residue fields, $j \in \mathcal{I}$. Borrowing from the theory of topological algebras, we will call the residue maps $\chi_j \colon \mathcal{A} \twoheadrightarrow \kappa_j$, $j \in \mathcal{I}$, also (formal) characters of $\mathcal{A}$ and we will denote by $\mathcal{X}_{\mathcal{I}}(\mathcal{A}) \coloneqq \{\chi_j : j \in \mathcal{I}\}$ the space[17] of (formal) characters of $\mathcal{A}$ with respect to $\mathcal{I}$, where we also put $\mathcal{X}(\mathcal{A}) \coloneqq \mathcal{X}_{\mathrm{Spm}(\mathcal{A})}(\mathcal{A})$ for the whole (formal) character space of $\mathcal{A}$. In other words, we have an obvious correspondence $\mathrm{Spm}(\mathcal{A}) \leftrightarrow \mathcal{X}(\mathcal{A})$. Following Definition 0.1.6, we put $\kappa_{\mathcal{I}} \overset{\text{def}}{=} \prod_{j\in\mathcal{I}} \kappa_j \in \mathsf{CAlg}_R$ for short. Furthermore, $\varnothing \neq \mathcal{I} \subseteq \mathrm{Spm}(\mathcal{A})$ also defines a *multi-character* or a *poly-character*

$$\begin{aligned} \chi_{\mathcal{I}} \coloneqq (\mathop{\times}_{j\in\mathcal{I}} \chi_j) \circ \Delta_{\mathcal{I}} \colon \mathcal{A} \hookrightarrow \mathcal{A}^{\mathcal{I}} &\to \kappa_{\mathcal{I}} \\ a &\mapsto (\chi_j(a))_{j\in\mathcal{I}} \end{aligned} \tag{0.1.51}$$

which is clearly an R-algebra homomorphism, and we have $\mathrm{Ker}(\chi_{\mathcal{I}}) = \bigcap \mathcal{I} \overset{\text{def}}{=} \bigcap_{j\in\mathcal{I}} \mathfrak{M}_j$. We remark that even for $\mathcal{I}$ only countably infinite, $\mathrm{Spm}(\kappa_{\mathcal{I}})$ can be quite complicated[18], but this will be of no consequence to us at present.

Definition 0.1.42 (Spectral Closures & Complements)**:** Let $\mathcal{A} \in \mathsf{CAlg}_R$, $X \subseteq \mathcal{A}$ arbitrary, and fix $\mathcal{I} \neq \varnothing$.

(1) Uniform $\mathcal{I}$-spectral closure of X:
$$\widehat{X} \coloneqq \widehat{X}_{\mathcal{I}} \coloneqq X^{\chi_{\mathcal{I}}} = X + \mathrm{Ker}(\chi_{\mathcal{I}}).$$

(2) Uniform $\mathcal{I}$-spectral complement of X:
$$\breve{X} \coloneqq \breve{X}_{\mathcal{I}} \coloneqq \chi_{\mathcal{I}}^{-1}(\chi_{\mathcal{I}}(X)^c) = \chi_{\mathcal{I}}^{-1}(\chi_{\mathcal{I}}(X))^c \overset{\text{def}}{=} \widehat{X}^c = (X + \mathrm{Ker}(\chi_{\mathcal{I}}))^c.$$

(3) Spectral projections of X:
$$X_j^{\mathrm{sp}} \coloneqq \chi_j(X),\ j \in \mathcal{I}.$$

(4) Full $\mathcal{I}$-spectral projection of X:
$$X^{\mathrm{sp}} \coloneqq X_{\mathcal{I}}^{\mathrm{sp}} \coloneqq \prod_{j\in\mathcal{I}} X_j^{\mathrm{sp}}$$

(5) Individual spectral closures of X:
$$\hat{X}_j \coloneqq \chi_j^{-1}(X_j^{\mathrm{sp}}) = X + \mathfrak{M}_j,\ j \in \mathcal{I}.$$

(6) Point-wise $\mathcal{I}$-spectral closure of X:
$$\hat{X} \coloneqq \hat{X}_{\mathcal{I}} \coloneqq \bigcap_{j\in\mathcal{I}} X^{\chi_j} \overset{\text{def}}{=} \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(X_j^{\mathrm{sp}}) \overset{\text{def}}{=} \bigcap_{j\in\mathcal{I}} \hat{X}_j = \bigcap_{j\in\mathcal{I}} (X + \mathfrak{M}_j) = \chi_{\mathcal{I}}^{-1}(X^{\mathrm{sp}}).$$

(7) Point-wise $\mathcal{I}$-spectral complement of X:
$$\check{X} \coloneqq \check{X}_{\mathcal{I}} \coloneqq \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(\chi_j(X)^c) \overset{\text{def}}{=} \bigcap_{j\in\mathcal{I}} \chi_j^{-1}((X_j^{\mathrm{sp}})^c) = \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(X_j^{\mathrm{sp}})^c \overset{\text{def}}{=} \bigcap_{j\in\mathcal{I}} \hat{X}_j^c = \bigcap_{j\in\mathcal{I}} (X + \mathfrak{M}_j)^c.$$

Remarks:

(1) One can even go as far as repeating the above definitions with $\mathrm{Spec}(\mathcal{A})$ in place of $\mathrm{Spm}(\mathcal{A})$ since we still get a reduced codomain, but this strays even further from our goals later on.

[17] calling it a space is perhaps a bit of an overstatement since we won't be needing any additional structure (e.g. topology) on it, so it's more of a *collection* at this point rather than an actual space;

[18] for a concrete example, $\mathrm{Spm}(\mathbb{R}^{\mathbb{N}})$ is in one-to-one correspondence with the space of ultrafilters on $\mathbb{N}$, i.e. the Stone–Čech compactification $\beta\mathbb{N}$;

(2) We have $a \in \widehat{X} \Leftrightarrow \exists\, x \in X\ \forall\, j \in \mathcal{I}\colon \chi_j(a) = \chi_j(x)$, hence the name "uniform".
(3) We have $a \in \hat{X} \Leftrightarrow \forall\, j \in \mathcal{I}\ \exists\, x_j \in X\colon \chi_j(a) = \chi_j(x_j)$, hence the name "point-wise" (in the sense of $\mathrm{Spm}(R)$).
(4) As expected, $\widehat{\varnothing} = \varnothing$, $\widecheck{\varnothing} = \mathcal{A}$, $\hat{\varnothing} = \varnothing$, $\check{\varnothing} = \mathcal{A}$.

The following is mostly set-theoretical non-sense, that, however, quickly piles up and can be easily confusable, so for bookkeeping purposes we create a careful record of it here, only to go on with our lives and forget about it.

Lemma 0.1.43 (Basic Properties)**:** Let $\mathcal{A} \in \mathsf{CAlg}_R$ and let $\varnothing \neq \mathcal{I} \subseteq \mathrm{Spm}(\mathcal{A})$ be an indexing subset.

(1) Basic properties of $\widehat{(-)}$:
   (i) $\widehat{(-)}$ is a closure operator.
   (ii) If $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathrm{Spm}(\mathcal{A})$, then $\widehat{X}_{\mathcal{I}} \supseteq \widehat{X}_{\mathcal{J}}$.
   (iii) $\forall\, j \in \mathcal{I}\colon \chi_j(\widehat{X}) = X_j^{\mathrm{sp}}$, and $\widehat{X}$ is the biggest subset of $\mathcal{A}$ satisfying $\chi_{\mathcal{I}}(\widehat{X}) = \chi_{\mathcal{I}}(X)$.
(2) Basic properties of $\hat{(-)}$:
   (i) $\hat{(-)}$ is a closure operator.
   (ii) If $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathrm{Spm}(\mathcal{A})$, then $\hat{X}_{\mathcal{I}} \supseteq \hat{X}_{\mathcal{J}}$.
   (iii) $\forall\, j \in \mathcal{I}\colon \chi_j(\hat{X}) = X_j^{\mathrm{sp}}$, and $\hat{X}$ is the biggest subset of $\mathcal{A}$ satisfying $\forall\, j \in \mathcal{I}\colon \chi_j(\hat{X}) \subseteq X_j^{\mathrm{sp}}$.
(3) Relation between $\widehat{X}$ and $\hat{X}$: $\widehat{X} \subseteq \hat{X}$, hence $\hat{\widehat{X}} = \widehat{\hat{X}} = \hat{X}$. In particular, the ideal-wise spectral closure is uniformly spectrally closed.
(4) Basic properties of $\widecheck{X}$:
   (i) If $X \subseteq Y$, then $\widecheck{X} \supseteq \widecheck{Y}$.
   (ii) If $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathrm{Spm}(\mathcal{A})$, then $\widecheck{X}_{\mathcal{I}} \subseteq \widecheck{X}_{\mathcal{J}}$.
   (iii) $\forall\, j \in \mathcal{I}\colon (X_j^{\mathrm{sp}})^c \overset{\mathrm{def}}{=} \chi_j(X)^c \subseteq \chi_j(\widecheck{X}) \subseteq \chi_j(X^c)$.
(5) Basic properties of $\check{X}$:
   (i) If $X \subseteq Y$, then $\check{X} \supseteq \check{Y}$.
   (ii) If $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathrm{Spm}(\mathcal{A})$, then $\check{X}_{\mathcal{I}} \supseteq \check{X}_{\mathcal{J}}$.
   (iii) $\check{X}$ is the biggest subset of $\mathcal{A}$ satisfying $\forall\, j \in \mathcal{I}\colon \chi_j(\check{X}) \subseteq (X_j^{\mathrm{sp}})^c$.
(6) Comparison of complements:
   (i) We have:

$$\check{\hat{X}} = \hat{\check{X}} = \check{X} = \widehat{\widecheck{X}} = \widecheck{\widehat{X}} \subseteq \hat{X}^c = \widecheck{\hat{X}} \subseteq \widecheck{X} \overset{\mathrm{def}}{=} \widehat{X}^c = \widehat{\widecheck{X}} = \widecheck{\widehat{X}} \subseteq X^c \subseteq \widehat{X^c} \subseteq \widehat{X^c} \tag{0.1.52}$$

   and

$$\ldots \subseteq \widecheck{X} \subseteq \hat{\widecheck{X}} \subseteq \widehat{X^c}. \tag{0.1.53}$$

   In particular:
   - $\check{X}$ is already both ideal-wise and uniformly spectrally closed;
   - $\widecheck{X}$ is already uniformly spectrally closed;
   - $\widecheck{(-)} = (-)^c$ on ideal-wise spectrally closed subsets.

   (ii) If $\chi_{\mathcal{I}}$ is injective, then:

$$\ldots \subseteq \widecheck{X} \overset{\mathrm{def}}{=} \widehat{X}^c = X^c = \widehat{X^c} \subseteq \hat{\widecheck{X}} = \widehat{X^c}. \tag{0.1.54}$$

(7) Comparison of double complements:
   (i) We have:

$$\widecheck{X}^c \subseteq \widecheck{X^c} \subseteq X \subseteq \widecheck{\widecheck{X}} = \widehat{X} \subseteq \hat{X} \subseteq \widecheck{\widecheck{X}} \subseteq \check{X}^c = \widecheck{\check{X}} \tag{0.1.55}$$

and

$$\widecheck{X}^c \subseteq \check{\breve{X}} \subseteq \breve{\breve{X}} = \widehat{X} \subseteq \dots \tag{0.1.56}$$

(ii) If $\chi_{\mathcal{I}}$ is injective, then:

$$\widecheck{X}^c = \check{\breve{X}} \subseteq \breve{X^c} = X = \breve{X}^c \stackrel{\text{def}}{=} \widehat{X} \subseteq \dots \tag{0.1.57}$$

Proof: To (1):

(i) Since $\chi_{\mathcal{I}}|X \twoheadrightarrow \chi_{\mathcal{I}}(X)$ is surjective (even though $\chi_{\mathcal{I}}$ itself need not to), we get $X \subseteq \chi_{\mathcal{I}}^{-1}(\chi_{\mathcal{I}}(X)) \stackrel{\text{def}}{=} \widehat{X}$. Furthermore, $X \subseteq Y \Rightarrow \chi_{\mathcal{I}}(X) \subseteq \chi_{\mathcal{I}}(Y) \Rightarrow \chi_{\mathcal{I}}^{-1}(\chi_{\mathcal{I}}(X)) \subseteq \chi_{\mathcal{I}}^{-1}(\chi_{\mathcal{I}}(Y)) \stackrel{\text{def}}{\Leftrightarrow} \widehat{X} \subseteq \widehat{Y}$. Finally, $\widehat{\widehat{X}} \stackrel{\text{def}}{=} \chi_{\mathcal{I}}^{-1}(\chi_{\mathcal{I}}(\chi_{\mathcal{I}}^{-1}(\chi_{\mathcal{I}}(X)))) = \chi_{\mathcal{I}}^{-1}(\chi_{\mathcal{I}}(X)) \stackrel{\text{def}}{=} \widehat{X}$.

(ii) This is clear: $a \in \widehat{X}_{\mathcal{J}} \stackrel{\text{def}}{\Leftrightarrow} \chi_{\mathcal{J}}(a) \in \chi_{\mathcal{J}}(X) \Leftrightarrow \exists\, x \in X\ \forall\, j \in \mathcal{J}\colon \chi_j(a) = \chi_j(x)$, in particular $\exists\, x \in X\ \forall\, j \in \mathcal{I} \subseteq \mathcal{J}\colon \chi_j(a) = \chi_j(x)$, i.e. $a \in \widehat{X}_{\mathcal{I}}$.

(iii) By (i) we have $X \subseteq \widehat{X}$, hence $\forall\, j \in \mathcal{I}\colon \chi_j(\widehat{X}) \supseteq \chi_j(X) \stackrel{\text{def}}{=} X_j^{\text{sp}}$. We thus need to show that $\forall\, j \in \mathcal{I}\colon \chi_j(\widehat{X}) \subseteq X_j^{\text{sp}}$. Fix $j \in \mathcal{I}$ and let $\bar{a}_j \in \chi_j(\widehat{X})$, i.e. $\exists\, a_j \in \widehat{X} \subseteq \mathcal{A}\colon \bar{a}_j = \chi_j(a_j)$. But $a_j \in \widehat{X} \stackrel{\text{def}}{=} \chi_{\mathcal{I}}^{-1}(\chi_{\mathcal{I}}(X))$ just means that $\exists\, x_j \in X\ \forall\, k \in \mathcal{I}\colon \chi_k(a_j) = \chi_k(x_j)$, in particular $\exists\, x_j \in X\colon \chi_j(a_j) = \chi_j(x_j)$, hence $\exists\, x_j \in X\colon \bar{a}_j = \chi_j(x_j)$, i.e. $\bar{a}_j \in \chi_j(X) \stackrel{\text{def}}{=} X_j^{\text{sp}}$. Since $\bar{a}_j \in \chi_j(\widehat{X})$ was arbitrary, we have $\chi_j(\widehat{X}) \subseteq X_j^{\text{sp}}$ as desired. Finally, $\widehat{X} \subseteq \mathcal{A}$ is the biggest subset with the property $\chi_{\mathcal{I}}(\widehat{X}) = \chi_{\mathcal{I}}(X)$ by definition.

To (2):

(i) Since $\chi_j|X \twoheadrightarrow \chi_j(X)$ is surjective, we have $X \subseteq \chi_j^{-1}(\chi_j(X))$, $j \in \mathcal{I}$, hence $X \subseteq \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(\chi_j(X)) \stackrel{\text{def}}{=} \hat{X}$. Next, if $X \subseteq Y$, then $\forall\, j \in \mathcal{I}\colon \chi_j(X) \subseteq \chi_j(Y) \Rightarrow \forall\, j \in \mathcal{I}\colon \chi_j^{-1}(\chi_j(X)) \subseteq \chi_j^{-1}(\chi_j(Y)) \Rightarrow \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(\chi_j(X)) \subseteq \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(\chi_j(Y)) \stackrel{\text{def}}{\Leftrightarrow} \hat{X} \subseteq \hat{Y}$. Finally, since $\hat{X} \stackrel{\text{def}}{=} \bigcap_{j\in\mathcal{I}} \hat{X}_j$, we have $\forall\, j \in \mathcal{I}\colon \hat{X} \subseteq \hat{X}_j$, hence $\forall\, j \in \mathcal{I}\colon \chi_j(\hat{X}) \subseteq \chi_j(\hat{X}_j) \stackrel{\text{def}}{=} X_j^{\text{sp}}$ and $\forall\, j \in \mathcal{I}\colon \chi_j^{-1}(\chi_j(\hat{X})) \subseteq \chi_j^{-1}(X_j^{\text{sp}})$. Thus, $\hat{\hat{X}} \stackrel{\text{def}}{=} \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(\chi_j(\hat{X})) \subseteq \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(X_j^{\text{sp}}) \stackrel{\text{def}}{=} \hat{X}$, i.e. $\hat{\hat{X}} \subseteq \hat{X}$. On the other hand, by the previous considerations we have $X \subseteq \hat{X}$, hence $\hat{X} \subseteq \hat{\hat{X}}$, therefore $\hat{\hat{X}} = \hat{X}$.

(ii) This is is clear: $a \in \hat{X}_{\mathcal{J}} \stackrel{\text{def}}{\Leftrightarrow} \forall\, j \in \mathcal{J}\colon \chi_j(a) \in \chi_j(X)$, in particular $\forall\, j \in \mathcal{I}\colon \chi_j(a) \in \chi_j(X)$, i.e. $a \in \hat{X}_{\mathcal{I}}$.

(iii) From the proof of (i) we have $\forall\, j \in \mathcal{I}\colon \chi_j(\hat{X}) \subseteq X_j^{\text{sp}}$. Conversely, since $X \subseteq \hat{X}$ by (i), we get $\forall\, j \in \mathcal{I}\colon X_j^{\text{sp}} \stackrel{\text{def}}{=} \chi_j(X) \subseteq \chi_j(\hat{X})$. Thus, $\forall\, j \in \mathcal{I}\colon \chi_j(\hat{X}) = X_j^{\text{sp}}$. The last claim is clear from the definition: if $Y \subseteq \mathcal{A}$ has the property $\forall\, j \in \mathcal{I}\colon \chi_j(Y) \subseteq X_j^{\text{sp}}$, then $Y \subseteq \hat{X}$.

To (3): By (1)(iii) we have $\forall\, j \in \mathcal{I}\colon \chi_{\mathcal{I}}(\widehat{X}) \subseteq X_j^{\text{sp}}$, hence $\forall\, j \in \mathcal{I}\colon \widehat{X} \subseteq \chi_j^{-1}(X_j^{\text{sp}})$, therefore $\widehat{X} \subseteq \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(X_j^{\text{sp}}) \stackrel{\text{def}}{=} \hat{X}$. Furthermore, by the closure operator properties we get $X \subseteq \widehat{X} \subseteq \hat{X} \Rightarrow \hat{X} \subseteq \hat{\widehat{X}} \subseteq \hat{\hat{X}} = \hat{X}$, i.e. $\hat{\widehat{X}} = \hat{X}$, and $\widehat{X} \subseteq \hat{X} \Rightarrow \hat{X} \subseteq \widehat{\hat{X}} \subseteq \hat{\hat{X}} = \hat{X}$, i.e. $\widehat{\hat{X}} = \hat{X}$ as well.

To (4):

(i) This is clear: if $X \subseteq Y$, then $\widehat{X} \subseteq \widehat{Y}$ by (1), hence $\widecheck{X} \stackrel{\text{def}}{=} \widehat{X}^c \supseteq \widehat{Y}^c \stackrel{\text{def}}{=} \widecheck{Y}$.

(ii) If $\mathcal{I} \subseteq J$, then $\widehat{X}_{\mathcal{I}} \supseteq \widehat{X}_{\mathcal{J}}$ by (1), hence $\widecheck{X}_{\mathcal{I}} \stackrel{\text{def}}{=} \widehat{X}_{\mathcal{I}}^c \subseteq \widehat{X}_{\mathcal{J}}^c \stackrel{\text{def}}{=} \widecheck{X}_{\mathcal{J}}$.

(iii) Since $X \subseteq \widehat{X}$ by closure properties, we have $\widecheck{X} \stackrel{\text{def}}{=} \widehat{X}^c \subseteq X^c$, hence $\chi_j(\widecheck{X}) \subseteq \chi_j(X^c)$. On the other hand, since χ_j is surjective, we have $\chi_j(\widecheck{X}) \stackrel{\text{def}}{=} \chi_j(\widehat{X}^c) \supseteq \chi_j(\widehat{X})^c \stackrel{(1)}{=} (X_j^{\text{sp}})^c$, i.e. $\chi_j(\widecheck{X}) \supseteq (X_j^{\text{sp}})^c$, $j \in \mathcal{I}$.

To (5):

(i) We have $X \subseteq Y \Rightarrow \forall\, j \in \mathcal{I}\colon \chi_j(X) \subseteq \chi_j(Y) \Rightarrow \forall\, j \in \mathcal{I}\colon \chi_j^{-1}(\chi_j(X))^c \supseteq \chi_j^{-1}(\chi_j(Y))^c \Rightarrow \check{X} \stackrel{\text{def}}{=} \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(\chi_j(X))^c \supseteq \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(\chi_j(Y))^c \stackrel{\text{def}}{=} \check{Y}$.

(ii) Since $\mathcal{J} \supseteq \mathcal{I}$, we have $\check{X}_{\mathcal{J}} \overset{\text{def}}{=} \bigcap_{j\in\mathcal{J}} \chi_j^{-1}(\chi_j(X))^c \subseteq \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(\chi_j(X))^c \overset{\text{def}}{=} \check{X}_{\mathcal{I}}$.

(iii) We have again from the definition that $\forall\, j \in \mathcal{I}$: $\check{X} \subseteq \chi_j^{-1}(\chi_j(X)^c)$, i.e. $\forall\, j \in \mathcal{I}$: $\chi_j(\check{X}) \subseteq \chi_j(X)^c \overset{\text{def}}{=} (X_j^{\text{sp}})^c$. It is clear that by definition $\check{X}$ is the biggest subset of $\mathcal{A}$ with this property.

To (6):

(i) We split the inclusion chains into successive distinct parts as follows:

- $\check{\widehat{X}} = \widehat{\check{X}} = \check{X}$: Since $X \subseteq \widehat{X}$, we have $\check{\widehat{X}} \subseteq \check{X}$ by inclusion reversal. Conversely, suppose that there exists $a \in \check{X}$ such that $a \in \check{\widehat{X}}^c \overset{\text{def}}{=} \big(\bigcap_{j\in\mathcal{I}} \chi_j^{-1}(\chi_j(\widehat{X}))^c\big)^c = \bigcup_{j\in\mathcal{I}} \chi_j^{-1}(\chi_j(\widehat{X}))$, i.e. $\exists\, j \in \mathcal{I}$: $\chi_j(a) \in \chi_j(\widehat{X}) \overset{(2)}{=} X_j^{\text{sp}}$. But $\chi_j(a) \in \chi_j(\widehat{X}) \subseteq (X_j^{\text{sp}})^c$ by (5), a contradiction. It follows that $\check{X} \subseteq \check{\widehat{X}}$, hence $\check{\widehat{X}} = \check{X}$. Next, $\check{X} \subseteq \widehat{\check{X}}$ holds by the closure property. On the other hand, by (2) and (5) we have $\forall\, j \in \mathcal{I}$: $\chi_j(\widehat{\check{X}}) = \chi_j(\check{X}) \subseteq (X_j^{\text{sp}})^c$, but $\check{X}$ is the biggest subset of $\mathcal{A}$ with the last property, hence $\widehat{\check{X}} \subseteq \check{X}$ as well. It follows that $\widehat{\check{X}} = \check{X}$.
- $\check{X} = \widecheck{\widehat{X}} = \widehat{\widecheck{X}}$: Since $X \subseteq \widehat{X}$ by the closure property, we have $\widecheck{\widehat{X}} \subseteq \check{X}$ by inclusion reversal. However, $\widehat{X} \subseteq \hat{X}$ by (3), hence $\check{X} = \check{\hat{X}} \subseteq \widecheck{\widehat{X}}$ by inclusion reversal. Thus, $\widecheck{\widehat{X}} = \check{\hat{X}} = \check{X}$. Lastly, $\widehat{\widecheck{X}} = \widehat{\hat{\check{X}}} \overset{(3)}{=} \hat{\check{X}} = \check{X}$.
- $\check{X} \subseteq \hat{X}^c = \widecheck{\hat{X}} \subseteq \widecheck{X}$: We have $\check{X} \overset{\text{def}}{=} \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(\chi_j(X))^c = \big(\bigcup_{j\in\mathcal{I}} \chi_j^{-1}(\chi_j(X))\big)^c \subseteq \big(\bigcap_{j\in\mathcal{I}} \chi_j^{-1}(\chi_j(X))\big)^c \overset{\text{def}}{=} \hat{X}^c$, and moreover $\widecheck{\hat{X}} \overset{\text{def}}{=} \widehat{\hat{X}}^c \overset{(3)}{=} \hat{X}^c$. Finally, since $\widehat{X} \subseteq \hat{X}$ by (3), we get $\hat{X}^c \subseteq \widehat{X}^c \overset{\text{def}}{=} \widecheck{X}$.
- $\widecheck{X} = \widehat{\widecheck{X}} = \widecheck{\widehat{X}}$: We have $\chi_{\mathcal{I}}(\widecheck{X}) \overset{\text{def}}{=} \chi_{\mathcal{I}}(\widehat{X}^c) \overset{\text{def}}{=} \chi_{\mathcal{I}}(\chi_{\mathcal{I}}^{-1}(\chi_{\mathcal{I}}(X)^c)) \subseteq \chi_{\mathcal{I}}(X)^c$, hence $\widehat{\widecheck{X}} \overset{\text{def}}{=} \chi_{\mathcal{I}}^{-1}(\chi_{\mathcal{I}}(\widecheck{X})) \subseteq \chi_{\mathcal{I}}^{-1}(\chi_{\mathcal{I}}(X)^c) \overset{\text{def}}{=} \widecheck{X}$, but $\widecheck{X} \subseteq \widehat{\widecheck{X}}$ by closure properties, therefore $\widehat{\widecheck{X}} = \widecheck{X}$. Furthermore, $\widecheck{\widehat{X}} \overset{\text{def}}{=} \widehat{\widehat{X}}^c \overset{(1)}{=} \widehat{X}^c \overset{\text{def}}{=} \widecheck{X}$ as well.
- $\widecheck{X} \subseteq X^c \subseteq \widehat{X^c} \subseteq \hat{X^c}$: Indeed, we have $\widecheck{X} \subseteq X^c$ from the proof of (4)(iii) and further $X^c \subseteq \widehat{X^c} \subseteq \hat{X^c}$ by closure properties and (3).
- Finally, since $\widecheck{X} \subseteq X^c$, we get $\widecheck{X} \subseteq \widehat{\widecheck{X}} \subseteq \widehat{X^c}$ by closure properties.

(ii) If $\chi_{\mathcal{I}}$ is injective, then $\chi_{\mathcal{I}}(X)^c \supseteq \chi_{\mathcal{I}}(X^c) \Rightarrow \chi_{\mathcal{I}}^{-1}(\chi_{\mathcal{I}}(X))^c = \chi_{\mathcal{I}}^{-1}(\chi_{\mathcal{I}}(X)^c) \supseteq \chi_{\mathcal{I}}^{-1}(\chi_{\mathcal{I}}(X^c))$, i.e. $\widecheck{X} \overset{\text{def}}{=} \widehat{X}^c \supseteq \widehat{X^c}$, hence $\widecheck{X} = X^c = \widehat{X^c}$ by (i) (or directly), and consequently, $\widehat{\widecheck{X}} = \widehat{X^c}$, where $\widehat{X^c} \subseteq \hat{X^c}$ by (i).

To (7):

(i) By (6) we have $\check{X} \subseteq \widecheck{X}$, hence $\widecheck{X^c} \subseteq \widecheck{X^c}$. Next, $X^c \subseteq \widehat{X^c}$ by the closure property, hence $\widecheck{X^c} \overset{\text{def}}{=} \widehat{X^c}^c \subseteq X$. Thus, $\widecheck{X^c} \subseteq \widecheck{X^c} \subseteq X \subseteq \widehat{X} \subseteq \hat{X}$, and moreover $\widecheck{\widecheck{X}} \overset{\text{def}}{=} \widehat{\widecheck{X}}^c \overset{(6)}{=} \widecheck{X}^c \overset{\text{def}}{=} (\widehat{X}^c)^c = \widehat{X}$. Furthermore, $a \in \hat{X} \overset{\text{def}}{\Leftrightarrow} \forall\, j \in \mathcal{I}$: $\chi_j(a) \in X_j^{\text{sp}} \subseteq \chi_j(\check{X})^c$ by (5), hence $a \in \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(\chi_j(\check{X})^c) \overset{\text{def}}{=} \widecheck{\check{X}}$, that is, $\hat{X} \subseteq \widecheck{\check{X}}$. Next, putting $Y := \check{X}$, we have by (6) that $\widecheck{Y} \subseteq Y^c$, i.e. $\widecheck{\check{X}} \subseteq \check{X}^c$. Finally, $\widecheck{\check{X}} \overset{\text{def}}{=} \widehat{\check{X}}^c \overset{(6)}{=} \check{X}^c$, this completes the first chain of inclusions. Regarding the second chain of inclusions, since $\check{Y} \subseteq \widecheck{Y}$ by (6), taking $Y := \widecheck{X}$ gives $\check{\widecheck{X}} \subseteq \widecheck{\widecheck{X}} = \widehat{X}$ by (i). Furthermore, $\widecheck{X} \subseteq X^c$ by (6), hence $\widecheck{X^c} \subseteq \check{\widecheck{X}}$ by inclusion reversal, or $\widecheck{X^c} \subseteq \check{\widecheck{X}} \subseteq \widecheck{\widecheck{X}} = \widehat{X}$ in total.

(ii) If χ_I is injective, then $\widecheck{X} = X^c = \widehat{X^c}$ by (6), hence $\check{\widecheck{X}} = \widecheck{X^c}$ and $\widecheck{X^c} = X = \widehat{X} \overset{\text{def}}{=} \widecheck{X}^c$ with $\widecheck{X^c} \subseteq \widecheck{X^c}$ by (i).

$\square$

Remarks:

(1) It is clear from the proofs that the above relations have little to do with commutative algebra,

but are entirely set-theoretical in nature.

(2) Observe that if $\mathcal{I} \subseteq \mathcal{J}$, then the complements $\overset{\smile}{X}$ and $\check{X}$ behave entirely oppositely to each other.

(3) We will be primarily interested in $\hat{X}$ and $\check{X}$ rather than $\overset{\frown}{X}$ and $\overset{\smile}{X}$. They will reappear again more crucially in Section 2.1 and onwards.

(4) The "estimate" $\check{X} \subseteq \hat{X}^c$ in Eq. (0.1.52) is somewhat generous since by Eq. (0.1.55) we have the "refinement" $\hat{X} \subseteq \overset{\smile}{\overset{\smile}{\check{X}}} \subseteq \check{X}^c$.

(5) In (6)(i) there is in general no relation between $\overset{\frown}{\hat{\overset{\smile}{X}}}$ and X^c or $\overset{\frown}{X^c}$ without further assumptions. Symmetrically, in (7)(i) there is likewise no relation between $\overset{\smile}{\check{\overset{\frown}{X}}}$ and X or $\overset{\smile}{X^c}$ in general.

(6) We remark that in general these notions really ought to be investigated in conjunction with properties of $\mathcal{A}$, but this goes far beyond the scope of the present work.

Lemma-Definition 0.1.44 (Spectral-relative units, radicals, classes)**:** Let $\mathcal{A} \in \mathsf{CAlg}_R$, let $\varnothing \neq \mathcal{I} \subseteq \mathrm{Spm}(\mathcal{A})$ be a collection of maximal ideals understood as an indexing subset, and $a \in \mathcal{A}$.

(1) Multiplicative submonoid with respect to $\mathcal{I}$:

$$\mathcal{A}_{\mathcal{I}}^{\times} \coloneqq \mathrm{U}_{\chi_{\mathcal{I}}}(\mathcal{A}) \overset{\text{def}}{=} \mathcal{A}_{\chi_{\mathcal{I}}}^{\times} \overset{\text{def}}{=} \chi_{\mathcal{I}}^{-1}(\kappa_{\mathcal{I}}^{\times}) = \bigcap_{j \in \mathcal{I}} \mathfrak{M}_j^c \supseteq \widehat{\mathcal{A}^{\times}} \supseteq \overset{\frown}{\mathcal{A}^{\times}} \supseteq \mathcal{A}^{\times}. \tag{0.1.58}$$

(2) Singular cone with respect to $\mathcal{I}$:

$$\begin{aligned} \mathrm{Sing}_{\mathcal{I}}(\mathcal{A}) &\coloneqq \mathrm{Sing}_{\chi_{\mathcal{I}}}(\mathcal{A}) = \bigcup \mathcal{I} \overset{\text{def}}{=} \bigcup_{j \in \mathcal{I}} \mathfrak{M}_j \subseteq \mathrm{Sing}(\mathcal{A}) \subseteq \overset{\frown}{\mathrm{Sing}(\mathcal{A})} \subseteq \widehat{\mathrm{Sing}(\mathcal{A})} = \\ &= \begin{cases} \mathrm{Sing}(\mathcal{A}), \text{ if } \mathcal{A} \text{ is local} \\ \mathcal{A}, \text{ otherwise.} \end{cases} \end{aligned} \tag{0.1.59}$$

Moreover:

$$\overset{\smile}{\mathrm{Sing}(\mathcal{A})} = \begin{cases} \mathcal{A}^{\times}, \text{ if } \mathcal{A} \text{ is local} \\ \varnothing, \text{ otherwise.} \end{cases} \tag{0.1.60}$$

Furthermore:

$$\overset{\frown}{\mathrm{Sing}(\mathcal{A})} = \begin{cases} \mathrm{Sing}(\mathcal{A}), \text{ if } \mathcal{I} = \mathrm{Spm}(\mathcal{A}) \\ \mathcal{A}, \text{ if } \mathcal{I} \text{ is a finite proper subset of } \mathrm{Spm}(\mathcal{A}) \\ \mathrm{Sing}(\mathcal{A}) \text{ or } \mathcal{A}, \text{ if } \mathcal{I} \text{ is an infinite proper subset of } \mathrm{Spm}(\mathcal{A}), \end{cases} \tag{0.1.61}$$

hence

$$\overset{\smile}{\mathrm{Sing}(\mathcal{A})} \overset{\text{def}}{=} \overset{\frown}{\mathrm{Sing}(\mathcal{A})}^{c} = \begin{cases} \mathcal{A}^{\times}, \text{ if } \mathcal{I} = \mathrm{Spm}(\mathcal{A}) \\ \varnothing, \text{ if } \mathcal{I} \text{ is a finite proper subset of } \mathrm{Spm}(\mathcal{A}) \\ \mathcal{A}^{\times} \text{ or } \varnothing, \text{ if } \mathcal{I} \text{ is an infinite proper subset of } \mathrm{Spm}(\mathcal{A}). \end{cases} \tag{0.1.62}$$

In particular, if $\#\,\mathrm{Spm}(\mathcal{A}) < \infty$, then:

$$\begin{aligned} \overset{\frown}{\mathrm{Sing}(\mathcal{A})} &= \begin{cases} \mathrm{Sing}(\mathcal{A}), \text{ if } \mathcal{I} = \mathrm{Spm}(\mathcal{A}) \\ \mathcal{A}, \text{ if } \mathcal{I} \neq \mathrm{Spm}(\mathcal{A}) \end{cases} \\ \overset{\smile}{\mathrm{Sing}(\mathcal{A})} &= \begin{cases} \mathcal{A}^{\times}, \text{ if } \mathcal{I} = \mathrm{Spm}(\mathcal{A}) \\ \varnothing, \text{ if } \mathcal{I} \neq \mathrm{Spm}(\mathcal{A}). \end{cases} \end{aligned} \tag{0.1.63}$$

(3) Jacobson radical with respect to $\mathcal{I}$:

$$\mathfrak{J}_{\mathcal{I}}(\mathcal{A}) \coloneqq \mathfrak{J}_{\chi_{\mathcal{I}}}(\mathcal{A}) \overset{\text{def}}{=} \chi_{\mathcal{I}}^{-1}(\mathfrak{J}(\kappa_{\mathcal{I}})) = \operatorname{Ker}(\chi_{\mathcal{I}}) = \bigcap \mathcal{I} \overset{\text{def}}{=} \bigcap_{j\in\mathcal{I}} \mathfrak{M}_j = \widehat{\mathfrak{J}(\mathcal{A})} = \widehat{\mathfrak{J}(\mathcal{A})} \supseteq \mathfrak{J}(\mathcal{A}). \tag{0.1.64}$$

Moreover:

$$\widetilde{\mathfrak{J}(\mathcal{A})} = \mathcal{A}_{\mathcal{I}}^{\times}. \tag{0.1.65}$$

(4) Nilradical with respect to $\mathcal{I}$:

$$\operatorname{nil}_{\mathcal{I}}(\mathcal{A}) \coloneqq \operatorname{nil}_{\chi_{\mathcal{I}}}(\mathcal{A}) \overset{\text{def}}{=} \chi_{\mathcal{I}}^{-1}(\operatorname{nil}(\kappa_{\mathcal{I}})) = \operatorname{Ker}(\chi_{\mathcal{I}}) = \mathfrak{J}_{\mathcal{I}}(\mathcal{A}) = \widehat{\operatorname{nil}(\mathcal{A})} = \widehat{\operatorname{nil}(\mathcal{A})} \supseteq \operatorname{nil}(\mathcal{A}). \tag{0.1.66}$$

Moreover:

$$\widetilde{\operatorname{nil}(\mathcal{A})} = \mathcal{A}_{\mathcal{I}}^{\times} = \widetilde{\mathfrak{J}(\mathcal{A})}. \tag{0.1.67}$$

(5) Singular “class” of $a \in \mathcal{A}$ with respect to $\mathcal{I}$:

$$[a]_{\mathcal{I}} \coloneqq a + \operatorname{Sing}_{\mathcal{I}}(\mathcal{A}) \subseteq [a]_{\text{sing}}. \tag{0.1.68}$$

Moreover:

$$\begin{aligned} \widehat{[a]_{\text{sing}}} &= a + \widehat{\operatorname{Sing}(\mathcal{A})} = \begin{cases} [a]_{\text{sing}}, \text{ if } \mathcal{A} \text{ is local} \\ \mathcal{A}, \text{ otherwise} \end{cases} \\ \widetilde{[a]_{\text{sing}}} &= a + \widetilde{\operatorname{Sing}(\mathcal{A})} = \begin{cases} [a]^{\times}, \text{ if } \mathcal{A} \text{ is local} \\ \varnothing, \text{ otherwise.} \end{cases} \end{aligned} \tag{0.1.69}$$

Furthermore:

$$\widehat{[a]_{\text{sing}}} = a + \widehat{\operatorname{Sing}(\mathcal{A})} = \begin{cases} [a]_{\text{sing}}, \text{ if } \mathcal{I} = \operatorname{Spm}(\mathcal{A}) \\ \mathcal{A}, \text{ if } \mathcal{I} \text{ is a finite proper subset of } \operatorname{Spm}(\mathcal{A}) \\ [a]_{\text{sing}} \text{ or } \mathcal{A}, \text{ if } \mathcal{I} \text{ is an infinite proper subset of } \operatorname{Spm}(\mathcal{A}), \end{cases} \tag{0.1.70}$$

hence

$$\begin{aligned} \widetilde{[a]_{\text{sing}}} &\overset{\text{def}}{=} \widehat{[a]_{\text{sing}}}^{\,c} = a + \widetilde{\operatorname{Sing}(\mathcal{A})} = \\ &= \begin{cases} [a]^{\times}, \text{ if } \mathcal{I} = \operatorname{Spm}(\mathcal{A}) \\ \varnothing, \text{ if } \mathcal{I} \text{ is a finite proper subset of } \operatorname{Spm}(\mathcal{A}) \\ [a]^{\times} \text{ or } \varnothing, \text{ if } \mathcal{I} \text{ is an infinite proper subset of } \operatorname{Spm}(\mathcal{A}). \end{cases} \end{aligned} \tag{0.1.71}$$

In particular, if $\operatorname{Spm}(\mathcal{A}) < \infty$, then:

$$\begin{aligned} \widehat{[a]_{\text{sing}}} &= \begin{cases} [a]_{\text{sing}}, \text{ if } \mathcal{I} = \operatorname{Spm}(\mathcal{A}) \\ \mathcal{A}, \text{ if } \mathcal{I} \neq \operatorname{Spm}(\mathcal{A}) \end{cases} \\ \widetilde{[a]_{\text{sing}}} &= \begin{cases} [a]^{\times}, \text{ if } \mathcal{I} = \operatorname{Spm}(\mathcal{A}) \\ \varnothing, \text{ if } \mathcal{I} \neq \operatorname{Spm}(\mathcal{A}). \end{cases} \end{aligned} \tag{0.1.72}$$

(6) Multiplicative “class” of $a \in \mathcal{A}$ with respect to $\mathcal{I}$:

$$[a]_{\mathcal{I}}^{\times} \coloneqq a + \mathcal{A}_{\mathcal{I}}^{\times} \supseteq [a]^{\times}. \tag{0.1.73}$$

Moreover, we have $\widehat{[a]^{\times}} = a + \widehat{\mathcal{A}^{\times}}$ and $\widehat{[a]^{\times}} = a + \widehat{\mathcal{A}^{\times}}$ as well as $\widetilde{[a]^{\times}} = a + \widetilde{\mathcal{A}^{\times}}$ and $\widetilde{[a]^{\times}} = a + \widetilde{\mathcal{A}^{\times}}$.

Proof: To (1): We have $a \in \mathcal{A}_{\mathcal{I}}^{\times} \Leftrightarrow \chi_{\mathcal{I}}(a) \in \kappa_{\mathcal{I}}^{\times} \Leftrightarrow \forall\, j \in \mathcal{I}\colon \chi_j(a) \in \kappa_j^{\times} \Leftrightarrow \forall\, j \in \mathcal{I}\colon a \in \mathfrak{M}_j^c$, i.e. $\mathcal{A}_{\mathcal{I}}^{\times} = \bigcap_{j\in\mathcal{I}} \mathfrak{M}_j^c$. Furthermore, we only have $\forall\, j \in \mathcal{I}\colon \chi_j(\mathcal{A}^{\times}) \subseteq \kappa_j^{\times}$, hence only $\widehat{\mathcal{A}^{\times}} \overset{\text{def}}{=} \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(\chi_j(\mathcal{A}^{\times})) \subseteq \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(\kappa_j^{\times}) = \bigcap_{j\in\mathcal{I}} \mathfrak{M}_j^c = \mathcal{A}_{\mathcal{I}}^{\times}$.

To (2): We have $\mathrm{Sing}_{\mathcal{I}}(\mathcal{A}) \overset{\text{def}}{=} (\mathcal{A}_{\mathcal{I}}^{\times})^c = (\bigcap_{j\in\mathcal{I}} \mathfrak{M}_j^c)^c = \bigcup_{j\in\mathcal{I}} \mathfrak{M}_j$ by (1). If $\mathcal{A} = (\mathcal{A}, \mathfrak{M})$ is local, then $\widehat{\mathrm{Sing}(\mathcal{A})} \overset{\text{def}}{=} \widehat{\mathfrak{M}} \overset{\text{def}}{=} \chi_{\mathcal{A}}^{-1}(\chi_{\mathcal{A}}(\mathfrak{M})) = \mathrm{Ker}(\chi_{\mathcal{A}}) = \mathfrak{M} \overset{\text{def}}{=} \mathrm{Sing}(\mathcal{A})$ and $\widecheck{\mathrm{Sing}(\mathcal{A})} \overset{\text{def}}{=} \widecheck{\mathfrak{M}} \overset{\text{def}}{=} \chi_{\mathcal{A}}^{-1}(\chi_{\mathcal{A}}(\mathfrak{M}))^c \overset{\text{def}}{=} \widehat{\mathfrak{M}}^c = \mathfrak{M}^c = \mathcal{A}^{\times}$. Now suppose that $\mathcal{A}$ is not local and write $\mathrm{Spm}(\mathcal{A}) =: \{\mathfrak{M}_i\}_i$, where $\#\,\mathrm{Spm}(\mathcal{A}) \geq 2$. If $i = j$, then $\chi_j^{-1}(\chi_j(\mathfrak{M}_i)) = \mathfrak{M}_i$ and $\chi_j^{-1}(\chi_j(\mathfrak{M}_i))^c = \mathfrak{M}_i^c$. However, if $i \neq j$, then $\chi_j^{-1}(\chi_j(\mathfrak{M}_i)) = \mathfrak{M}_i + \mathfrak{M}_j = \mathcal{A}$ by maximality, hence $\chi_j^{-1}(\chi_j(\mathfrak{M}_i))^c = \varnothing$. Thus, $\widehat{\mathrm{Sing}(\mathcal{A})} \overset{\text{def}}{=} \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(\chi_j(\bigcup_i \mathfrak{M}_i)) = \bigcap_{j\in\mathcal{I}}(\bigcup_i \chi_j^{-1}(\chi_j(\mathfrak{M}_i))) = \bigcap_{j\in\mathcal{I}} \mathcal{A} = \mathcal{A}$ and $\widecheck{\mathrm{Sing}(\mathcal{A})} \overset{\text{def}}{=} \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(\chi_j(\bigcup_i \mathfrak{M}_i))^c = \bigcap_{j\in\mathcal{I}}(\bigcap_i \chi_j^{-1}(\chi_j(\mathfrak{M}_i))^c) = \varnothing$ as $\#\,\mathrm{Spm}(\mathcal{A}) \geq 2$. Similarly, since $\chi_{\mathcal{I}}^{-1}(\chi_{\mathcal{I}}(\mathfrak{M}_i)) = \mathrm{Ker}(\chi_{\mathcal{I}}) + \mathfrak{M}_i = \mathfrak{J}_{\mathcal{I}}(\mathcal{A}) + \mathfrak{M}_i$ (see (3)), we have $\widehat{\mathrm{Sing}(\mathcal{A})} = \chi_{\mathcal{I}}^{-1}(\chi_{\mathcal{I}}(\bigcup_i \mathfrak{M}_i)) = \bigcup_i \chi_{\mathcal{I}}^{-1}(\chi_{\mathcal{I}}(\mathfrak{M}_i)) = \bigcup_i(\mathfrak{J}_{\mathcal{I}}(\mathcal{A}) + \mathfrak{M}_i) = \big(\bigcup_{i\in\mathcal{I}}(\mathfrak{J}_{\mathcal{I}}(\mathcal{A}) + \mathfrak{M}_i)\big) \cup \big(\bigcup_{i\in\mathcal{I}^c}(\mathfrak{J}_{\mathcal{I}}(\mathcal{A}) + \mathfrak{M}_i)\big) = \mathrm{Sing}_{\mathcal{I}}(\mathcal{A}) \cup \big(\bigcup_{i\in\mathcal{I}^c}(\mathfrak{J}_{\mathcal{I}}(\mathcal{A}) + \mathfrak{M}_i)\big)$ as $\forall\, i \in \mathcal{I}\colon \mathfrak{J}_{\mathcal{I}}(\mathcal{A}) + \mathfrak{M}_i = \mathfrak{M}_i$. In particular, if $\mathcal{I} = \mathrm{Spm}(\mathcal{A})$, we get $\mathcal{I}^c = \varnothing$ and $\widehat{\mathrm{Sing}(\mathcal{A})} = \mathrm{Sing}_{\mathrm{Spm}(\mathcal{A})}(\mathcal{A}) \overset{\text{def}}{=} \mathrm{Sing}(\mathcal{A})$. Now suppose that $\mathcal{I}^c \neq \varnothing$. If $\#\mathcal{I} < \infty$ and $i \in \mathcal{I}^c$, then $\mathfrak{J}_{\mathcal{I}}(\mathcal{A}) \nsubseteq \mathfrak{M}_i$ by prime avoidance (see e.g. [AM69, Prop. 1.11]) and maximality, hence $\mathfrak{J}_{\mathcal{I}}(\mathcal{A}) + \mathfrak{M}_i = \mathcal{A}$, implying $\widehat{\mathrm{Sing}(\mathcal{A})} = \mathcal{A}$. However, when $\mathcal{I}$ is infinite, we have two possibilities:

- if $\exists\, i \in \mathcal{I}^c\colon \mathfrak{J}_{\mathcal{I}}(\mathcal{A}) \nsubseteq \mathfrak{M}_i$, then we have again $\mathfrak{J}_{\mathcal{I}}(\mathcal{A}) + \mathfrak{M}_i = \mathcal{A}$, hence $\widehat{\mathrm{Sing}(\mathcal{A})} = \mathcal{A}$;
- if $\forall\, i \in \mathcal{I}^c\colon \mathfrak{J}_{\mathcal{I}}(\mathcal{A}) \subseteq \mathfrak{M}_i$, then as before $\forall\, i \in \mathcal{I}^c\colon \mathfrak{J}_{\mathcal{I}}(\mathcal{A}) + \mathfrak{M}_i = \mathfrak{M}_i$, hence $\widehat{\mathrm{Sing}(\mathcal{A})} = \mathrm{Sing}_{\mathcal{I}}(\mathcal{A}) \cup \mathrm{Sing}_{\mathcal{I}^c}(\mathcal{A}) = \mathrm{Sing}(\mathcal{A})$.

In summary, if $\mathcal{I}$ is a proper infinite subset of $\mathrm{Spm}(\mathcal{A})$, then $\widehat{\mathrm{Sing}(\mathcal{A})} \in \{\mathrm{Sing}(\mathcal{A}), \mathcal{A}\}$ and in such generality either one is possible.

To (3): Since $\mathfrak{J}(\kappa_{\mathcal{I}}) \overset{\text{def}}{=} \mathfrak{J}(\prod_{j\in\mathcal{I}} \kappa_j) = \prod_{j\in\mathcal{I}} \mathfrak{J}(\kappa_j) = 0$, we have $\mathfrak{J}_{\mathcal{I}}(\mathcal{A}) = \mathrm{Ker}(\chi_{\mathcal{I}}) = \bigcap_{j\in\mathcal{I}} \mathfrak{M}_j$. Furthermore, since $\chi_j\colon \mathcal{A} \twoheadrightarrow \kappa_j$ is surjective, we have $\chi_j(\mathfrak{J}(\mathcal{A})) \subseteq \mathfrak{J}(\kappa_j) = 0$, $j \in \mathcal{I}$, hence $\widehat{\mathfrak{J}(\mathcal{A})} \overset{\text{def}}{=} \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(\mathfrak{J}(\mathcal{A})) = \bigcap_{j\in\mathcal{I}} \mathrm{Ker}(\chi_j) = \bigcap_{j\in\mathcal{I}} \mathfrak{M}_j = \mathfrak{J}_{\mathcal{I}}(\mathcal{A})$. Similarly, $\widecheck{\mathfrak{J}(\mathcal{A})} \overset{\text{def}}{=} \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(\chi_j(\mathfrak{J}(\mathcal{A})))^c = \bigcap_{j\in\mathcal{I}} \mathfrak{M}_j^c = \mathcal{A}_{\mathcal{I}}^{\times}$ by (1). Finally, set $\mathcal{B} \coloneqq \chi_{\mathcal{I}}(\mathcal{A}) \subseteq \kappa_{\mathcal{I}}$ and denote by $\pi_j\colon \kappa_{\mathcal{I}} \twoheadrightarrow \kappa_j$, $j \in \mathcal{I}$, the canonical projections. Without loss of generality we can assume $\mathcal{A} \neq 0$, hence $\mathcal{B} \neq 0$. We show that $\mathfrak{J}(\mathcal{B}) = 0$. To this end, we have a commutative diagram

$$\begin{tikzcd}
 & & \kappa_{\mathcal{I}} \arrow[dd, two heads, "\pi_j"] \\
\mathcal{A} \arrow[urr, two heads, "\chi_{\mathcal{I}}"] \arrow[r, two heads, "\chi'_{\mathcal{I}}"] \arrow[drr, two heads, "\chi_j"'] & \mathcal{B} \arrow[ur, hook, "\subseteq_{\mathcal{B}}"'] \arrow[dr, dashed, two heads, "\pi_j^{\mathcal{B}}"] & \\
 & & \kappa_j
\end{tikzcd} \tag{0.1.74}$$

for every $j \in \mathcal{I}$, where $\chi'_{\mathcal{I}}$ denotes the *image corestriction* of $\chi_{\mathcal{I}}$ and the induced map $\pi_j^{\mathcal{B}} \coloneqq \pi_j \circ \subseteq_{\mathcal{B}}$ is indeed surjective. Thus, $\mathcal{B} \subseteq_{\mathcal{B}} \kappa_{\mathcal{I}}$ is a *subdirect product*, hence $\mathfrak{J}(\mathcal{B}) \subseteq \mathfrak{J}(\kappa_{\mathcal{I}}) = 0$ by Lemma 0.1.38. Since $\chi'_{\mathcal{I}}\colon \mathcal{A} \twoheadrightarrow \mathcal{B}$ is surjective, it follows that $\chi_{\mathcal{I}}(\mathfrak{J}(\mathcal{A})) \overset{\text{def}}{=} \chi'_{\mathcal{I}}(\mathfrak{J}(\mathcal{A})) \subseteq \mathfrak{J}(\mathcal{B}) = 0$, hence $\widehat{\mathfrak{J}(\mathcal{A})} \overset{\text{def}}{=} \chi_{\mathcal{I}}^{-1}(\chi_{\mathcal{I}}(\mathfrak{J}(\mathcal{A}))) = \mathrm{Ker}(\chi_{\mathcal{I}}) = \mathfrak{J}_{\mathcal{I}}(\mathcal{A})$ as well.

To (4): We have $\mathrm{nil}(\kappa_{\mathcal{I}}) = \prod_{j\in\mathcal{I}} \mathrm{nil}(\kappa_j) = 0$, hence $\mathrm{nil}_{\mathcal{I}}(\mathcal{A}) = \mathrm{Ker}(\chi_{\mathcal{I}}) = \mathfrak{J}_{\mathcal{I}}(\mathcal{A})$ by (3). Moreover, since κ_j, $j \in \mathcal{I}$, are in particular reduced, we have $\widehat{\mathrm{nil}(\mathcal{A})} \overset{\text{def}}{=} \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(\chi_j(\mathrm{nil}(\mathcal{A}))) = \bigcap_{j\in\mathcal{I}} \mathrm{Ker}(\chi_j) = \mathfrak{J}_{\mathcal{I}}(\mathcal{A})$ and $\widecheck{\mathrm{nil}(\mathcal{A})} \overset{\text{def}}{=} \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(\chi_j(\mathrm{nil}(\mathcal{A})))^c = \bigcap_{j\in\mathcal{I}} \mathrm{Ker}(\chi_j)^c = \bigcap_{j\in\mathcal{I}} \mathfrak{M}_j^c = \mathcal{A}_{\mathcal{I}}^{\times}$. Similarly, since $\kappa_{\mathcal{I}}$ too is reduced, we have $\widehat{\mathrm{nil}(\mathcal{A})} \overset{\text{def}}{=} \chi_{\mathcal{I}}^{-1}(\chi_{\mathcal{I}}(\mathrm{nil}(\mathcal{A}))) = \mathrm{Ker}(\chi_{\mathcal{I}}) = \mathfrak{J}_{\mathcal{I}}(\mathcal{A})$.

To (5): We have $b \in \widehat{[a]_{\mathrm{sing}}} \overset{\text{def}}{\Leftrightarrow} \forall\, j \in \mathcal{I}\colon \chi_j(b) \in \chi_j(a + \mathrm{Sing}(\mathcal{A})) = \chi_j(a) + \chi_j(\mathrm{Sing}(\mathcal{A})) \Leftrightarrow \forall\, j \in \mathcal{I}\colon \chi_j(b - a) \in \chi_j(\mathrm{Sing}(\mathcal{A})) \Leftrightarrow \forall\, j \in \mathcal{I}\colon b - a \in \chi_j^{-1}(\chi_j(\mathrm{Sing}(\mathcal{A}))) \Leftrightarrow b - a \in \bigcap_{j\in\mathcal{I}} \chi_j^{-1}(\chi_j(\mathrm{Sing}(\mathcal{A}))) \overset{\text{def}}{=} \widehat{\mathrm{Sing}(\mathcal{A})}$, i.e. $b \in a + \widehat{\mathrm{Sing}(\mathcal{A})}$, and the first equality follows from (2). Analogously for $\widecheck{[a]_{\mathrm{sing}}}$. Furthermore, $\widehat{[a]_{\mathrm{sing}}} \overset{\text{def}}{=} a + \mathrm{Sing}(\mathcal{A}) + \mathfrak{J}_{\mathcal{I}}(\mathcal{A}) \overset{\text{def}}{=} a + \widehat{\mathrm{Sing}(\mathcal{A})}$, which implies the remaining statements in (5) by way of (2).

To (6): Analogous to (5).

As before, any remaining stuff follows immediately from Definition 0.1.33 and Lemma 0.1.43. □

Remarks:

(1) If $X \subseteq \mathcal{A}$, then in the above notation we simply have $\widehat{X} \overset{\text{def}}{=} X + \mathfrak{J}_{\mathcal{I}}(\mathcal{A})$, hence $\breve{X} \overset{\text{def}}{=} (X + \mathfrak{J}_{\mathcal{I}}(\mathcal{A}))^c$.

(2) Observe that in this generality it would be difficult to obtain more meaningful descriptions of $\widehat{\mathcal{A}^\times} = \bigcap_{j\in\mathcal{I}}(\mathcal{A}^\times + \mathfrak{M}_j)$ and $\widetilde{\mathcal{A}^\times} = \bigcap_{j\in\mathcal{I}}(\mathcal{A}^\times + \mathfrak{M}_j)^c$ and similarly for $\widehat{\mathcal{A}^\times} = \mathcal{A}^\times + \mathfrak{J}_{\mathcal{I}}(\mathcal{A}) = \mathcal{A}^\times + \bigcap_{j\in\mathcal{I}} \mathfrak{M}_j$ and $\breve{\mathcal{A}^\times} = \widehat{\mathcal{A}^\times}^c = (\mathcal{A}^\times + \mathfrak{J}_{\mathcal{I}}(\mathcal{A}))^c = (\mathcal{A}^\times + \bigcap_{j\in\mathcal{I}} \mathfrak{M}_j)^c$, but we will do so in more concreteness in the next Section 0.1.10.

(3) If $\mathcal{B} \overset{\text{def}}{=} \chi_{\mathcal{I}}(\mathcal{A}) \subseteq \kappa_{\mathcal{I}}$ were closed under *pseudo-inverses*, then $\mathcal{B}$ would be Von Neumann regular, hence we would automatically have $\mathfrak{J}(\mathcal{B}) = 0$. However, in our special case we got away with the much simpler fact that $\mathcal{B} \overset{\text{def}}{=} \chi_{\mathcal{I}}(\mathcal{A})$ is a *subdirect product* of fields.

(4) Finally, recall that in the commutative case $\mathcal{A}/\operatorname{nil}(\mathcal{A})$ is always a subdirect product of integral domains and $\mathcal{A}/\mathfrak{J}(\mathcal{A})$ is always a subdirect product of fields.

0.1.10. Finite Products of (Commutative) Local Algebras

Here Section 0.1.7 and Section 0.1.9 finally start to converge together, more precisely by way of Lemma-Definition 0.1.37 and Lemma-Definition 0.1.44, respectively. We remark, however, that at this stage we don't need the full Artinian property yet.

Corollary 0.1.45:

(1) Factorization of Finitely Decomposable Representations of Finite "Commutative" Products: Let $\mathcal{A} \in \mathsf{CAlg}_R$ and $M \in \mathsf{Mod}(\mathcal{A})$ via $\rho\colon \mathcal{A} \to \operatorname{End}_R(M)$. If $\mathcal{A} = \prod_{j=1}^n \mathcal{A}_j$ in CAlg_R and M is finitely decomposable in $\mathsf{Mod}(\mathcal{A})$, then there exists a complete decomposition ${}_{\mathcal{A}}M = \bigoplus_{k=1}^m {}_{\mathcal{A}}M_k$ and a fiber-grouping index mapping $\tau := \tau_\rho\colon [m] \to [n]$, $k \mapsto \tau(k)$, depending on said decomposition and ρ, such that ρ can be written as

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\rho^{\mathrm{cor}}"] \arrow[d, "\Pi_\rho"'] & \prod_{k=1}^m \operatorname{End}_R(M_k) \arrow[r, hook, "\iota_{\mathrm{cor}}"] & \operatorname{End}_R(M) \\ \prod_{k=1}^m \mathcal{A}_{\tau(k)} \arrow[ur, dashed, "\exists_1\, \overline{\rho^{\mathrm{cor}}}"'] & & \end{tikzcd} \tag{0.1.75}$$

where:

- $\Pi_\rho := \Pi_{(\tau_\rho)}$ is a projection selection morphism, in a standard form, associated to the representation ρ;
- $\overline{\rho^{\mathrm{cor}}} = \times_{k=1}^m \bar{\rho}_k$ for some representations $\bar{\rho}_k\colon \mathcal{A}_{\tau(k)} \to \operatorname{End}_R(M_k)$ such that $\rho_k = \bar{\rho}_k \circ \operatorname{pr}^{\mathcal{A}}_{\tau(k)}$, $1 \le k \le m$, where $\rho^{\mathrm{cor}} = (\rho_1, \ldots, \rho_m)$ as usual.

Moreover:

$$\operatorname{Ker}(\rho) = \begin{cases} \bigcap_{k\in\tau^{-1}(j)} \operatorname{Ker}(\bar{\rho}_k), & \text{if } j \in \tau([m]) \\ \mathcal{A}_j, & \text{otherwise.} \end{cases} \tag{0.1.76}$$

(2) Factorization of Morphisms from Finite Products to Finite Products of Local Algebras: Let $\mathcal{A} = \prod_{j=1}^n \mathcal{A}_j$ in Alg_R and let $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ be a morphism in Alg_R. If $\mathcal{B}$ is a finite product of local R-algebras, then there exists a rearrangement $\mathcal{B} \cong \prod_{k=1}^m (\mathcal{B}_k, \mathfrak{n}_k)$ and an index parametrization $\tau := \tau_\varphi\colon [m] \to [n]$, $k \mapsto \tau(k)$, depending on said rearrangement and φ, such that φ can be

written as

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\varphi"] \arrow[d, "\Pi_\varphi"'] & \mathcal{B} \cong \prod_{k=1}^m (\mathcal{B}_k, \mathfrak{n}_k) \\ \prod_{k=1}^m \mathcal{A}_{\tau(k)} \arrow[ur, dashed, "\exists_1 \bar{\varphi}"'] & \end{tikzcd} \tag{0.1.77}$$

where:

- $\Pi_\varphi := \Pi_{(\tau_\varphi)}$ is a projection selection morphism, in a standard form, associated to the morphism φ;
- $\bar{\varphi} = \times_{k=1}^m \bar{\varphi}_k$ for some morphisms $\bar{\varphi}_k \colon \mathcal{A}_{\tau(k)} \to (\mathcal{B}_k, \mathfrak{n}_k)$ such that $\varphi_k = \bar{\varphi}_k \circ \mathrm{pr}^{\mathcal{A}}_{\tau(k)}$, $1 \le k \le m$, with $(\varphi_1, \dots, \varphi_m)$ being the "components" of φ with respect to the rearrangement.

Moreover:

$$\mathrm{Ker}(\varphi) = \begin{cases} \bigcap\limits_{k \in \tau^{-1}(j)} \mathrm{Ker}(\bar{\varphi}_k), \text{ if } j \in \tau([m]) \\ \mathcal{A}_j, \text{ otherwise.} \end{cases} \tag{0.1.78}$$

Proof: To (1): This is just Corollary 0.1.29.

To (2): Any local ring is a block, hence the claim follows from Lemma 0.1.27. □

Let $\mathcal{A} \in \mathsf{CAlg}_R$. If $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocCAlg}_R$, we will also write $\chi_{\mathcal{A}} \colon \mathcal{A} \twoheadrightarrow \kappa$ for its unique character. A key observation is that the induced homomorphism $\chi_{\mathcal{A}}^\times := \mathrm{U}(\chi_{\mathcal{A}}) \colon \mathcal{A}^\times \twoheadrightarrow \kappa^\times$ of units is surjective by locality of $\mathcal{A}$. Given $X \subseteq (\mathcal{A}, \mathfrak{m}, \kappa)$, it has *trivial projection closure(s)* $\widetilde{X} = X^{\mathrm{pr}_I} = X$ (since $|I| = 1$ by virtue of $(\mathcal{A}, \mathfrak{m}, \kappa)$ being a block) and *canonical spectral closure(s)* $\widehat{X} = \widehat{X} \overset{\text{def}}{=} \chi_{\mathcal{A}}^{-1}(\chi_{\mathcal{A}}(X)) = X + \mathfrak{m}$ with *canonical spectral complement(s)* $\check{X} = \widehat{X}^c = \widehat{X}^c \overset{\text{def}}{=} \check{X}$. Moreover, clearly $\mathrm{Sing}(\mathcal{A}) \overset{\text{def}}{=} \mathfrak{m}$. Furthermore, it is immediate that:

- $\widehat{\mathcal{A}^\times} = \mathcal{A}^\times + \mathfrak{m} = \mathcal{A}^\times$, hence $\widecheck{\mathcal{A}^\times} = \mathfrak{m}$;
- $\widehat{\mathfrak{m}} = \mathfrak{m}$, hence $\check{\mathfrak{m}} = \mathcal{A}^\times$;
- $\widehat{\mathrm{nil}(\mathcal{A})} \overset{\text{def}}{=} \chi_{\mathcal{A}}^{-1}(\chi_{\mathcal{A}}(\mathrm{nil}(\mathcal{A}))) = \mathrm{Ker}(\chi_{\mathcal{A}}) = \mathfrak{m}$ since $\mathrm{nil}(\mathcal{A}) \subseteq \mathfrak{m}$, hence $\widecheck{\mathrm{nil}(\mathcal{A})} = \mathcal{A}^\times$.

Finally, if $a \in \mathcal{A}$, then $[a]_{\mathrm{sing}} \overset{\text{def}}{=} a + \mathfrak{m} \overset{\text{def}}{=} [a] \in \mathcal{A}/\mathfrak{m} = \kappa$ and $[a]^\times \overset{\text{def}}{=} a + \mathcal{A}^\times = a + \mathcal{A} \setminus \mathfrak{m}$.

Now suppose that $\mathcal{A} = \prod_{j=1}^n (\mathcal{A}_j, \mathfrak{m}_j, \kappa_j)$ in CAlg_R. Unlike the local case, this is arguably the simplest class of examples, where the different versions of projection closures, spectral closures, and spectral complements, respectively, no longer agree. We have $\mathrm{Spm}(\mathcal{A}) = \{\mathfrak{M}_j : 1 \le j \le n\}$, where $\mathfrak{M}_j = \mathcal{A}_1 \times \cdots \times \mathcal{A}_{j-1} \times \mathfrak{m}_j \times \mathcal{A}_{j+1} \times \cdots \times \mathcal{A}_n$, $1 \le j \le n$, whence $\mathfrak{J}(\mathcal{A}) = \mathfrak{m}_1 \times \cdots \times \mathfrak{m}_n$. Furthermore, we have factorizations

$$\begin{tikzcd} \mathcal{A} \arrow[r, two heads, "\mathrm{pr}_j"] \arrow[dr, two heads, "\chi_j"'] & \mathcal{A}_j \arrow[d, two heads, "\chi_{\mathcal{A}_j}"] \\ & \kappa_j \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathcal{A}^\times \arrow[r, two heads, "\mathrm{pr}_j^\times"] \arrow[dr, two heads, "\chi_j^\times"'] & \mathcal{A}_j^\times \arrow[d, two heads, "\chi_{\mathcal{A}_j}^\times"] \\ & \kappa_j^\times \end{tikzcd} \tag{0.1.79}$$

in particular $X_j^{\mathrm{sp}} = \chi_{\mathcal{A}_j}(X_j)$, $1 \le j \le n$. We get a natural bijective correspondence

$$\mathcal{P}([n]) \leftrightarrow \mathcal{P}(\mathrm{Spm}(\mathcal{A})),$$

where we make the **identification** $I \mapsto \mathcal{I} := \{\mathfrak{M}_j : j \in I\}$. The main source for I-s is of course Corollary 0.1.45 via $I = I_\rho \overset{\text{def}}{=} \tau_\rho([m])$ or $I = I_\varphi \overset{\text{def}}{=} \tau_\varphi([m])$ (cf. Definition 0.1.32). Accordingly, we will write

$$\mathcal{X}_I(\mathcal{A}) \overset{\text{def}}{=} \mathcal{X}_{\mathcal{I}}(\mathcal{A}) \overset{\text{def}}{=} \{\chi_j : j \in I\} \tag{0.1.80}$$

for the corresponding subspace of (formal) characters. The corresponding *multi-character* is simply given as

$$\begin{tikzcd} \mathcal{A} \arrow[r, two heads, "\mathrm{pr}_I"] \arrow[dr, "\chi_I"'] & \prod_{j\in I}\mathcal{A}_j \arrow[d, "\times_{j\in I}\chi_{\mathcal{A}_j}"] \\ & \prod_{j\in I}\kappa_j \end{tikzcd} \quad \text{inducing} \quad \begin{tikzcd} \mathcal{A}^\times \arrow[r, two heads, "\mathrm{pr}_I^\times"] \arrow[dr, "\chi_I^\times"'] & \prod_{j\in I}\mathcal{A}_j^\times \arrow[d, "\times_{j\in I}\chi_{\mathcal{A}_j}^\times"] \\ & \prod_{j\in I}\kappa_j^\times \end{tikzcd} \tag{0.1.81}$$

where $(a_1, \ldots, a_n) \mapsto (a_j)_{j\in I} \mapsto (a_j \bmod \mathfrak{m}_j)_{j\in I}$. In line with Section 0.1.7, it is convenient at this point to introduce also the following short notation:

$$\mathfrak{m}_{(I)} := \prod_{j=1}^{n} \begin{cases} \mathfrak{m}_j, \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise} \end{cases} \lhd \mathcal{A} \tag{0.1.82}$$

Clearly, if $|I| = 1$, then $\mathfrak{m}_{(I)}$ coincides with some $\mathfrak{M}_j \in \mathrm{Spm}(\mathcal{A})$, but otherwise it is not maximal. In particular,

$$\mathrm{Ker}(\chi_I) = \mathfrak{m}_{(I)}. \tag{0.1.83}$$

If $X \subseteq \mathcal{A}$, then we have the relation $X^{\mathrm{sp}} \overset{\text{def}}{=} X_I^{\mathrm{sp}} = \chi_I(X_I)$ as well as

$$\widetilde{X} \overset{\text{def}}{=} X_{(I)} \overset{\text{def}}{=} \prod_{j=1}^{n} \begin{cases} X_j, \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise} \end{cases} \tag{0.1.84}$$

and

$$\widehat{X} \overset{\text{def}}{=} X + \mathrm{Ker}(\chi_I) = X + \mathfrak{m}_{(I)}. \tag{0.1.85}$$

Furthermore, we can write explicitly

$$\widehat{X} = \prod_{j=1}^{n} \begin{cases} \chi_{\mathcal{A}_j}^{-1}(X_j^{\mathrm{sp}}), \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise} \end{cases} \overset{\text{def}}{=} \prod_{j=1}^{n} \begin{cases} \widehat{X_j}, \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise} \end{cases} = \prod_{j=1}^{n} \begin{cases} X_j + \mathfrak{m}_j, \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise,} \end{cases} \tag{0.1.86}$$

where indeed $\widehat{X_j} \overset{\text{def}}{=} \chi_{\mathcal{A}_j}^{-1}(\chi_{\mathcal{A}_j}(X_j)) \overset{\text{def}}{=} \chi_{\mathcal{A}_j}^{-1}(\chi_{\mathcal{A}_j}(\mathrm{pr}_j(X))) = \chi_{\mathcal{A}_j}^{-1}(X_j^{\mathrm{sp}})$ is the *canonical* spectral closure of X_j in the local R-algebra $(\mathcal{A}_j, \mathfrak{m}_j, \kappa_j)$ and the *individual* spectral closure of X is then given by $\widehat{X}_j = \mathcal{A}_1 \times \cdots \times \mathcal{A}_{j-1} \times \widehat{X_j} \times \mathcal{A}_{j+1} \times \cdots \times \mathcal{A}_n$ (cf. Definition 0.1.42), $1 \le j \le n$. Similarly, we can write explicitly

$$\check{X} = \prod_{j=1}^{n} \begin{cases} \chi_{\mathcal{A}_j}^{-1}((X_j^{\mathrm{sp}})^c), \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise} \end{cases} \overset{\text{def}}{=} \prod_{j=1}^{n} \begin{cases} \widehat{X_j}^{\,c}, \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise} \end{cases} = \prod_{j=1}^{n} \begin{cases} (X_j + \mathfrak{m}_j)^c, \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise.} \end{cases} \tag{0.1.87}$$

Thus we have the simple relation

$$\widehat{X} = \widetilde{X} + \mathfrak{m}_{(I)} \overset{\text{def}}{=} X_{(I)} + \mathfrak{m}_{(I)}. \tag{0.1.88}$$

Furthermore, in view of Corollary 0.1.45 we can similarly write without loss of generality

$$\mathrm{Ker}(\rho) = \prod_{j=1}^{n} \begin{cases} \mathfrak{a}_j, \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise} \end{cases} =: \mathfrak{a}_{(I)} \tag{0.1.89}$$

too, where $\forall\, j \in I\colon \mathfrak{a}_j \lhd \mathcal{A}_j$ and $\mathfrak{a}_j \subseteq \mathfrak{m}_j$ by locality of $\mathcal{A}_j$.

Lemma 0.1.46:

(1) We have:

$$\begin{aligned} &X \subseteq X^\rho \subseteq \widehat{X} \subseteq \widehat{X} \\ &X \subseteq X^\rho \subseteq (\widetilde{X})^\rho = \widetilde{X^\rho} \subseteq \widehat{X}. \end{aligned} \tag{0.1.90}$$

(2) We have:

$$X \subseteq X^{\mathrm{pr}_I} \subseteq \widetilde{X} \subseteq (\widetilde{X})^\rho = \widetilde{X^\rho} \subseteq \widehat{X}. \tag{0.1.91}$$

(3) We have:

$$\widehat{X} = \widehat{\widetilde{X}} = \widetilde{\widehat{X}} = \widehat{\widehat{X}} = \widetilde{\widetilde{X}} = \widehat{X^\rho} = (\widehat{X})^\rho. \tag{0.1.92}$$

(4) We have:

$$\widehat{X^\rho} = (\widehat{X})^\rho = \widehat{X}. \tag{0.1.93}$$

Proof: To (1): Indeed $X^\rho = X + \mathfrak{a}_{(I)} \subseteq X + \mathfrak{m}_{(I)} \overset{\text{def}}{=} \widehat{X} \subseteq X_{(I)} + \mathfrak{m}_{(I)} \overset{\text{def}}{=} \widehat{X}$ and $\widetilde{X^\rho} \overset{\text{def}}{=} (X + \mathfrak{a}_{(I)})_{(I)} = X_{(I)} + \mathfrak{a}_{(I)} = (\widetilde{X})^\rho \subseteq X_{(I)} + \mathfrak{m}_{(I)} \overset{\text{def}}{=} \widehat{X}$.

To (2): Indeed $X^{\mathrm{pr}_I} \overset{\text{def}}{=} X + \mathrm{Ker}(\mathrm{pr}_I) \subseteq X_{(I)} + \mathrm{Ker}(\mathrm{pr}_I) = X_{(I)} \overset{\text{def}}{=} \widetilde{X}$.

To (3): We have $\widetilde{\widetilde{X}} \overset{\text{def}}{=} (X + \mathfrak{m}_{(I)})_{(I)} = X_{(I)} + \mathfrak{m}_{(I)} \overset{\text{def}}{=} \widehat{\widetilde{X}} = \widehat{X}$. Also, $\widehat{X^\rho} \overset{\text{def}}{=} (X + \mathfrak{a}_{(I)})_{(I)} + \mathfrak{m}_{(I)} = X_{(I)} + \mathfrak{a}_{(I)} + \mathfrak{m}_{(I)} = X_{(I)} + \mathfrak{m}_{(I)} \overset{\text{def}}{=} \widehat{X}$ and $(\widehat{X})^\rho \overset{\text{def}}{=} X_{(I)} + \mathfrak{m}_{(I)} + \mathfrak{a}_{(I)} = X_{(I)} + \mathfrak{m}_{(I)} \overset{\text{def}}{=} \widehat{X}$.

To (4): We have $\widehat{X^\rho} \overset{\text{def}}{=} (X + a_{(I)}) + \mathfrak{m}_{(I)} = X + \mathfrak{m}_{(I)} \overset{\text{def}}{=} \widehat{X}$ and $(\widehat{X})^\rho \overset{\text{def}}{=} (X + \mathfrak{m}_{(I)}) + \mathfrak{a}_{(I)} = X + \mathfrak{m}_{(I)} \overset{\text{def}}{=} \widehat{X}$. □

Remarks:

(1) Like in the case of $\widetilde{X}$, both $\widehat{X}$ and $\check{X}$ clearly depend on the choice of a product decomposition for $\mathcal{A}$ as they are generally not invariant under permutation isomorphisms.

(2) For example, X will be taken by an open subset $U \subseteq \mathcal{A}$ or the support of a (singular) 1-cycle Γ, see Section 2.1 and onwards.

Lemma 0.1.47 (Agreement between I-notions and $\mathcal{I}$-notions)**:** Let $\mathcal{A} = \prod_{j=1}^n (\mathcal{A}_j, \mathfrak{m}_j, \kappa_j)$ in CAlg_R, $\varnothing \neq I \subseteq [n]$, and $\mathcal{I} = \{\mathfrak{M}_j : j \in I\} \subseteq \mathrm{Spm}(\mathcal{A})$.

(1) Relative multiplicative submonoid:

$$\mathcal{A}^\times_{(I)} = \widetilde{\mathcal{A}^\times} \overset{\text{def}}{=} \prod_{j=1}^n \begin{cases} \mathcal{A}_j^\times, \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise} \end{cases} = \widehat{\mathcal{A}^\times} = \widehat{\mathcal{A}^\times} = \begin{cases} \mathfrak{m}_j^c, \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise} \end{cases} = \bigcap_{j \in I} \mathfrak{M}_j^c \overset{\text{def}}{=} \mathcal{A}^\times_{\mathcal{I}} \supseteq \mathcal{A}^\times \tag{0.1.94}$$

and

$$\check{\mathcal{A}^\times} = \prod_{j=1}^n \begin{cases} \mathfrak{m}_j, \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise} \end{cases} \tag{0.1.95}$$

(2) Relative singular cone:

$$\mathrm{Sing}_{(I)}(\mathcal{A}) = \mathrm{Sing}_{\mathcal{I}}(\mathcal{A}) = \bigcup_{j \in I} \mathfrak{M}_j = \check{\mathcal{A}^\times}. \tag{0.1.96}$$

Furthermore:

$$\begin{aligned} \widehat{\mathrm{Sing}(\mathcal{A})} &= \begin{cases} \mathfrak{m}, \text{ if } \mathcal{A} = (\mathcal{A}, \mathfrak{m}, \kappa) \\ \mathcal{A}, \text{ otherwise} \end{cases} \\ \check{\mathrm{Sing}(\mathcal{A})} &= \begin{cases} \mathcal{A}^\times, \text{ if } \mathcal{A} = (\mathcal{A}, \mathfrak{m}, \kappa) \\ \varnothing, \text{ otherwise} \end{cases} \end{aligned} \tag{0.1.97}$$

and

$$\widehat{\mathrm{Sing}(\mathcal{A})} = \begin{cases} \mathrm{Sing}(\mathcal{A}), \text{ if } I = [n] \\ \mathcal{A}, \text{ if } I \neq [n] \end{cases}$$
$$\widetilde{\mathrm{Sing}(\mathcal{A})} = \begin{cases} \mathcal{A}^\times, \text{ if } I = [n] \\ \varnothing, \text{ if } I \neq [n]. \end{cases} \tag{0.1.98}$$

(3) Relative Jacobson radical:

$$\begin{aligned} \mathfrak{J}_{(I)}(\mathcal{A}) = \widetilde{\mathfrak{J}(\mathcal{A})} \overset{\text{def}}{=} \mathfrak{m}_{(I)} &:= \prod_{j=1}^{n} \begin{cases} \mathfrak{m}_j, \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise} \end{cases} = \widetilde{\mathcal{A}^\times} = \bigcap_{j\in I} \mathfrak{M}_j = \\ &= \mathfrak{J}_{\mathcal{I}}(\mathcal{A}) = \widehat{\mathfrak{J}(\mathcal{A})} = \widehat{\mathfrak{J}(\mathcal{A})} \supseteq \mathfrak{J}(\mathcal{A}) \end{aligned} \tag{0.1.99}$$

and

$$\widetilde{\mathfrak{J}(\mathcal{A})} = \mathcal{A}_{\mathcal{I}}^\times = \mathcal{A}_{(I)}^\times. \tag{0.1.100}$$

(4) Relative nilradical:

$$\mathrm{nil}_{(I)}(\mathcal{A}) = \widetilde{\mathrm{nil}(\mathcal{A})} \overset{\text{def}}{=} \begin{cases} \mathrm{nil}(\mathcal{A}_j), \text{if } j \in I \\ \mathcal{A}_j, \text{otherwise} \end{cases} \tag{0.1.101}$$

but only

$$\mathrm{nil}_{\mathcal{I}}(\mathcal{A}) = \widehat{\mathrm{nil}(\mathcal{A})} = \widehat{\mathrm{nil}(\mathcal{A})} = \mathfrak{J}_{(I)}(\mathcal{A}) = \prod_{j=1}^{n} \begin{cases} \mathfrak{m}_j, \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise} \end{cases} \supseteq \mathrm{nil}_{(I)}(\mathcal{A}) \supseteq \mathrm{nil}(\mathcal{A}). \tag{0.1.102}$$

Moreover:

$$\widetilde{\mathrm{nil}(\mathcal{A})} = \widetilde{\mathfrak{J}(\mathcal{A})} = \mathcal{A}_{\mathcal{I}}^\times = \mathcal{A}_{(I)}^\times. \tag{0.1.103}$$

(5) Relative singular "class" of $a \in \mathcal{A}$:

$$[a]_{(I)} = [a]_{\mathcal{I}} = a + \mathrm{Sing}_{(I)}(\mathcal{A}). \tag{0.1.104}$$

Moreover:

$$\widehat{[a]_{\mathrm{sing}}} = \begin{cases} [a]_{\mathrm{sing}}, \text{ if } \mathcal{A} = (\mathcal{A}, \mathfrak{m}, \kappa) \\ \mathcal{A}, \text{ otherwise} \end{cases}$$
$$\widetilde{[a]_{\mathrm{sing}}} = \begin{cases} [a]^\times, \text{ if } \mathcal{A} = (\mathcal{A}, \mathfrak{m}, \kappa) \\ \varnothing, \text{ otherwise.} \end{cases} \tag{0.1.105}$$

and

$$\widehat{[a]_{\mathrm{sing}}} = \begin{cases} [a]_{\mathrm{sing}}, \text{ if } I = [n] \\ \mathcal{A}, \text{ if } I \neq [n] \end{cases}$$
$$\widetilde{[a]_{\mathrm{sing}}} = \begin{cases} [a]^\times, \text{ if } I = [n] \\ \varnothing, \text{ if } I \neq [n]. \end{cases} \tag{0.1.106}$$

(6) Relative multiplicative "class" of $a = (a_1, \dots, a_n) \in \mathcal{A}$:

$$[a]_{(I)}^\times = [a]_{\mathcal{I}}^\times = \prod_{j=1}^{n} \begin{cases} a_j + \mathcal{A}_j^\times, \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise} \end{cases} \overset{\text{def}}{=} \prod_{j=1}^{n} \begin{cases} [a_j]^\times, \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise} \end{cases} = \widehat{[a]^\times} = \widehat{[a]^\times}. \tag{0.1.107}$$

Moreover:

$$\widetilde{[a]^{\times}} = \prod_{j=1}^{n} \begin{cases} a_j + \mathfrak{m}_j, \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise} \end{cases} \overset{\text{def}}{=} \prod_{j=1}^{n} \begin{cases} [a_j]_{\text{sing}}, \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise} \end{cases} \tag{0.1.108}$$

and

$$\widetilde{[a]^{\times}} = [a]_{(I)}. \tag{0.1.109}$$

Proof: These characterizations mostly follow from Lemma-Definition 0.1.37, Lemma-Definition 0.1.44, and the previous discussion. More precisely:

To (1): By the preceeding discussion we have $\widehat{(\mathcal{A}^{\times})_j} = \widehat{\mathcal{A}_j^{\times}} = \mathcal{A}_j^{\times}$, thus the descriptions of $\widehat{\mathcal{A}^{\times}}$ and $\widetilde{\mathcal{A}^{\times}}$ follow from Eq. (0.1.86) and Eq. (0.1.87), respectively. That $\widehat{\mathcal{A}^{\times}}$ admits the same description follows from Eq. (0.1.81).

To (2): The first part $\mathrm{Sing}_{(I)}(\mathcal{A}) = \mathrm{Sing}_{\mathcal{I}}(\mathcal{A})$ follows from $\mathcal{A}^{\times}_{(I)} = \mathcal{A}^{\times}_{\mathcal{I}}$ by taking the complement. Furthermore, by (1) we have $\widetilde{\mathcal{A}^{\times}} \overset{\text{def}}{=} \widehat{\mathcal{A}^{\times}}^{c} = (\bigcap_{j\in I} \mathfrak{M}_j^c)^c = \bigcup_{j\in I} \mathfrak{M}_j = \mathrm{Sing}_{(I)}(\mathcal{A})$. The remaining characterizations in (2) follow from Lemma-Definition 0.1.44 since $\mathrm{Spm}(\mathcal{A}) < \infty$.

To (3): The left side follows from Lemma-Definition 0.1.37 since $\mathfrak{J}(\mathcal{A}_j) = \mathfrak{m}_j$ by locality of $\mathcal{A}_j$, $1 \leq j \leq n$, and the right hand side is just Lemma-Definition 0.1.44. They meet because clearly $\bigcap_{j\in I} \mathfrak{M}_i = \mathfrak{m}_{(I)} \overset{(1)}{=} \widetilde{\mathcal{A}^{\times}}$. The second statement is also just Lemma-Definition 0.1.44 and (1) again.

To (4): The inclusion $\mathrm{nil}_{\mathcal{I}}(\mathcal{A}) \supseteq \mathrm{nil}_{(I)}(\mathcal{A})$ proves itself from the two explicit descriptions.

To (5): The first part follows from (2) by definition. The last part follows from Lemma-Definition 0.1.44 as $\mathrm{Spm}(\mathcal{A}) < \infty$.

To (6): The first and second description follow from (1) and Lemma-Definition 0.1.44 (6). Furthermore, by (2) and Lemma-Definition 0.1.44 (6) we have $\widetilde{[a]^{\times}} = a + \widetilde{\mathcal{A}^{\times}} = a + \mathrm{Sing}_{(I)}(\mathcal{A}) \overset{\text{def}}{=} [a]_{(I)}$. □

Remark: The only "mistmatch" is (4), where we only have $\mathrm{nil}_{\mathcal{I}}(\mathcal{A}) \supseteq \mathrm{nil}_{(I)}(\mathcal{A})$. But that will disappear in the next Section 0.1.12 too.

In the special setting of this section, along with $\Pi_{(\tau)}$ (see Definition 0.1.25) one can also associate to a given $\tau\colon [m] \to [n]$ an obvious residue selection homomorphism in a completely analogous way, which for the purposes of the later analysis and consistency we call slightly differently:

Definition 0.1.48 (Spectral Selection Morphism)**:** Let $\mathcal{A} = \prod_{j=1}^{n}(\mathcal{A}_j, \mathfrak{m}_j, \kappa_j)$, $m \in \mathbb{N}$, and let $\tau\colon [m] \to [n]$ be an index parametrization.

(1) We put

$$\Sigma := \Sigma_{(\tau)} := (\chi_{\tau(1)}, \ldots, \chi_{\tau(m)}),$$

that is,

$$\Sigma_{(\tau)}\colon \mathcal{A} \to \prod_{k=1}^{m} \kappa_{\tau(k)}$$
$$(a_1, \ldots, a_n) \mapsto (\bar{a}_{\tau(1)}, \ldots, \bar{a}_{\tau(m)}),$$

where $\bar{a}_j := \chi_{\mathcal{A}_j}(a_j) \in \mathcal{A}_j/\mathfrak{m}_j = \kappa_j$, $1 \leq j \leq n$.

(2) Given a morphism $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ or an (anti-)representation $\lambda\colon \mathcal{A} \to \mathrm{End}_R(M)$ with corresponding

index parametrization τ_φ or τ_λ, respectively, in the sense of Section 0.1.5, we also put for short

$$\Sigma_\varphi := \Sigma_{(\tau_\varphi)} \text{ and } \Sigma_\lambda := \Sigma_{(\tau_\lambda)}.$$

Remarks:

(1) Analogously to Lemma-Definition 0.1.26, we can describe $\Sigma_{(\tau)}$ as

$$\mathcal{A} \xrightarrow{\chi_I} \kappa_I \xhookrightarrow{\delta_I} \textstyle\prod_{j\in I} \kappa_j^{m_j},$$

where $\delta_I := \times_{j\in I}\delta_j$ with $\delta_j \colon \kappa_j \hookrightarrow \kappa_j^{m_j}$, $\bar{a}_j \mapsto (\bar{a}_j, \dots, \bar{a}_j)$, $j \in I$, denoting the corresponding diagonal inclusions.

(2) The spectral selection morphism $\Sigma_{(\tau)} \colon \mathcal{A} \to \prod_{k=1}^m \kappa_{\tau(k)}$ is self-evidently a special case of a morphism between finite products of commutative local R-algebras, hence Corollary 0.1.45 applies: accordingly, its standard factorization is simply given by

$$\Sigma_{(\tau)} = (\times_{k=1}^m \chi_{\mathcal{A}_{\tau(k)}}) \circ \Pi_{(\tau)}.$$

0.1.11. The Residue Field

Lemma 0.1.49: Let $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocCAlg}_R$. If $M \in \mathsf{Mod}(\mathcal{A})$ is simple, then ${}_{\mathcal{A}}M \cong {}_{\mathcal{A}}\kappa$ in $\mathsf{Mod}(\mathcal{A})$. In particular, ${}_{\mathcal{A}}\kappa$ itself is a simple, hence an indecomposable $\mathcal{A}$-module.

Proof: Let $0 \neq \mu \in M$ and define a homomorphism of $\mathcal{A}$-modules $\hat{\mu} \colon \mathcal{A} \to M$, $a \mapsto a\mu$. Then $\hat{\mu} \neq 0$ since $\hat{\mu}(1_{\mathcal{A}}) = \mu \neq 0$. Hence $\operatorname{Im}(\hat{\mu}) \subseteq M$ is a non-trivial $\mathcal{A}$-submodule. By simplicity of M, it follows that $\hat{\mu}$ is surjective, thus $M \cong \mathcal{A}/\operatorname{Ker}(\hat{\mu})$, which is then also simple and therefore $\operatorname{Ker}(\hat{\mu}) = \mathfrak{m}$ by the correspondence for ideals of quotient rings. In other words, $M \cong \mathcal{A}/\mathfrak{m} \overset{\text{def}}{=} \kappa$ in $\mathsf{Mod}(\mathcal{A})$. □

Suppose now that $R = K$ is a field and $(\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocCAlg}_K$. Then we get a field extension κ/K via $K \hookrightarrow \mathcal{A} \twoheadrightarrow \kappa$, which is finite if $\mathcal{A} \in \mathsf{FdCAlg}_K$ (and the converse will also hold when $\mathcal{A}$ is local Artinian, see Lemma 0.1.58 later). Moreover, $\mathcal{A} \cong \kappa \oplus_K \mathfrak{m}$ in Vec_K since every SES $0 \to \mathfrak{m} \hookrightarrow \mathcal{A} \twoheadrightarrow \kappa \to 0$ of (possibly infinite-dimensional) K-vector spaces splits (assuming AC), hence $\dim_K \mathcal{A} = [\kappa : K] + \dim_K \mathfrak{m}$ (with standard conventions for ∞). Despite $\mathcal{A} \cong \kappa \oplus_K \mathfrak{m}$, we have that $\mathcal{A}$ is indecomposable over itself (as local rings are blocks, see Section 0.1.2), in particular ${}_{\mathcal{A}}\mathcal{A} \neq {}_{\mathcal{A}}\kappa \oplus_{\mathcal{A}} {}_{\mathcal{A}}\mathfrak{m}$. One can also see this directly as follows: $\forall\, z_1, z_2 \in \kappa\ \forall\, X_1, X_2 \in \mathfrak{m}\colon (z_1 \oplus X_1)(z_2 \oplus X_2) = (z_1 z_2) \oplus (z_1 X_2 + z_2 X_1 + X_1 X_2)$ with $z_1 X_2 + z_2 X_1 + X_1 X_2 \neq (z_1 \oplus X_1) X_2$ unless $X_1 = 0$ or $z_2 = 0$. Regardless of the indecomposability of ${}_{\mathcal{A}}\mathcal{A}$, note that ${}_{\mathcal{A}}\mathfrak{m}$ itself may or may not be decomposable in $\mathsf{Mod}(\mathcal{A})$:

Example:

(1) Decomposable ${}_{\mathcal{A}}\mathfrak{m}$: if $\mathcal{A} := K[x, y]/(x, y)^2$, then $\mathfrak{m} = (\bar{x}, \bar{y}) = K\bar{x} \oplus_K K\bar{y} = (\bar{x}) \oplus_{\mathcal{A}} (\bar{y})$.

(2) Indecomposable ${}_{\mathcal{A}}\mathfrak{m}$: if $\mathcal{A} := K[x]/(x^n)$, then $\mathfrak{m} = (\bar{x})$ is clearly *uniserial* and hence cannot be decomposable. □

As alluded in the Introduction, a key aspect of the analytic theory is the existence of a retraction diagram

$$\begin{tikzcd} \kappa \arrow[r, hook, "\iota_{\mathcal{A}}"] \arrow[dr, "\mathrm{id}_\kappa"'] & \mathcal{A} \arrow[d, two heads, "\chi_{\mathcal{A}}"] \\ & \kappa \end{tikzcd}$$

so we want to take a moment to discuss it. The issue can be posed as follows: given $(\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocCRing}$, when does the canonical quotient map $\chi_{\mathcal{A}} \colon \mathcal{A} \twoheadrightarrow \kappa$ admit a *section* $\iota_{\mathcal{A}} \colon \kappa \to \mathcal{A}$? Note that this is a stronger condition than merely requiring $\mathcal{A} \in \mathsf{LocCAlg}_\kappa$ via *some* $\iota_{\mathcal{A}} \colon \kappa \hookrightarrow \mathcal{A}$ since in general $\theta_{\mathcal{A}} := \chi_{\mathcal{A}} \circ \iota_{\mathcal{A}} \in \operatorname{End}_{\mathsf{CRing}}(\kappa)$ need not be an automorphism of κ (so far these are only homomorphisms of rings, not of κ-algebras):

$$\begin{tikzcd} \kappa \arrow[r, hook, "\iota_{\mathcal{A}}"] \arrow[dr, dashed, "\theta_{\mathcal{A}}"'] & \mathcal{A} \arrow[d, "\chi_{\mathcal{A}}"] \\ & \kappa \end{tikzcd}$$

As one can check, this diagram would also hold in $\mathsf{LocCAlg}_\kappa$, *provided* ${}_\kappa({}_{\mathcal{A}}\kappa)$ is taken with the κ-vector space structure given by restriction of scalars along $\theta_{\mathcal{A}}$ (since $\chi_{\mathcal{A}}$ then also becomes κ-linear[19]), which may disagree with the tautological vector space structure of κ. However, if in fact $\theta_{\mathcal{A}} \in \mathrm{Aut}(\kappa) \setminus \{\mathrm{id}_\kappa\}$, then replacing $\iota_{\mathcal{A}}$ by $\iota'_{\mathcal{A}} := \iota_{\mathcal{A}} \circ \theta_{\mathcal{A}}^{-1} \colon \kappa \hookrightarrow \mathcal{A}$ clearly gives now an honest retraction diagram in $\mathsf{LocCAlg}_\kappa$. A shortlist of fields κ, for which $\mathrm{End}(\kappa) = \mathrm{Aut}(\kappa)$ holds, includes $\mathbb{F}_q$, $\mathbb{Q}$, $\mathbb{R}$, as well as any (finite or not) *algebraic extension* thereof, in particular $\mathbb{C}/\mathbb{R}$, so in these cases we *automatically* have the equivalence $(\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocCAlg}_\kappa \Leftrightarrow \mathcal{A}$ admits a retraction diagram in $\mathsf{LocCAlg}_\kappa$. Moving on, the problem of existence of a section of $\chi_{\mathcal{A}}$ in $\mathsf{LocCRing}$ is intimately connected with the structure theory of *complete local rings*. To this end, recall that $(\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocCRing}$ is said to have a *coefficient field* if $\mathcal{A}$ contains (as a subring) an isomorphic copy of κ itself via some $\iota_{\mathcal{A}} \colon \kappa \hookrightarrow \mathcal{A}$ such that $\theta_{\mathcal{A}} \in \mathrm{Aut}(\kappa)$, see [Stacks, Tag 0326] for the more general context of coefficient *rings*. We begin with the following simpler observation:

Lemma 0.1.50 (Characteristic in Local Rings)**:** Let $(\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocCRing}$ be a local ring.

(1) We have:
 (i) If $\mathrm{char}(\kappa) = 0$, then $\mathrm{char}(\mathcal{A}) = 0$.
 (ii) If $\mathrm{char}(\mathcal{A}) > 0$, then $\mathrm{char}(\mathcal{A}) \geq \mathrm{char}(\kappa) > 0$.

(2) If $\mathcal{A}$ is equicharacteristic, then $\mathcal{A} \supseteq$ prime field of κ.

(3) If $\mathcal{A} \supseteq K$ a field, then $\mathcal{A}$ is equicharacteristic.

(4) In particular, $\mathrm{char}(\kappa) = 0 \Leftrightarrow \mathcal{A} \supseteq \mathbb{Q}$.

Proof: To (1): Let $q := \mathrm{char}(\mathcal{A}) > 0$. Then $0 = \chi_{\mathcal{A}}(q) = q\chi_{\mathcal{A}}(1_{\mathcal{A}}) = q1_\kappa$. If either $\mathrm{char}(\kappa) = 0 < q$ or $\mathrm{char}(\kappa) > q$, we get a contradiction.

To (2): By hypothesis, we have $\mathrm{char}(\mathcal{A}) = \mathrm{char}(\kappa) =: p$, where $p \in \{0, 2, 3, 5, \dots\}$. Since $\mathbb{Z}$ is an initial object in CRing, we have unique maps $\varphi \colon \mathbb{Z} \to \mathcal{A}$ and $\psi \colon \mathbb{Z} \to \kappa$ with the following diagram being *necessarily* commutative:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & \mathcal{A} \arrow[d, "\chi_{\mathcal{A}}"] \\ & \kappa \end{tikzcd}$$

Then $\mathrm{Ker}(\varphi) = p\mathbb{Z}$. If $p > 0$, then φ induces an inclusion $\bar{\varphi} \colon \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathcal{A}$, and we are done. If $p = 0$, then $\mathbb{Z} \overset{\varphi}{\hookrightarrow} \mathcal{A}$ and $\mathbb{Z} \setminus \{0\} \subseteq \mathcal{A} \setminus \mathfrak{m} = \mathcal{A}^\times$ as $\mathrm{Ker}(\psi) = p\mathbb{Z} = 0$, hence $\mathcal{A} \supseteq \mathbb{Q}$.

To (3): Similarly to (2), we get a commutative diagram

$$\begin{tikzcd} & & \mathcal{A} \arrow[dd, "\chi_{\mathcal{A}}"] \\ \mathbb{Z} \arrow[urr] \arrow[r] \arrow[drr] & K \arrow[ur, hook] \arrow[dr, dashed] & \\ & & \kappa \end{tikzcd}$$

hence $\mathrm{char}(\mathcal{A}) = \mathrm{char}(K) = \mathrm{char}(\kappa)$.

[19] given a κ-algebra $\mathcal{A}$, algebra homomorphisms $\chi \colon \mathcal{A} \to \kappa$ are sometimes called *weights* of $\mathcal{A}$, especially in the representation theory of Lie algebras, however we will stick to the term "characters" since it matches better the topological and analytic *character* of the present work, which doesn't have much weight;

To (4): If $\mathrm{char}(\kappa) = 0$, then $\mathcal{A}$ is automatically of equicharacteristic 0 by (1), hence $\mathcal{A} \supseteq \mathbb{Q}$ by (2). Conversely, if $\mathcal{A} \supseteq \mathbb{Q}$, then $\mathrm{char}(\kappa) = \mathrm{char}(\mathcal{A}) = \mathrm{char}(\mathbb{Q}) = 0$ by (3). □

Remark: Even though $\mathrm{char}(\kappa) = 0 \Rightarrow \mathrm{char}(\mathcal{A}) = 0$, one still speaks of "equicharacteristic 0" since the converse is not true in general: for $(\mathbb{Z}_{(p)}, (p), \mathbb{F}_p)$ we have $\mathrm{char}(\mathbb{Z}_{(p)}) = 0$, but $\mathrm{char}(\mathbb{F}_p) = p$.

Lemma 0.1.51: Let $(\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocCRing}$ be a complete. Then $\mathcal{A}$ is Henselian.

Proof: See for instance [Stacks, Tag 04GE], in particular [Stacks, Tag 04GM]. □

Lemma 0.1.52: Let $(\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocCRing}$ be complete with $\mathrm{char}(\kappa) = 0$. Then $\mathcal{A}$ contains a maximal subfield, and any such field is a coefficient field for $\mathcal{A}$, hence in particular $\mathcal{A} \in \mathsf{LocCAlg}_\kappa$.

Proof: This is well-known and follows from Lemma 0.1.51 and Zorn's lemma, using $\mathcal{A} \supseteq \mathbb{Q}$ from Lemma 0.1.50. See also [Nag75, Thm. 31.1 on p. 106] or [ZS60, Cor. 2 on p. 280]. □

But one can do even better:

Lemma 0.1.53: Let $(\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocCRing}$ be equicharacteristic and complete. Then $\mathcal{A}$ contains a coefficient field, hence in particular $\mathcal{A} \in \mathsf{LocCAlg}_\kappa$.

Proof: This follows from Cohen Structure Theorem, see [Stacks, Tag 0323]. □

Remarks:
1. (1) Both Lemma 0.1.52 and Lemma 0.1.53 employ Zorn's lemma under the hood and are therefore non-constructive in general.
2. (2) The induced κ-structure on $(\mathcal{A}, \mathfrak{m}, \kappa)$ is in general not unique.

Definition 0.1.54 (Spectral Retraction Diagram)**:** Let $(\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocCRing}$ and let κ be a coefficient field for it. Then the maps $\iota_\mathcal{A}$ and $\chi_\mathcal{A}$ from

$$\begin{tikzcd} \kappa \arrow[r, hook, "\iota_\mathcal{A}"] \arrow[dr, "\mathrm{id}_\kappa"'] & \mathcal{A} \arrow[d, two heads, "\chi_\mathcal{A}"] \\ & \kappa \end{tikzcd} \tag{0.1.110}$$

in $\mathsf{LocCAlg}_\kappa$ will be called *spectral section* or *spectral inclusion/embedding* and *character*, or *spectral retraction*, or *spectral projection* of $\mathcal{A}$, respectively.

Remarks:
1. (1) This is in line with Section 0.1.9. Moreover, see also Eq. (0.1.112) for some further justification of the terms.
2. (2) Using the spectral embedding, the map $\chi_\mathcal{A}$ is indeed a projection operator since $\chi_\mathcal{A}^2 = \chi_\mathcal{A}$.
3. (3) If additionally $\kappa = \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, then the pair $(\chi_\mathcal{A}, \iota_\mathcal{A})$ also defines a *topological retraction* onto $\mathbb{K} \subseteq \mathcal{A}$. As we will see later, diagram (0.1.110) heavily determines what is going on in the analytic theory. In the case $\kappa = \mathbb{K} = \mathbb{C}$, one can thus also speak of a *Henselian lifting of complex analysis in the plane.* □

Now let $\mathcal{A} = \prod_{j=1}^n (\mathcal{A}_j, \mathfrak{m}_j, \kappa_j) \in \mathsf{Art}_K$ with κ_j being a coefficient field for $\mathcal{A}_j$, $1 \leq j \leq n$, and let $I \subseteq [n]$. Furthermore, recall from Section 0.1.2 the multi-projections $\mathrm{pr}_I \colon \mathcal{A} \twoheadrightarrow \mathcal{A}_I$ and the (non-unital) multi-inclusions $\varepsilon_I \colon \mathcal{A}_I \hookrightarrow \mathcal{A}$. Having coefficients fields at our disposal, we can add to that mapping package the following two:

Definition 0.1.55: With the above notations:

(1) Spectral Nonunital Inclusions:

$$\begin{aligned} \iota_j \coloneqq \iota_j^{\mathcal{A}} \coloneqq \varepsilon_j \circ \iota_{\mathcal{A}_j} \colon \kappa_j &\hookrightarrow \mathcal{A}_j \hookrightarrow \mathcal{A} \\ z_j &\mapsto (0, \dots, 0, z_j, 0, \dots, 0). \end{aligned}$$

(2) Spectral Nonunital Multi-Inclusions:

$$\begin{aligned} \iota_I \coloneqq \iota_I^{\mathcal{A}} \coloneqq \varepsilon_I \circ (\times_{j \in I} \iota_{\mathcal{A}_j}) \colon \kappa_I &\hookrightarrow \mathcal{A}_I \hookrightarrow \mathcal{A} \\ (z_j)_{j \in I} &\mapsto (z_1, \dots, z_n), \end{aligned}$$

where $z_j = 0$ whenever $j \notin I$.

0.1.12. Artinian Rings & Algebras I: Commutative Algebra

Remark (Terminology)**:** Given a field K, or more generally $R \in \mathsf{CRing}$, there is sometimes an (implied) distinction in the literature between *Artinian* K-algebras, i.e. K-algebras that are Artin(ian) rings, on the one hand and *Artin* K-algebras on the other, with the latter taken to specifically mean those *Artinian* K-algebras $\mathcal{A}$ that are also *finitely generated*[20] over K, i.e. $\dim_K \mathcal{A} < \infty$. In particular, there is a difference in the meaning of *Artin* between Artin *rings* and Artin *algebras*. For instance, [ARS97] defines $\mathcal{A} \in \mathsf{Alg}_R$ to be *Artin* iff $R \in \mathsf{CRing}$ is itself *Artin(ian)* and $\mathcal{A}$ is finite as an R-module. Note that *Artin* R-algebras are always *Artinian*. While it certainly makes a lot of sense to have some such short designation for *finitely generated Artinian algebras*, especially since incidentally these are the kind we are ultimately intersted in, we will **not** subscribe to that convention so as to minimize potential confusion (including our own) as well as typographical slip-ups. □

Given any $\mathcal{A} \in \mathsf{CAlg}_R$ and any $M \in \mathsf{Mod}(\mathcal{A})$, we will denote by $\ell_{\mathcal{A}}(M)$ its length as an $\mathcal{A}$-module, that is, by definition ${}_{\mathcal{A}}M$ has finite composition (i.e. has a composition series) if and only if $\ell_{\mathcal{A}}(M) < \infty$. When this is the case, Jordan–Hölder theorem applies, giving a set of *invariants* for ${}_{\mathcal{A}}M$. We now fix our base ring $R \in \mathsf{CRing}$ to be a field $R \coloneqq K$, where our main point of interest lies in the case $\mathcal{A} \in \mathsf{FdCAlg}_K$. Then, by virtue of finite dimensionality, $\mathcal{A}$ is Artinian, that is, every descending chain of ideals becomes stationary.

Definition 0.1.56: Let $R \in \mathsf{CRing}$

(1) Art_R will denote the full subcategory of CAlg_R of *Artinian* R-algebras.
(2) LocArt_R will denote the full subcategory of Art_R of *local Artinian* R-algebras.

We summarize the most important facts about Artinian rings in the following:

Lemma 0.1.57 (Artinian Rings)**:** Let $R \in \mathsf{CRing}$ and $\mathcal{A} \in \mathsf{CAlg}_R$. The following are equivalent:

(i) $\mathcal{A} \in \mathsf{Art}_R$.
(ii) $\ell_{\mathcal{A}}(\mathcal{A}) =: \ell < \infty$, i.e. $\mathcal{A}$ admits a composition series of ideals $\mathcal{A} =: \mathfrak{a}_\ell \supsetneqq \cdots \supsetneqq \mathfrak{a}_1 \supsetneqq \mathfrak{a}_0 \coloneqq 0$.
(iii) $\mathcal{A}$ is Noetherian and $\mathrm{Kdim}(\mathcal{A}) = 0$, i.e. $\mathrm{Spec}(\mathcal{A}) = \mathrm{Spm}(\mathcal{A})$. In particular, $\mathfrak{J}(\mathcal{A}) = \mathrm{nil}(\mathcal{A})$.
(iv) $\mathcal{A}$ is Noetherian and $\mathrm{Spec}(\mathcal{A})$ is discrete (and finite).
(v) $\mathcal{A} \cong \prod_{j=1}^n (\mathcal{A}_j, \mathfrak{m}_j, \kappa_j)$ uniquely up to index permutations, where $(\mathcal{A}_j, \mathfrak{m}_j, \kappa_j) \in \mathsf{LocArt}_R$, $1 \leq j \leq n$.

Proof: This is standard, see for instance [AM69, Ch. 8]. Observe, however, that the R-algebra structure is naturally inhereted by the local rings $(\mathcal{A}_j, \mathfrak{m}_j, \kappa_j)$, $1 \leq j \leq n$, so that the product is valid not only in CRing (as is the usual statement for Artin rings), but also in CAlg_R. □

In particular, it often suffices to consider only the case of local Artinian rings and K-algebras $\mathcal{A} =$

[20] recall that every (module-)finite R-algebra is algebra-finite (also called of finite type), and for *Artinian* K-algebras the converse holds as well, see [AM69, Ex. 8.3], hence for *Artinian* algebras *over a field* the distinction between "finitely generated *as a module*" and "finitely generated *as an algebra*" disappears;

$(\mathcal{A}, \mathfrak{m}, \kappa)$. This includes the construction of (counter-)examples.

Remarks: The characterization in the above Lemma 0.1.57 is optimal in the following sense:

(1) By [AM69, Ex. 8.2] finiteness of $\operatorname{Spec}(\mathcal{A})$ in (iv) is indeed optional (implied).

(2) Note that if $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \kappa)$ is Artinian, then $\mathfrak{m}$ is not just a *nil ideal*, i.e. $\mathfrak{m} \subseteq \operatorname{nil}(\mathcal{A}) = \bigcap \operatorname{Spec}(\mathcal{A})$, but is *itself nilpotent* of order $\leq \dim_K \mathcal{A}$. While every nilpotent ideal is automatically a nil ideal, the converse need not hold: if $\mathcal{A} := K[x_1, x_2, \ldots]/(x_1^2, x_2^2, \ldots)$ and $\mathfrak{m} := (\bar{x}_1, \bar{x}_2, \ldots)$, then $\mathfrak{m}$ is clearly a nil (maximal) ideal, but not nilpotent, since $\forall\, k \in \mathbb{N}\colon \mathfrak{m}^k \ni \bar{x}_1\bar{x}_2 \ldots \bar{x}_k \neq 0$. Moreover, we have the general fact that any $\mathcal{A} \in \mathsf{CRing}$ that contains a *nil* maximal ideal $\mathfrak{m}$ satisfies $\operatorname{Spec}(\mathcal{A}) = \operatorname{Spm}(\mathcal{A}) = \{\mathfrak{m}\}$ (hence finite and discrete), in particular $(\mathcal{A}, \mathfrak{m})$ is local of $\operatorname{Kdim}(\mathcal{A}) = 0$. Thus, the above $\mathcal{A}$ is also an example for a ring with all these properties that is also not Noetherian (since $\mathfrak{m}$ is not finitely generated) and hence not Artinian. Nevertheless, *nility* of maximal ideals is sufficient to ensure that every morphism between local rings is *automatically local*, which is otherwise not necessarily the case. In particular, every morphism $(\mathcal{A}, \mathfrak{m}, \kappa) \xrightarrow{\varphi} (\mathcal{B}, \mathfrak{n}, \lambda)$ of local Artinian rings is automatically local, i.e. $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$. This will be crucial down the road.

(3) In continuation of (2), requiring further $\mathfrak{m}$ to be *nilpotent* instead of only *nil* is still not enough: if we take $\mathcal{A} := K[x_1, x_2, \ldots]/(x_1, x_2, \ldots)^2$ and $\mathfrak{m} := (\bar{x}_1, \bar{x}_2, \ldots)$ again, then $\mathfrak{m} \in \operatorname{Spm}(\mathcal{A})$ as $\mathcal{A}/\mathfrak{m} \cong K$ and it is not difficult to see that $\mathfrak{m}^2 = 0$, but $\mathfrak{m}$ is clearly once again not finitely generated. This shows that indeed one cannot drop the Noetherian hypothesis in (iii) and (iv).

(4) Furthermore, one cannot drop the $\operatorname{Kdim}(\mathcal{A}) = 0$ hypothesis in (iii) either, since there exist Noetherian rings with *nil* Jacobson radicals $\mathfrak{J}(\mathcal{A}) \neq 0$, but of *positive* Krull dimension. Indeed, consider $\mathcal{A} := \mathbb{C}[x, y]/(x^2)$ and $\pi\colon \mathcal{A} \twoheadrightarrow \mathcal{A}/(\bar{x}) \cong \mathbb{C}[y]$. Since $\bar{x} \in \operatorname{nil}(\mathcal{A})$, we have automatically $\forall\, \mathfrak{p} \in \operatorname{Spec}(\mathcal{A})\colon \mathfrak{p} \ni \bar{x}$. By the correspondence for prime ideals of quotient rings, we thus get induced ($\subseteq$-preserving) bijections $\operatorname{Spec}(f)\colon \operatorname{Spec}(\mathbb{C}[y]) \xrightarrow{\sim} \operatorname{Spec}(\mathcal{A})$, $\operatorname{Spm}(f)\colon \operatorname{Spm}(\mathbb{C}[y]) \xrightarrow{\sim} \operatorname{Spm}(\mathcal{A})$, $\mathfrak{q} \mapsto \pi^{-1}(\mathfrak{q})$, hence $\operatorname{Spm}(\mathcal{A}) = \{(\bar{x}, \bar{y} - c) : c \in \mathbb{C}\}$ (as $\mathbb{C}$ is algebraically closed) and $\operatorname{Spec}(\mathcal{A}) = \{(\bar{x})\} \cup \operatorname{Spm}(\mathcal{A})$. In particular, $\operatorname{Kdim}(\mathcal{A}) = \operatorname{Kdim}(\mathbb{C}[y]) = 1$ (given explicitly by the chains $(\bar{x}) \subset (\bar{x}, \bar{y} - c)$, $c \in \mathbb{C}$) and $\mathfrak{J}(\mathcal{A}) = \bigcap_{c \in \mathbb{C}} (\bar{x}, \bar{y} - c) = (\bar{x}) \neq 0$ (since the elements of $(\bar{x}, \bar{y} - c)$ are of the form $\bar{x} f_c(\bar{y}) + g_c(\bar{y})$ for some $f_c(\bar{y}), g_c(\bar{y}) \in \mathbb{C}[\bar{y}]$, $c \in \mathbb{C}$), hence $\mathfrak{J}(\mathcal{A})^2 = 0$. Note that even though by (2) it is not possible to exhibit a *local* such example, there is in fact an abundance of such counter-examples: by the *Rabinowitsch form* of Hilbert's Nullstellensatz every finitely generated K-algebra $\mathcal{A}$ (automatically Noetherian by Hilbert's Basissatz) is *Jacobson*, hence in particular $\operatorname{nil}(\mathcal{A}) = \mathfrak{J}(\mathcal{A})$.

(5) A non-trivial local Artinian K-*algebra* need not have a non-trivial radical $\mathfrak{m}$: indeed, fields are precisely the local Artinian rings with trivial radicals. In particular, any field extension $K \subset L$ is a local Artinian K-algebra with trivial radical. This also helps narrow down the next point.

(6) We don't want to leave the impression that Artinian K-algebras are only finite-dimensional, though. Namely:
   - Infinite-dimensional Artinian K-algebras with trivial radical are precisely infinite field extensions $K \subset L$.
   - Infinite-dimensional Artinian K-algebras with non-trivial radicals: for instance, $L[x]/(x^2)$, where L/K is again any infinite extension.

   On the other hand, in a sense these are all types of infinite-dimensional Artinian K-algebras since the condition $[\kappa : K] = \infty$ is both sufficient (see Section 0.1.11) and necessary for infinite-dimensinality of Artinian K-algebras as we will show in Lemma 0.1.58 below.

Lemma 0.1.58: Let K be a field and $(\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocArt}_K$. Then:
(1) $\mathcal{A} \in \mathsf{LocArt}_\kappa$.
(2) In fact, $\mathcal{A} \in \mathsf{FdCAlg}_\kappa$ with $\dim_\kappa \mathcal{A} = \ell_\mathcal{A}(\mathcal{A})$.

Proof: To (1): Since $\mathfrak{m}$ is nilpotent, $\mathcal{A}$ is in particular a (Noetherian) *complete local ring.* As $\mathcal{A} \supseteq K$, $\mathcal{A}$ is also equicharacteristic and hence admits a coefficient field by Lemma 0.1.53.

To (2): Since $\mathcal{A}$ is Artinian, it admits a composition series of ideals $\mathcal{A} =: \mathfrak{a}_\ell \supsetneqq \mathfrak{m} =: \mathfrak{a}_{\ell-1} \supsetneqq \cdots \supsetneqq \mathfrak{a}_1 \supsetneqq \mathfrak{a}_0 := 0$, where $\ell := \ell_\mathcal{A}(\mathcal{A}) < \infty$ by definition. This means that $\forall\, 1 \leq i \leq \ell$: $\mathfrak{a}_i/\mathfrak{a}_{i-1} \in \mathsf{Mod}(\mathcal{A})$ is simple, hence by locality and Lemma 0.1.49 we have $\forall\, 1 \leq i \leq \ell$: $\mathfrak{a}_i/\mathfrak{a}_{i-1} \cong {}_\mathcal{A}\kappa$ in $\mathsf{Mod}(\mathcal{A})$, starting with $\mathfrak{a}_1 \cong {}_\mathcal{A}\kappa$. Since κ is also a *coefficient field* for $\mathcal{A}$ by (1), we have that ${}_\kappa({}_\mathcal{A}\kappa)$ is isomorphic to the tautological κ-vector space structure on κ (the κ-linear isomorphism being given by $\theta_\mathcal{A}$ itself, see Section 0.1.11), hence $\dim_\kappa {}_\kappa({}_\mathcal{A}\kappa) = 1$, which completes the proof. □

Corollary 0.1.59: If $(\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocArt}_\mathbb{Z}$ with $\operatorname{char}(\kappa) = 0$, then $\mathcal{A} \in \mathsf{FdCAlg}_\kappa$ and $\dim_\kappa \mathcal{A} = \ell_\mathcal{A}(\mathcal{A})$.

Proof: By Lemma 0.1.52 we have that κ is automatically a coefficient field for $\mathcal{A}$, and the claim now follows from Lemma 0.1.58 (2). □

Remarks:
(1) Here we see once again the conceptual difference between Artin *rings* and Artin K-*algebras*: the condition $\mathcal{A} \supset K$ is quite strong as it makes $\mathcal{A}$ equicharacteristic!
(2) $(\mathbb{Z}/(p^n), (\bar{p}), \mathbb{F}_p)$, $n > 1$, is an example of a local Artinian ring (since finite) that is not an $\mathbb{F}_p$-algebra, in particular $\chi\colon \mathbb{Z}/(p^n) \twoheadrightarrow \mathbb{F}_p$ does not admit a section. Indeed, $\mathbb{Z}/(p^n)$ is not even a well-defined $\mathbb{F}_p$-vector space, since $\operatorname{char}(\mathbb{F}_p) = p \neq p^n = \operatorname{char}(\mathbb{Z}/(p^n))$.

As we remarked in Section 0.1.10, the proofs of Lemma 0.1.52 and Lemma 0.1.53, and hence that of Lemma 0.1.58 and Corollary 0.1.59 are somewhat non-constructive. We include here two, more constructive proofs applicable to our special case of interest.

Lemma 0.1.60: Let $(\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{FdCAlg}_K$. If K is a perfect field, then $\mathcal{A} \in \mathsf{FdCAlg}_\kappa$, extending the K-algebra structure.

Proof: Since $\mathcal{A} = \kappa \oplus_K \mathfrak{m}$, it suffices to show that $\mathfrak{m}$ is a vector space over κ in a way that extends the K-linear structure. As K is a perfect field, the finite extension κ/K is separable. Thus, by the Primitive Element Theorem we have $\kappa = K[\bar{\alpha}]$ for some $0 \neq \bar{\alpha} \in \mathcal{A}/\mathfrak{m} \overset{\text{def}}{=} \kappa$. Now, pick a representative $\alpha \in \bar{\alpha}$, $\alpha \in \mathcal{A}^\times$, and define a K-linear action $\kappa \times \mathfrak{m} \to \mathfrak{m}$ via $\bar{\alpha} \cdot x := \alpha x$. This defines a κ-linear structure on $\mathfrak{m}$, *depending* on α and extending the previous K-linear structure. □

For the special case of $\kappa = \mathbb{C}$, we present a constructive proof[21] of Lemma 0.1.58 that unlike Lemma 0.1.60 does not need finite dimensionality of $\mathcal{A}$:

Lemma 0.1.61: Let $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C}) \in \mathsf{LocArt}_\mathbb{R}$. Then $\mathcal{A}$ is automatically a $\mathbb{C}$-algebra, extending the $\mathbb{R}$-algebra structure.

Proof: Since $\mathcal{A}$ is Artinian, we have $\operatorname{nil}(\mathcal{A}) = \mathfrak{m}$. We are going to plug nilpotent elements into formal power series to explicitly *lift*[22] a root of $P(t) := t^2 + 1 \in \mathbb{R}[t] \subset \mathcal{A}[t]$ from $\kappa = \mathbb{C}$ to $\mathcal{A}$. As $\kappa = \mathbb{C}$ is algebraically closed, there exists $a \in \mathcal{A}$ such that $\bar{a}^2 = -1$ in κ, i.e. $a^2 + 1 =: x \in \mathfrak{m}$ is nilpotent. Therefore, $-a^2 = 1 - x$ has an inverse $1 + y = 1 + x + x^2 + \ldots$ for $y \in \mathfrak{m}$, which in turn has a square root $1 + z = 1 + y/2 - y^2/8 + y^3/16 - \ldots$ for some $z \in \mathfrak{m}$. Now setting $a' := a(1+z)$, we get $a'^2 = a^2(1+z)^2 = a^2(1+y) = a^2/(1-x) = -1$ as desired. □

[21] I learned this proof from Jeremy Rickard, see [MSE01];
[22] cf. Hensel's lemma;

Lemma 0.1.62 (Uniqueness of the Complex Structure on $\mathcal{A}$)**:**

(1) If $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C})$, then the induced $\mathbb{C}$-structure on $\mathcal{A}$ is unique up to a sign (i.e. up to conjugation).

(2) If $\mathcal{A} = \prod_{j=1}^{n}(\mathcal{A}_j, \mathfrak{m}_j, \mathbb{C})$ and $J_1 \in \mathcal{A}_1, \ldots, J_n \in \mathcal{A}_n$ are corresponding complex structures, then $J = (\pm J_1, \ldots, \pm J_n)$ are all complex structures on $\mathcal{A}$.

Proof: To (1): By Corollary 0.1.59 there exists $J \in \operatorname{End}_{\mathcal{A}}(\mathcal{A}) = \mathcal{Z}(\mathcal{A}) = \mathcal{A}$ such that $J^2 = -1_{\mathcal{A}}$. Now consider the equation $Z^2 = -1$ in $\mathcal{A}$. Writing $Z = z \oplus X$ with $z \in \mathbb{C}$ (induced by J) and $X \in \mathfrak{m}$, we have $Z^2 = (x \oplus X)^2 = z^2 \oplus (2zX + X^2) = -1$, hence $2zX + X^2 = X(2z + X) = 0$ in $\mathfrak{m}$ and $z = \pm J$. But then $2z + X \in \mathcal{A}^{\times}$ (since $z \neq 0$), hence $X = 0$. Thus $\pm J$ are all the solutions to $Z^2 = -1$ in $\mathcal{A}$.

To (2): immediate from (1). □

Problem 2 (Galois Descent)**:** Does $(\mathcal{A}, \mathfrak{m}, \mathbb{C}) \in \mathsf{LocArt}_{\mathbb{R}}$ always have a real form and, if yes, is it unique? If yes, does this extend to other field extensions in place of $\mathbb{C}/\mathbb{R}$? □

Finally, we mention that there is a special class of local Artin rings that traditionally bears a different name in differential geometry and are usually defined in a somewhat clumsy, *ad hoc* way, but in view of Corollary 0.1.59 one can do better definition-wise:

Definition 0.1.63: A Weil algebra[23] is simply an object $(\mathcal{A}, \mathfrak{m}, \mathbb{R}) \in \mathsf{LocArt}_{\mathbb{Z}}$.

For details we refer the reader to [KMS93, Def. 35.2 on p. 298] and onwards. However, for a variety of reasons we propose that a better naming convention would be Artin–Weil $\mathbb{R}$-algebras (that is, also specifying the coefficient field), so that, following the obvious naming pattern, one could also say that the main setting of the present work is that of Artin–Weil $\mathbb{C}$-algebras. Carchedi and Roytenberg gave the following rather general definition:

Definition 0.1.64: Let $R \in \mathsf{CRing}$.

(1) $\mathcal{A} \in \mathsf{CAlg}_R$ is called a formal Weil R-algebra if we have a surjective homomorphism $\mathcal{A} \xrightarrow{\chi_{\mathcal{A}}} R$ in CAlg_R such that $\chi_{\mathcal{A}}$ admits a section in CAlg_R and $\operatorname{Ker}(\chi_{\mathcal{A}}) \lhd \mathcal{A}$ is nilpotent, i.e. $\mathcal{A}$ is a split nilpotent extension of R in CAlg_R.

(2) $\mathcal{A} \in \mathsf{CAlg}_R$ is called a Weil R-algebra if $\mathcal{A}$ is a formal Weil R-algebra that is also finitely generated in $\mathsf{Mod}(R)$.

We refer the reader to [CR13, Def. 2.39 on p. 16] for more details, but we would like to highlight once again that all homomorphisms in the above definition are taken in CAlg_R rather than CRing (so we have again a distinction between Weil *algebras* vs. "Weil *rings*" similar in spirit to that of Artin *algebras* vs. Artin *rings* from the beginning of the section). Note moreover that $\operatorname{Ker}(\chi_{\mathcal{A}})$ need not be even maximal (for example, this would be the case if R does not contain a field) and that a Weil R-algebra $\mathcal{A}$ need not be Artinian unless we require R itself to be Artinian.

Definition 0.1.65 (Height)**:** Let $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocArt}_R$ and $M \in \mathsf{Mod}(\mathcal{A})$.

(1) ${}_{\mathcal{A}}M$-Height of $\mathcal{A}$ (or $\mathfrak{m}$):
$$\nu_M(\mathfrak{m}) := \min\{k \in \mathbb{N} : \mathfrak{m}^k M = 0\} \overset{\text{def}}{=} \min\{k \in \mathbb{N} : \mathfrak{m}^k \subseteq \operatorname{Ann}_{\mathcal{A}}(M)\}.$$

(2) Height[24] of $\mathcal{A}$:
$$\nu := \nu(\mathcal{A}) := \nu_{\mathcal{A}}(\mathfrak{m}),$$

[23] not to be confused with the Weil algebra $W(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$;

[24] should not be confused with $\operatorname{ht}(\mathfrak{p})$, the *height of a prime ideal*, which is the Krull dimension of its localization;

i.e. the order/degree of nilpotency of $\mathfrak{m}$.

(3) Height of a morphism: if $(\mathcal{A}, \mathfrak{m}, \kappa) \xrightarrow{\varphi} (\mathcal{B}, \mathfrak{n}, \lambda)$ in LocArt_R, then

$$\nu_\varphi := \nu_{\mathcal{B}}(\mathfrak{m}).$$

Definition 0.1.66 (Width)**:** Let $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocArt}_\kappa$.

(1) Width(s) of $\mathcal{A}$:

$$d_j := \dim_\kappa \mathfrak{m}^j/\mathfrak{m}^{j+1},\ 1 \le j \le \nu - 1.$$

(2) Cumulative widths of $\mathcal{A}$:

$$l_j := 2 + \sum_{i=1}^{j-1} d_i,\ 1 \le j \le \nu.$$

Remarks:

(1) The terms "height" and "width" of $\mathcal{A}$ appear to go back to [Wei53] and are rather standard in the literature on manifolds over Weil algebras (incl. Weil bundles and Weil functors). Piggybacking on that, we have slightly extended them to include our setting.

(2) In particular, d_1 is the dimension of the *Zariski cotangent space* $\mathfrak{m}/\mathfrak{m}^2$ and $d_{\nu-1} = \dim_\kappa \mathfrak{m}^{\nu-1}$.

(3) Observe that $l_1 = 2$, $l_{j+1} - l_j = d_j \ge 1$, and $l_\nu = \dim_\kappa \mathcal{A} + 1$. □

Now let K be a field and $(\mathcal{A}, \mathfrak{m}, K) \in \mathsf{LocArt}_K$ (cosmetic change of notation). Since each $\mathfrak{m}^j$ is a finitely generated $\mathcal{A}$-module, each choice of K-basis for $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ lifts to a generating set (over $\mathcal{A}$) of $\mathfrak{m}^j$. In the case of $\mathfrak{m}/\mathfrak{m}^2$, such a generating set for $\mathfrak{m}$ is also sometimes called a *pseudo-basis* of $\mathfrak{m}$ (or $\mathcal{A}$). Given $(\mathcal{A}, \mathfrak{m}, K) \xrightarrow{\varphi} (\mathcal{B}, \mathfrak{n}, K)$ in LocArt_K, φ is automatically local and induces a K-linear map $\varphi_{\mathrm{cot}} \colon \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2$ of Zariski cotangent spaces. As usual, $(-)_{\mathrm{cot}}$ is a functor $\mathsf{LocArt}_K \to \mathsf{Vec}_K$. Furthermore, every $(\mathcal{A}, \mathfrak{m}, K) \in \mathsf{LocArt}_K$ is a finitely generated K-algebra. It follows from Nakayama's Lemma that the minimum of generators of $\mathcal{A}$ is precisely the *width* $\mu := \mu(\mathcal{A}) := d_1 \overset{\text{def}}{=} \dim_K \mathfrak{m}/\mathfrak{m}^2$. In particular, since in general $\mu > 0 = \mathrm{Kdim}(\mathcal{A})$, $\mathcal{A}$ is never a regular local ring unless it is a field. Using the width μ and the height ν, there is a somewhat canonical way to represent $\mathcal{A}$:

Definition 0.1.67: We will call $(K_{\mu,\nu}, \mathfrak{m}_{\mu,\nu}, K) \in \mathsf{LocArt}_K$ given by

$$K_{\mu,\nu} := K[x_1, \dots, x_\mu]/(x_1, \dots, x_\mu)^\nu$$

and $\mathfrak{m}_{\mu,\nu} := (\bar{x}_1, \dots, \bar{x}_\mu) \lhd K_{\mu,\nu}$ the *jet K-algebra*[25] of type (μ, ν). We also set $K_\nu := K_{1,\nu} \overset{\text{def}}{=} K[x]/(x^\nu)$.

Remark: Observe that indeed $\mu(\mathcal{A}_{\mu,\nu}) = \mu$ and $\nu(\mathcal{A}_{\mu,\nu}) = \nu$. □

Lemma 0.1.68: Let $(\mathcal{A}, \mathfrak{m}, K) \in \mathsf{LocArt}_K$ with $\mu := \mu(\mathcal{A})$ and $\nu := \nu(\mathcal{A})$ and let $\nu_0 \ge \nu$. Then:

(1) $(\mathcal{A}, \mathfrak{m}, K)$ is a quotient of $(K_{\mu,\nu_0}, \mathfrak{m}_{\mu,\nu_0}, K)$.

(2) Every such presentation is unique up to $\sigma \in \mathrm{Aut}_K(K_{\mu,\nu_0})$, i.e. we have a commutative diagram

$$\begin{tikzcd} K_{\mu,\nu_0} \arrow[rr, dashed, "\cong", "\sigma"'] \arrow[dr, two heads, "\pi_1"'] & & K_{\mu,\nu_0} \arrow[dl, two heads, "\pi_2"] \\ & \mathcal{A} & \end{tikzcd} \tag{0.1.111}$$

Proof: To (1): Let $\{\bar{a}_1, \dots, \bar{a}_\mu\}$ be a K-basis of $\mathfrak{m}/\mathfrak{m}^2$ giving $\mathcal{A} = K[a_1, \dots, a_\mu]$. Consider the corresponding *evaluation* epimorphism $\pi \colon K[x_1, \dots, x_\mu] \twoheadrightarrow \mathcal{A}$, $P \mapsto P(a_1, \dots, a_\mu)$. Since $\nu \le \nu_0$, we

[25] as it is indeed closely related to jets, but there doesn't appear to be a standard name for it despite its ubiquity;

have $(x_1, \ldots, x_\mu)^{\nu_0} \subseteq \operatorname{Ker}(\pi)$, hence π factors through K_{μ,ν_0} as

$$\begin{tikzcd} K[x_1, \ldots, x_\mu] \arrow[r, two heads, "\pi"] \arrow[d, two heads, "q"'] & \mathcal{A} \\ K_{\mu,\nu_0} \arrow[ur, dashed, two heads, "\bar{\pi}"'] & \end{tikzcd}$$

where q is the canonical quotient map. Thus, $\mathcal{A} \cong K_{\mu,\nu_0}/\operatorname{Ker}(\bar{\pi})$.

To (2): Write $K_{\mu,\nu_0} \overset{\text{def}}{=} K[\bar{x}_1, \ldots, \bar{x}_\mu]$ and define $a_i := \pi_1(\bar{x}_i)$, $1 \leq i \leq \mu$, hence $\mathcal{A} = K[a_1, \ldots, a_\mu]$. On the other hand, let $\bar{P}_1, \ldots, \bar{P}_\mu \in K_{\mu,\nu_0}$ be arbitrary such that $\pi_2(\bar{P}_i) = a_i$, $1 \leq i \leq \mu$, and consider the *substitution* homomorphism $\sigma\colon K_{\mu,\nu_0} \to K_{\mu,\nu_0}$, $\bar{x}_i \mapsto \bar{P}_i$, $1 \leq i \leq \mu$. We have $\pi_2 \circ \sigma \overset{\text{def}}{=} \pi_1$, hence by and functoriality of $(-)_{\text{cot}}$ we get a commutative diagram

$$\begin{tikzcd} \mathfrak{m}_{\mu,\nu_0}/\mathfrak{m}^2_{\mu,\nu_0} \arrow[rr, "\sigma_{\text{cot}}"] \arrow[dr, "(\pi_1)_{\text{cot}}"'] & & \mathfrak{m}_{\mu,\nu_0}/\mathfrak{m}^2_{\mu,\nu_0} \arrow[dl, "(\pi_2)_{\text{cot}}"] \\ & \mathfrak{m}/\mathfrak{m}^2 & \end{tikzcd}$$

in FdVec_K, where $\sigma_{\text{cot}}\colon [\bar{x}_i] \mapsto [\bar{P}_i]$, $1 \leq i \leq \mu$. But $\dim_K \mathfrak{m}_{\mu,\nu_0}/\mathfrak{m}^2_{\mu,\nu_0} \overset{\text{def}}{=} \mu \overset{\text{def}}{=} \dim_K \mathfrak{m}/\mathfrak{m}^2$, hence $(\pi_1)_{\text{cot}}$ and $(\pi_2)_{\text{cot}}$ are K-linear isomorphisms, so $\sigma_{\text{cot}} \in \operatorname{GL}(\mathfrak{m}_{\mu,\nu_0}/\mathfrak{m}^2_{\mu,\nu_0})$. Since $\{[\bar{x}_i] : 1 \leq i \leq \mu\}$ is a K-basis, it follows that so is $\{[\bar{P}_i] : 1 \leq i \leq \mu\}$, hence $\mathcal{A} = K[\bar{P}_1, \ldots, \bar{P}_\mu]$, that is, σ is surjective. Since σ is in particular K-linear and $\dim_K K_{\mu,\nu_0} < \infty$, it follows that σ is also injective, which completes the proof. □

Lemma 0.1.69 (Jordan–Hölder Basis)**:** Let $(\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocArt}_\kappa$ with $\dim_\kappa \mathcal{A} = d$. Then $\mathcal{A}$ has a basis $\{a_1 = 1_\mathcal{A}, a_2, \ldots, a_d\}$ with structure tensor $(\alpha^k_{j\ell})$ such that:
 (i) $\forall\, 1 \leq k, \ell \leq d$: $\alpha^k_{1\ell} = \alpha^k_{\ell 1} = \delta^k_\ell$;
 (ii) $\forall\, j, \ell \geq 2\ \forall\, k \leq \max\{j, \ell\}$: $\alpha^k_{j\ell} = \alpha^k_{\ell j} = 0$.

Proof: By Lemma 0.1.58 there exists a Jordan–Hölder composition series of ideals of length d, say, $\mathcal{A} =: \mathfrak{a}_1 \supsetneqq \mathfrak{m} =: \mathfrak{a}_2 \supsetneqq \cdots \supsetneqq \mathfrak{a}_d \supsetneqq \mathfrak{a}_{d+1} := 0$. Pick $a_1 := 1_\mathcal{A} \in \mathfrak{a}_1 \setminus \mathfrak{a}_2$, $a_2 \in \mathfrak{a}_2 \setminus \mathfrak{a}_3$, $a_i \in \mathfrak{a}_i \setminus \mathfrak{a}_{i+1}$, $1 \leq i \leq d$, and notice that the choice of a_1 yields (i). Without loss of generality suppose now that $2 \leq j \leq \ell$. Then $a_j a_\ell \in \mathfrak{a}_\ell$, hence $\forall\, k < \ell$: $\alpha^k_{j\ell} = 0$. We are left to check the case $k = \ell$. We have

$$(a_j - \alpha^\ell_{j\ell}) a_\ell = \sum_{k=\ell+1}^{d} \alpha^k_{j\ell} a_k \in \mathfrak{a}_{\ell+1}$$

If $\alpha^k_{j\ell} \neq 0$, then $a_j - \alpha^k_{j\ell} \in \mathcal{A}^\times$, therefore $a_\ell \in \mathfrak{a}_{\ell+1}$, a contradiction. This proves (ii). □

Remark: In particular, if $d \geq 2$, then $a_d^2 = 0$, and if $d \geq 3$, then $a_{d-1} a_d = 0$.

Corollary 0.1.70 (Triangular Form)**:** Let $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocArt}_\kappa$ with $\dim_\kappa \mathcal{A} = d$. Then $\mathcal{A}$ is triangulable. More precisely, it is isomorphic to a (commutative) κ-algebra of lower-triangular Toeplitz matrices of size d

$$\begin{pmatrix} x & & 0 \\ & \ddots & \\ * & & x \end{pmatrix}, \tag{0.1.112}$$

whose regular representation coincides with the matrix subalgebra itself.

Proof: This is immediate from Lemma 0.1.69, where also observe that the regular representation of an upper-triangular algebra is in fact lower-triangular. □

Remarks:

(1) For self-evident reasons we will also call this the *stable* regular representation of $\mathcal{A}$.

(2) In the special case of algebraically closed κ, another proof follows from [HK71, p. 199-204]: indeed, since κ is now algebraically closed, all elements of $\mathcal{A}$ are triangulable as their minimal polynomials split, and in fact they are simultaneously triangulable since they are mutually commuting, hence $\mathcal{A}$ itself is triangulable. Since any local ring is connected, triangulability forces $\mathcal{A}$ to be Toeplitz. $\square$

From a rather formal point of view, one criterion for niceness of a basis is the number of zeros among the structure constants. In this regard, the choice in Lemma 0.1.69 is not necessarily an optimal one. To improve on this, one might use the following:

Lemma 0.1.71: Let $(\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocArt}_\kappa$ with $\dim_\kappa \mathcal{A} = d$. Then $\mathcal{A}$ admits a κ-basis $\{a_1 := 1_\mathcal{A}, a_2, \ldots, a_d\}$ with structure tensor (α^i_{jk}) such that:

(i) $\forall\, 1 \le i, k \le d$: $\alpha^i_{1k} = \alpha^i_{k1} = \delta^i_k$;

(ii) $\forall\, 1 \le r, s \le \nu - 1\ \forall\, j \ge l_r\ \forall\, k \ge l_s\ \forall\, 1 \le i \le l_{\min\{r+s,\nu\}} - 1$: $\alpha^i_{jk} = 0$.

Proof: Consider the descending chain of ideals $\mathcal{A} \supsetneqq \mathfrak{m} \supsetneqq \mathfrak{m}^2 \supsetneqq \cdots \supsetneqq \mathfrak{m}^{\nu-1} \supsetneqq \mathfrak{m}^\nu = 0$. This is a decreasing chain of vector κ-subspaces of $\mathcal{A}$, therefore we can pick a basis $a_1 := 1_\mathcal{A} \in \mathcal{A} \setminus \mathfrak{m} = \mathcal{A}^\times$, $a_{l_1}, \ldots, a_{l_2-1} \in \mathfrak{m} \setminus \mathfrak{m}^2$, $a_{l_2}, \ldots, a_{l_3-1} \in \mathfrak{m}^2 \setminus \mathfrak{m}^3$ etc. Let $r, s \in \mathbb{N}$ and let $a_j \in \mathfrak{m}^r$ and $a_k \in \mathfrak{m}^s$, i.e. $j \ge l_r$ and $k \ge l_s$. Then $a_j a_k \in \mathfrak{m}^{r+s}$, therefore $\forall\, 1 \le i \le l_{\min\{r+s,\nu\}} - 1$: $\alpha^i_{jk} = 0$, where the min condition comes from the fact that $a_j a_k = 0$ if $r + s \ge \nu$. $\square$

Remarks:

(1) We still have $a_d^2 = 0$ if $d \ge 2$, and $a_{d-1} a_d = 0$ if $d \ge 3$. In fact $\forall\, j \ge l_r\ \forall\, k \ge l_{\nu-r}$: $a_j a_k = 0$ since $a_j \in \mathfrak{m}^r$ and $a_k \in \mathfrak{m}^{\nu-r}$.

(2) It is not always possible to improve on the number of zero structure constants by means of Lemma 0.1.71, for example if $d_1 = d_2 = \cdots = d_{\nu-1} = 1$ as in the case of $K[x]/(x^{\nu-1})$.

$\square$

Lemma 0.1.72 (Schur–Jacobson)**:** If K is an arbitrary field, then the maximal dimension of a commutative K-algebra of matrices of size n is $\lfloor n^2/4 \rfloor + 1$.

Proof: We refer the reader to [Jac44] for the classical source or to [Mir98] for a short proof. $\square$

Corollary 0.1.73: Let $\mathcal{A} = \prod_{j=1}^n (\mathcal{A}_j, \mathfrak{m}_j, \kappa) \in \mathsf{FdCAlg}_\kappa$ with $d_j := \dim_\kappa \mathcal{A}_j$, $1 \le j \le n$. Writing $Z = (z_1 \oplus X_1, \ldots, z_n \oplus X_n) \in \mathcal{A}$ for $z_j \in \kappa$ and $X_j \in \mathfrak{m}_j$, $1 \le j \le n$, we have:

(1) The spectrum of Z is given by $\sigma(Z) := \{z_1, \ldots, z_n\}$, where each z_j has algebraic multiplicity d_j, $1 \le j \le n$.

(2) The character space of $\mathcal{A}$ is given by $\mathcal{X}(\mathcal{A}) = \{\chi_j \colon Z \mapsto z_j \mid 1 \le j \le n\}$.

(3) Consequently, $\operatorname{tr}(Z) = \sum_{j=1}^n d_j \chi_j(Z)$ and $\det(Z) = \prod_{j=1}^n \chi_j(Z)^{d_j}$. $\square$

Definition 0.1.74: Let $\mathcal{A}$ be a Weil $\mathbb{K}$-algebra. A Weil $\mathcal{A}$-module is simply an object in $\mathsf{FgMod}(\mathcal{A})$.

Lemma 0.1.75 (Localizations of Artinian rings)**:** Let $\mathcal{A} \cong \prod_{j=1}^n (\mathcal{A}_j, \mathfrak{m}_j, \kappa_j)$ be an Artinian ring and let $S \subseteq \mathcal{A}$ be a multiplicative system. Then $S^{-1}\mathcal{A} \cong \prod_{j \in J} \mathcal{A}_j$ for some $J \subseteq [n]$.

Proof: First suppose that $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \kappa)$ and let $S \subseteq \mathcal{A}$ be a multiplicative system. Then either $S \subseteq \mathcal{A}^\times$ or $S \cap \mathfrak{m} \ne \varnothing$, where $\mathfrak{m} = \operatorname{nil}(\mathcal{A})$ since $\mathcal{A}$ is Artinian, i.e. $\exists\, f \in S$: $f^k = 0$ for some $k \in \mathbb{N}$. Hence either $S^{-1}\mathcal{A} \cong \mathcal{A}$ or $S^{-1}\mathcal{A} \cong 0$. For the general case set $S_j := \operatorname{pr}_j(S)$, $1 \le j \le n$. Then $S_j \subseteq \mathcal{A}_j$, $1 \le j \le n$, are themselves multiplicative systems. One easily checks that $S^{-1}\mathcal{A} \cong \prod_{j \in J} S_j^{-1}\mathcal{A}_j$, which implies the claim. $\square$

Lemma 0.1.76 (Polynomials over $\mathcal{A}$)**:** Let $(\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocArt}_\kappa$ and $P(Z) := \sum_{k=0}^n a_k Z^k \in \mathcal{A}[Z]$. Then:

(1) $P \in \mathcal{A}[Z]^\times \Leftrightarrow a_0 \in \mathcal{A}^\times$ and $a_1, \dots, a_n \in \mathfrak{m}$.
(2) $\operatorname{nil}(\mathcal{A}[Z]) = \mathfrak{m}[Z]$.
(3) $\operatorname{Ann}_{\mathcal{A}[Z]}(P) \neq 0 \Leftrightarrow \operatorname{Ann}_{\mathcal{A}}(P) \neq 0$.
(4) $\mathcal{A}[Z]$ is Noetherian.

Proof: To (1)–(3): These are general facts valid for any $R \in \mathsf{CRing}$ with $\operatorname{nil}(R)$ in place of $(\mathcal{A}, \mathfrak{m} = \operatorname{nil}(\mathcal{A}))$, see [AM69, Ex. 2 after Ch. 1].

To (4): This follows from Hilbert's Basissatz since $\mathcal{A}$ is itself Noetherian. $\square$

Lemma 0.1.77 (Hilbert's Basissatz for Power Series)**:** Let $\mathcal{A} \in \mathsf{CRing}$ be Noetherian. Then $\mathcal{A}[[Z_1, \dots, Z_m]]$ is also Noetherian.

Proof: This is a standard fact in Commutative Algebra, but we give here a short proof for completeness sake because we haven't found an exposition we like. By induction it suffices to prove the claim for $S := \mathcal{A}[[Z]]$. Let $I \lhd \mathcal{A}[[Z]]$ and let $\mathfrak{a}$ be the set of lowest degree coefficients of elements from I. It is easily verified that $\mathfrak{a} \lhd \mathcal{A}$. Since $\mathcal{A}$ is Noetherian, $\mathfrak{a}$ is finitely generated, say, $\mathfrak{a} = (a_1, \dots, a_n)$ for some $a_1, \dots, a_n \in \mathcal{A}$. Let $f_i(Z) := a_i Z^m + \cdots \in I$, $1 \leq i \leq n$, where we have assumed without loss of generality that the degree after each a_i is the same (multiply with suitable powers of Z as I is an ideal). We are going to show that $I \cap (Z^m) = (f_1, \dots, f_n)$. Let $g(Z) = bZ^m + \cdots \in I \cap (Z^m)$. Then $b \in \mathfrak{a}$, hence $b = \sum_{i=1}^n b_i a_i$ for some $b_i \in \mathcal{A}$, and $g(Z) - \sum_{i=1}^n b_i f_i(Z) \in I \cap (Z^{m+1})$. Likewise $\exists\, b_1', \dots, b_n' \in \mathcal{A}$: $g(Z) - \sum_{i=1}^n b_i f_i(Z) - \sum_{i=1}^n b_i' Z f_i(Z) \in I \cap (Z^{m+2})$. Iteratively it follows that $g(Z) = \sum_{i=1}^n h_i(Z) f_i(Z)$ for some $h_i(Z) \in \mathcal{A}[[Z]]$, $1 \leq i \leq n$. Thus $I \cap (Z^m)$ is a finitely generated S-module. But since we have inclusions $I \cap (Z^m) \hookrightarrow (Z^m)$ and $I \hookrightarrow S$ of S-modules and hence as $\mathcal{A}$-modules, we also have a well-defined inclusion $I/(I \cap (Z^m)) \hookrightarrow S/(Z^m) \cong \mathcal{A}^m$ as $\mathcal{A}$-modules. But $\mathcal{A}^m$ is a Noetherian $\mathcal{A}$-module, hence $I/(I \cap (Z^m))$ is a finitely generated $\mathcal{A}$-module, ergo a finitely generated S-module. This together with $I \cap (Z^m)$ itself being finitely generated S-module implies that the whole I is a finitely generated S-module, which proves the claim. $\square$

Much of the following is valid for any $\mathcal{A} = (\mathcal{A}, \mathfrak{m}) \in \mathsf{LocCRing}$ with $\mathfrak{m}$ nilpotent. Recall though that if more generally $\mathcal{A} \in \mathsf{CRing}$ contains a *nil* maximal ideal $\mathfrak{m}$, then automatically $(\mathcal{A}, \mathfrak{m}) \in \mathsf{LocCRing}$ with $\operatorname{Kdim}(\mathcal{A}) = 0$, in which case we have the implications $\mathcal{A}$ Noetherian $\Leftrightarrow$ $\mathcal{A}$ Artinian $\Rightarrow$ $\mathfrak{m}$ actually nilpotent (see discussion at the beginning of Section 0.1.12).

Lemma 0.1.78 (Formal Power Series over $\mathcal{A}$)**:** Let $\mathcal{A} \in \mathsf{CRing}$ and let $f(Z) := \sum_{k=0}^\infty a_k Z^k \in \mathcal{A}[[Z]]$. Then:

(1) $f \in \mathcal{A}[[Z]]^\times \Leftrightarrow a_0 \in \mathcal{A}^\times$.
(2) If $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocCRing}$, then $(\mathcal{A}[[Z]], \langle \mathfrak{m}, Z \rangle, \kappa) \in \mathsf{LocCRing}$ as well.
(3) If $(\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocArt}$, then $\operatorname{nil}(\mathcal{A}[[Z]]) = \mathfrak{m}[[Z]]$.
(4) If $(\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocArt}$, then $\operatorname{Ann}_{\mathcal{A}[[Z]]}(f) \neq 0 \Leftrightarrow \operatorname{Ann}_{\mathcal{A}}(f) \neq 0$.
(5) $\mathcal{A}[[Z_1, \dots, Z_m]]$ is Noetherian for any $m \in \mathbb{N}$.

Proof: To (1): This is standard.

To (2): By (1) we have $\mathcal{A}[[Z]] \setminus \mathcal{A}[[Z]]^\times = \{f \in \mathcal{A}[[Z]] : a_0 \in \mathfrak{m}\} = \mathfrak{m} + Z\mathcal{A}[[Z]]$. We claim that $\mathfrak{m} + Z\mathcal{A}[[Z]] = \langle \mathfrak{m}, Z \rangle \lhd \mathcal{A}[[Z]]$, this would already imply $\mathcal{A}[[Z]] \in \mathsf{LocCRing}$. "$\subseteq$" is obvious, and for the converse one has $\mathfrak{m}\mathcal{A}[[Z]] \subseteq \mathfrak{m}[[Z]] = \mathfrak{m} + Z\mathfrak{m}[[Z]] \subseteq \mathfrak{m} + Z\mathcal{A}[[Z]]$. Finally, it is immediate that $\mathcal{A}[[Z]]/(\mathfrak{m} + Z\mathcal{A}[[Z]]) \cong \mathcal{A}/\mathfrak{m} = \kappa$.

To (3): The first part of the inclusion $\operatorname{nil}(\mathcal{A}[[Z]]) \subseteq (\operatorname{nil}(\mathcal{A}))[[Z]] \overset{\text{def}}{=} \mathfrak{m}[[Z]]$ holds for any $\mathcal{A} \in \mathsf{CRing}$

(see [AM69, Ex. 3 after Ch. 5]). Conversely, since $\mathfrak{m} \lhd \mathcal{A}$ is itself nilpotent, one sees that $\mathfrak{m}[[Z]] \subseteq \mathrm{nil}(\mathcal{A}[[Z]])$.

To (4): By [Fie71] this holds more generally for any Noetherian $\mathcal{A} \in \mathsf{CRing}$.

To (5): This follows from Lemma 0.1.77 since $\mathcal{A}$ is Noetherian. ☐

Remarks:

(1) Note that $\mathfrak{m}[[Z]] \subsetneqq \langle \mathfrak{m}, Z\rangle$ always in (3) as $Z \notin \mathfrak{m}[[Z]]$. In particular, even if $(\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocArt}$, $\mathcal{A}[[Z]]$ is not Artinian in stark contast to $(\mathcal{A}((Z)), \mathfrak{m}((Z)), \kappa((Z)))$, since we then have $\mathrm{nil}(\mathcal{A}[[Z]]) \subsetneqq \langle Z, \mathfrak{m}\rangle = \mathfrak{J}(\mathcal{A}[[Z]])$.

(2) In fact, [Fie71] shows more generally that if $\mathcal{A} \in \mathsf{CRing}$ is Noetherian, then $\mathrm{nil}(\mathcal{A}[[Z]]) = (\mathrm{nil}(\mathcal{A}))[[Z]]$.

Lemma 0.1.79 (Lagrange Inversion, Formal Inverse Function Theorem)**:** Let $\mathcal{A} \in \mathsf{CRing}$ and let $f := \sum_{k=0}^{\infty} a_k Z^k \in \mathcal{A}[[Z]]$ such that $f(0) \equiv a_0 = 0$ and $f'(0) \equiv a_1 \in \mathcal{A}^\times$. Then f has a unique formal compositional inverse $g := \sum_{k=1}^{\infty} b_k Z^k \in \mathcal{A}[[Z]]$, $g(0) \equiv b_0 = 0$, where $b_1 = 1/a_1$. That is, $f(g(Z)) = g(f(Z)) = Z$ in $\mathcal{A}[[Z]]$.

Proof: For example see [Sta99]. ☐

Remarks:

(1) A left compositional inverse is automatically also a right compositional inverse, and vice versa: if $f(g(Z)) = Z$ and $g(h(Z)) = Z$, then $g(f(Z)) = g(f(g(h(Z)))) = g(h(Z)) = Z$. However, is it easy to see that existence of a left-inverse implies existence of a right-inverse and vice versa?

(2) In characteristic 0 all coefficients in the Lagrange inversion can be stated with the help of (formal) residua.

Lemma 0.1.80 (Formal Implicit Function Theorem)**:** Let $F(Z, W) := \sum_{i,j=0}^{\infty} a_{ij} Z^i W^j \in \mathcal{A}[[Z, W]]$ such that $F(Z, 0) = 0$ and $\frac{\partial F}{\partial W}(0, 0) \equiv a_{0,1} \in \mathcal{A}^\times$. Then there exists a unique $g \in \mathcal{A}[[Z]]$ such that $g(0) = 0$ and $F(Z, g(Z)) = 0$ in $\mathcal{A}[[Z]]$.

Proof: Put $S := \mathcal{A}[[Z]]$, i.e. $\mathcal{A}[[Z, W]] = \mathcal{A}[[Z]][[W]] = S[[W]]$, $\tilde{F}(W) := F(Z, W) \in S[[W]]$, and note that $\tilde{F}(0) = F(Z, 0) = 0$ in S and the free coefficient of $\frac{\mathrm{d}\tilde{F}}{\mathrm{d}W}$ is a power series in $\mathcal{A}[[Z]]$. Since $\frac{\partial F}{\partial W}(0, 0) \in \mathcal{A}^\times$ is the free coefficient of $\frac{\mathrm{d}\tilde{F}}{\mathrm{d}W}(0) \in \mathcal{A}[[Z]]$, it follows that $\frac{\mathrm{d}\tilde{F}}{\mathrm{d}W}(0) \in \mathcal{A}[[Z]]^\times = S^\times$. Thus, by Lemma 0.1.79, there exists a unique $\tilde{G}(W) \in S[[W]]$, $G(Z, W) := \tilde{G}(W) \in \mathcal{A}[[Z, W]]$, such that $\tilde{G}(0) = 0$ and $\tilde{F}(\tilde{G}(W)) = W$, i.e. $F(Z, G(Z, W)) = W$. Now put $g(Z) := G(Z, 0) \in \mathcal{A}[[Z]]$. ☐

Lemma 0.1.81 (IBN)**:** Let $0 \neq \mathcal{A} \in \mathsf{CAlg}_R$ and $m, n \in \mathbb{N}$ such that $\mathcal{A}^m \cong \mathcal{A}^n$ in $\mathsf{Mod}(\mathcal{A})$. Then $m = n$.

Proof: This is well-known, but we include the proof here for completeness sake since it is very short. By Zorn's lemma $\mathcal{A}$ has a maximal ideal $\mathfrak{m}$. Let $\kappa := \mathcal{A}/\mathfrak{m}$ and suppose that $f\colon \mathcal{A}^m \xrightarrow{\cong} \mathcal{A}^n$ in $\mathsf{Mod}(\mathcal{A})$. Then $\mathrm{id} \otimes f\colon \kappa \otimes_{\mathcal{A}} \mathcal{A}^m \xrightarrow{\cong} \kappa \otimes_{\mathcal{A}} \mathcal{A}^n$ is an isomorphism of finite-dimensional vector spaces over κ, hence $m = n$. ☐

Remarks:

(1) Thus the "R-dimension" of a module-finite, module-free, commutative R-algebra $\mathcal{A}$ is a well-defined quantity, usually referred to as rank of the algebra (as in rank of the underlying free R-module).

(2) Commutativity is essential here. There exist non-commutative rings that do not have the IBN property. ☐

0.1.13. Artinian Rings & Algebras II: Representation Data

This section is essentially a summary of Section 0.1.4 through Section 0.1.10 applied to our special case of interest $\mathcal{A} \in \mathsf{FdCAlg}_K$. But before we do that, we should establish certain fundamental facts about $\mathsf{FdMod}(\mathcal{A})$.

Lemma 0.1.82: Let K be a field and $\mathcal{A} \in \mathsf{FdCAlg}_K$. Then $\mathsf{FdMod}(\mathcal{A}) = \mathsf{FgMod}(\mathcal{A})$. Furthermore, for $M \in \mathsf{Mod}(\mathcal{A})$ the following are equivalent:

(i) M is Noetherian;

(ii) M is Artinian;

(iii) $\ell_{\mathcal{A}}(M) < \infty$;

(iv) $M \in \mathsf{FgMod}(\mathcal{A}) = \mathsf{FdMod}(\mathcal{A})$.

Proof: If $M \in \mathsf{FdMod}(\mathcal{A})$ with $\dim_K M = m$, then any K-basis $\{x_1, \ldots, x_m\} \subset M$ is clearly a generating set for M over $\mathcal{A}$. Conversely, if $M \in \mathsf{FgMod}(\mathcal{A})$ and $\{y_1, \ldots, y_n\} \subset M$ are generators over $\mathcal{A}$, then $\{a_i \cdot y_j : 1 \leq i \leq d, 1 \leq j \leq n\} \subset M$ is a finite generating set for M over K, where $\{a_1, \ldots, a_d\}$ is any K-basis of $\mathcal{A}$. Hence $\dim_K M < \infty$. Moreover, in general we only have that $\ell_{\mathcal{A}}(M) < \infty \Leftrightarrow M$ is *both* Artinian and Noetherian. However, since $\mathcal{A} \in \mathsf{FdCAlg}_K$, then in particular $\mathcal{A} \in \mathsf{Art}_{\mathbb{Z}}$, hence $\mathcal{A}$ is semiprimary, and by the Akizuki–Hopkins–Levitzki Theorem (see [Lam01, Thm. 4.15 on p. 55]) we have that M Noetherian $\Leftrightarrow$ M Artinian. This establishes (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). Finally, (i) $\Rightarrow$ (iv) $\Rightarrow$ (i),(ii),(iii) is clear. □

Remark: Some might say that FdMod is the differential geometer's definition, whereas FgMod is the algebraist's definition. □

Since $\ell_{\mathcal{A}}(M) < \infty$ for $M \in \mathsf{FdMod}(\mathcal{A})$, it follows from the Krull–Schmidt–Remak–Azumaya Theorem ([Lam01, Thm. 19.21 and Cor. 19.22 on p. 288]) that M admits a *complete decomposition* ${}_{\mathcal{A}}M = \bigoplus_{k=1}^{m} {}_{\mathcal{A}}M_k$, where m is uniquely determined and so are the isomorphism types $[{}_{\mathcal{A}}M_k]$, $1 \leq k \leq m$, up to permutation. This decomposition is actually *strong*: since $\ell_{\mathcal{A}}(-)$ is additive on SESes, we have $\ell_{\mathcal{A}}(M_k) < \infty$ too, hence by Fitting's Lemma each ${}_{\mathcal{A}}M_k$ is also *strongly indecomposable*, $1 \leq k \leq m$. That is, $\mathrm{End}_{\mathcal{A}}(M_k)$ is a *local* (non-commutative) K-algebra with *nil* maximal ideal $\mathfrak{J}(\mathrm{End}_{\mathcal{A}}(M_k))$ which is in fact *nilpotent* with $\mathfrak{J}(\mathrm{End}_{\mathcal{A}}(M_k))^{\ell_{\mathcal{A}}(M_k)} = 0$, $1 \leq k \leq m$, see [Lam01, Rem. 19.18 on p. 286]. Moreover, we have $\dim_K \mathrm{End}_{\mathcal{A}}(M_k) \leq \dim_K \mathrm{End}_K(M_k) < \infty$, hence $\mathrm{End}_{\mathcal{A}}(M_k)$ is itself a *non-commutative* local Artinian K-algebra, $1 \leq k \leq m$. If $\rho_k\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M_k)$, $1 \leq k \leq m$, denote the corresponding representations, then $\rho^{\mathrm{mor}} \overset{\mathrm{def}}{=} (\rho_1^{\mathrm{im}}, \ldots, \rho_n^{\mathrm{im}})$. As $\mathcal{A}$ is commutative, each ρ_k in fact factorizes as $\rho_k\colon \mathcal{A} \twoheadrightarrow \rho_k(\mathcal{A}) \overset{\mathrm{def}}{=} \mathcal{B}_k \hookrightarrow \mathrm{End}_{\mathcal{A}}(M_k) \hookrightarrow \mathrm{End}_{\mathbb{K}}(M_k)$, hence $\mathcal{B}_k$ is necessarily itself local (for example by Lemma 0.1.57 and the fact that $\mathrm{End}_{\mathcal{A}}(M_k)$ cannot contain any non-trivial idempotents as ${}_{\mathcal{A}}M_k$ is indecomposable), $1 \leq k \leq m$. It follows that the *convenient corestriction* of ρ has the form

$$\rho^{\mathrm{mor}}\colon \mathcal{A} \to \mathcal{B} \overset{\mathrm{def}}{=} \prod_{k=1}^{m} (\mathcal{B}_k, \mathfrak{n}_k, \lambda_k) \tag{0.1.113}$$

and ρ factorizes as

$$\rho\colon \mathcal{A} \xrightarrow{\rho^{\mathrm{mor}}} \prod_{k=1}^{m} (\mathcal{B}_k, \mathfrak{n}_k, \lambda_k) \hookrightarrow \prod_{k=1}^{m} \mathrm{End}_{\mathcal{A}}(M_k) \hookrightarrow \prod_{k=1}^{m} \mathrm{End}_K(M_k) \xhookrightarrow{\iota_{\mathrm{cor}}} \mathrm{End}_K(M) \tag{0.1.114}$$

with the local factors of $\mathcal{B} \in \mathsf{FdCAlg}_K$ matching precisely the direct product of the diagonal endomorphism K-algebras. This is more advantageous than working with $\rho^{\mathrm{im}}\colon \mathcal{A} \twoheadrightarrow \rho(\mathcal{A})$ instead. In principle, this allows everywhere to consider the representation setting as a special case of the morphism setting.

Corollary 0.1.83 (Standard Factorizations)**:** Let $\mathcal{A} = \prod_{j=1}^{n}(\mathcal{A}_j, \mathfrak{m}_j, \kappa_j) \in \mathsf{FdCAlg}_K$.

(1) Representation Setting: if $M \in \mathsf{FdMod}(\mathcal{A})$ via $\rho\colon \mathcal{A} \to \mathrm{End}_K(M)$, then there exist a unique complete (strong) decomposition $M = \bigoplus_{k=1}^{m} M_k$ in $\mathsf{FdMod}(\mathcal{A})$ and an index parametrization $\tau := \tau_\rho\colon [m] \to [n]$, $k \mapsto \tau(k)$, depending on said decomposition and ρ, such that ρ can be written as

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\rho^{\mathrm{cor}}"] \arrow[d, "\Pi_\rho"'] & \prod_{k=1}^{m} \mathrm{End}_K(M_k) \arrow[r, hook, "\iota_{\mathrm{cor}}"] & \mathrm{End}_K(M) \\ \prod_{k=1}^{m} \mathcal{A}_{\tau(k)} \arrow[ur, dashed, "\exists_1\, \overline{\rho^{\mathrm{cor}}}"'] & & \end{tikzcd} \tag{0.1.115}$$

where:

- $\Pi_\rho := \Pi_{(\tau_\rho)}$ is a projection selection morphism, in a standard form, associated to the representation ρ;
- $\overline{\rho^{\mathrm{cor}}} = \times_{k=1}^{m} \bar{\rho}_k$ for some representations $\bar{\rho}_k\colon \mathcal{A}_{\tau(k)} \to \mathrm{End}_K(M_k)$ such that $\rho_k = \bar{\rho}_k \circ \mathrm{pr}^{\mathcal{A}}_{\tau(k)}$, $1 \le k \le m$, where $\rho^{\mathrm{cor}} = (\rho_1, \ldots, \rho_m)$ as usual.

Moreover:

$$\mathrm{Ker}(\rho) = \begin{cases} \bigcap\limits_{k \in \tau^{-1}(j)} \mathrm{Ker}(\bar{\rho}_k), \text{ if } j \in \tau([m]) \\ \mathcal{A}_j, \text{ otherwise.} \end{cases} \tag{0.1.116}$$

(2) Morphism Setting: if $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ in FdCAlg_K, then there exists an isomorphism (block decomposition) $\mathcal{B} \cong \prod_{k=1}^{m}(\mathcal{B}_k, \mathfrak{n}_k, \lambda_k)$ in FdCAlg_K and an index parametrization $\tau := \tau_\varphi\colon [m] \to [n]$, $k \mapsto \tau(k)$, depending on said isomorphism and φ, such that

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\varphi"] \arrow[d, "\Pi_\varphi"'] & \mathcal{B} \cong \prod_{k=1}^{m}(\mathcal{B}_k, \mathfrak{n}_k, \lambda_k) \\ \prod_{k=1}^{m} \mathcal{A}_{\tau(k)} \arrow[ur, dashed, "\exists_1\, \bar{\varphi}"'] & \end{tikzcd} \tag{0.1.117}$$

where:

- $\Pi_\varphi := \Pi_{(\tau_\varphi)}$ is a projection selection morphism, in a standard form, associated to the morphism φ;
- $\bar{\varphi} = \times_{k=1}^{m} \bar{\varphi}_k$ for some morphisms $\bar{\varphi}_k\colon (\mathcal{A}_{\tau(k)}, \mathfrak{m}_{\tau(k)}, \kappa_{\tau(k)}) \to (\mathcal{B}_k, \mathfrak{n}_k, \lambda_k)$ in FdCAlg_K such that $\varphi_k = \bar{\varphi}_k \circ \mathrm{pr}^{\mathcal{A}}_{\tau(k)}$, $1 \le k \le m$, with $(\varphi_1, \ldots, \varphi_m)$ being the "components" of φ with respect to the block decomposition of $\mathcal{B}$.

Moreover:

$$\mathrm{Ker}(\varphi) = \begin{cases} \bigcap\limits_{k \in \tau^{-1}(j)} \mathrm{Ker}(\bar{\varphi}_k), \text{ if } j \in \tau([m]) \\ \mathcal{A}_j, \text{ otherwise.} \end{cases} \tag{0.1.118}$$

Finally, both factorizations are compatible with each other in the sense that $I_{\rho^{\mathrm{mor}}} = I_\rho$ and $I_{\varphi^{\mathrm{rep}}} = I_\varphi$.

Proof: To (1): Here are all the various ways to make the conclusion:

(i) Since M is finitely decomposable in $\mathsf{Mod}(\mathcal{A})$, the claim follows from Lemma 0.1.30.

(ii) Since $\mathcal{A}$ is *commutative* and M is finitely decomposable in $\mathsf{Mod}(\mathcal{A})$, the claim also follows from Corollary 0.1.45 (1) (based on Corollary 0.1.29).

(iii) Since $\mathcal{A}$ is *commutative* and $\mathrm{End}_{\mathcal{A}}(M_k)$ (or $\mathcal{B}_k$), $1 \le k \le m$, are local (hence blocks), the claim also follows from Eq. (0.1.114) and Lemma 0.1.27, where observe that an *(external) isomorphic* rearrangement of $\prod_{k=1}^{m} \mathrm{End}_{\mathcal{A}}(M_k)$ simply corresponds to an *internal* rearrangement of ${}_{\mathcal{A}}M = \bigoplus_{k=1}^{m} {}_{\mathcal{A}}M_k$. This is the same as deducing (1) from (2).

It is clear that argument (i) is the most general among the three as it uses the least amount of assumptions and theory.

To (2): By Lemma 0.1.57, $\mathcal{B}$ admits a block decomposition into a product of local Artinian R-algebras. Thus the claim follows from Corollary 0.1.45 (based on Lemma 0.1.27). $\square$

Remark: The condition in Corollary 0.1.24 of ${}_{\mathcal{A}}\mathcal{B}$ being indecomposable does not hold in general even in LocArt_κ and so one still cannot deduce (2) from (1) by way of Corollary 0.1.31. Indeed, consider $\mathcal{A} := K[t]/(t^2)$, $\mathcal{B} := K[x,y]/(x,y)^2$, and define $\varphi\colon \mathcal{A} \hookrightarrow \mathcal{B}$, $\bar{t} \mapsto \bar{x}$. Putting $M_1 := \mathrm{span}_K\{1_{\mathcal{B}}, \bar{x}\}$ and $M_2 := \mathrm{span}_K\{\bar{y}\}$, we clearly have $\mathcal{B} = M_1 \oplus_K M_2$. Since $\bar{t}M_1 \overset{\text{def}}{=} \bar{x}M_1 \subseteq M_1$ as $\bar{x}^2 = 0$ and $\bar{t}M_2 \overset{\text{def}}{=} \bar{x}M_2 \subseteq M_2$ as $\bar{x}\bar{y} = 0$, we have that M_1 and M_2 are in fact $\mathcal{A}$-submodules of ${}_{\mathcal{A}}\mathcal{B}$, hence $\mathcal{B} = M_1 \oplus_{\mathcal{A}} M_2$ as well. $\square$

Now, by Definition 0.1.32 by way of Lemma-Definition 0.1.26 and in accordance with Corollary 0.1.83 ρ and φ give rise to the distinguished indexing subsets $I_\rho \subseteq [n]$ and $I_\varphi \subseteq [n]$, respectively. Furthermore, while ρ and φ give rise to *relative notions* as defined in Section 0.1.6, I_ρ and I_φ in turn give rise to corresponding *product notions* as per Section 0.1.7. In the setup of Section 0.1.10 we have established a canonical correspondence between the "core" *product notions* and the "core" *spectral notions*, more precisely in Lemma 0.1.47. This correspondence also allows us to *consistently* associate I_ρ and I_φ to some *further spectral notions* as introduced in Section 0.1.9 and *available only in spectral context*, most importantly *spectral closures* and *spectral complements*. **We are now going to show that the *relative* notions induced by ρ and φ converge with the *product/spectral* notions induced by I_ρ and I_φ.** This will also settle our final working definitions henceforth.

Lemma-Definition 0.1.84: Let $\mathcal{A} = \prod_{j=1}^n (\mathcal{A}_j, \mathfrak{m}_j, \kappa_j) \in \mathsf{FdCAlg}_K$, $M \in \mathsf{FdMod}(\mathcal{A})$ given by $\rho\colon \mathcal{A} \to \mathrm{End}_K(M)$, and $X \subseteq \mathcal{A}$.

(1) ρ-Closure (ρ-Saturation):

$$X^\rho \overset{\text{def}}{=} X + \mathrm{Ker}(\rho). \tag{0.1.119}$$

(2) Projection closure with respect to ρ:

$$\widetilde{X} := \widetilde{X}_\rho := \widetilde{X}_{(I_\rho)} = \begin{cases} X_j, \text{ if } j \in I_\rho \\ \mathcal{A}_j, \text{ otherwise.} \end{cases} \tag{0.1.120}$$

(3) Character space associated to ρ:

$$\mathcal{X}_\rho(\mathcal{A}) := \mathcal{X}_{I_\rho}(\mathcal{A}) \overset{\text{def}}{=} \{\chi_j : j \in I_\rho\}. \tag{0.1.121}$$

(4) Multi-Character with respect to ρ:

$$\begin{aligned} \chi_\rho := \chi_{I_\rho}\colon \mathcal{A} &\twoheadrightarrow \prod_{j \in I_\rho} \kappa_j \\ (a_1, \ldots, a_n) &\mapsto (a_j \bmod \mathfrak{m}_j)_{j \in I_\rho}. \end{aligned} \tag{0.1.122}$$

(5) Uniform spectral closure of X with respect to ρ:

$$\widehat{X} := \widehat{X}_\rho := \widehat{X}_{I_\rho} \overset{\text{def}}{=} X^{\chi_\rho}. \tag{0.1.123}$$

(6) Uniform spectral complement of X with respect to ρ:

$$\breve{X} := \breve{X}_\rho := \breve{X}_{I_\rho} \overset{\text{def}}{=} \chi_\rho^{-1}(\chi_\rho(X)^c) \overset{\text{def}}{=} (\widehat{X})^c. \tag{0.1.124}$$

(7) Point-wise spectral closure of X with respect to ρ:

$$\hat{X} := \hat{X}_\rho := \hat{X}_{I_\rho} = \prod_{j=1}^n \begin{cases} X_j^{\mathrm{sp}} \times \mathfrak{m}_j, \text{ if } j \in I_\rho \\ \mathcal{A}_j, \text{ otherwise.} \end{cases} \tag{0.1.125}$$

(8) Point-wise spectral complement of X with respect to ρ:

$$\check{X} := \check{X}_\rho := \check{X}_{I_\rho} = \prod_{j=1}^{n} \begin{cases} (X_j^{\mathrm{sp}})^c \times \mathfrak{m}_j, \text{ if } j \in I_\rho \\ \mathcal{A}_j, \text{ otherwise.} \end{cases} \tag{0.1.126}$$

Analogously for $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ in FdCAlg_K and I_φ.

Proof: Immediate from the original definitions. □

Lemma 0.1.85 (Preimages of index-relative $\mathrm{U}(-)$ and $\mathrm{nil}(-)$)**:** Let K be a field and $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ a morphism in FdCAlg_K with accompanying block decompositions $\mathcal{A} = \prod_{j=1}^n(\mathcal{A}_j, \mathfrak{m}_j, \kappa_j)$ and $\mathcal{B} = \prod_{k=1}^m(\mathcal{B}_k, \mathfrak{n}_k, \lambda_k)$ φ-adapted. Then for $\varnothing \neq J \subseteq [m]$ we have:
(1) $\varphi^{-1}(\mathcal{B}^\times_{(J)}) = \mathcal{A}^\times_{(\tau_\varphi(J))}$;
(2) $\varphi^{-1}(\mathrm{nil}_{(J)}(\mathcal{B})) = \mathrm{nil}_{(\tau_\varphi(J))}(\mathcal{A})$.

Proof: By Lemma-Definition 0.1.26 and Corollary 0.1.83, we can write $\varphi = (\times_{k=1}^m \bar{\varphi}_k) \circ \Pi_\tau \overset{\text{def}}{=} (\times_{k=1}^m \bar{\varphi}_k) \circ (\times_{j\in I}\Delta_j) \circ \mathrm{pr}_I$ for $\tau := \tau_\varphi \colon [m] \to [n]$, $I := I_\varphi \overset{\text{def}}{=} \tau_\varphi([m])$ (see Definition 0.1.32), and $\bar{\varphi}_k \colon \mathcal{A}_{\tau(k)} \to \mathcal{B}_k$, $1 \leq k \leq m$. By locality of the K-algebras and *nility* of the corresponding maximal ideals, we have $\bar{\varphi}_k^{-1}(\mathfrak{n}_k) = \mathfrak{m}_{\tau(k)}$ and $\bar{\varphi}_k^{-1}(\mathcal{B}_k^\times) = \mathcal{A}^\times_{\tau(k)}$, $1 \leq k \leq m$. We only prove (1) since (2) follows completely analogously. To this end, we have

$$(\times_{k=1}^m \bar{\varphi}_k)^{-1}(\mathcal{B}^\times_{(J)}) = \prod_{k=1}^{m} \begin{cases} \mathcal{A}^\times_{\tau(k)}, \text{ if } k \in J \\ \mathcal{A}_{\tau(k)}, \text{ otherwise.} \end{cases}$$

Furthermore, for any $p_j, q_j \in \mathbb{N}_0$ with $m_j = p_j + q_j$ we have

$$\Delta_j^{-1}\big((\mathcal{A}_j^\times)^{p_j} \times \mathcal{A}_j^{q_j}\big) = \begin{cases} \mathcal{A}_j^\times, \text{ if } p_j \geq 1 \\ \mathcal{A}_j, \text{ otherwise.} \end{cases}$$

But $p_j = |J \cap \tau^{-1}(j)|$, in particular $p_j \geq 1 \Leftrightarrow j \in \tau(J)$, $j \in I$. Thus

$$(\times_{j\in I}\Delta_j)^{-1}\big((\times_{k=1}^m \bar{\varphi}_k)^{-1}(\mathcal{B}^\times_{(J)})\big) = \prod_{j\in I} \begin{cases} \mathcal{A}_j^\times, \text{ if } j \in \tau(J) \\ \mathcal{A}_j, \text{ otherwise.} \end{cases}$$

Therefore

$$\varphi^{-1}(\mathcal{B}^\times_{(J)}) = \mathrm{pr}_I^{-1}\left(\prod_{j\in I} \begin{cases} \mathcal{A}_j^\times, \text{ if } j \in \tau(J) \\ \mathcal{A}_j, \text{ otherwise} \end{cases}\right) = \prod_{j=1}^{n} \begin{cases} \mathcal{A}_j^\times, \text{ if } j \in \tau(J) \\ \mathcal{A}_j, \text{ otherwise.} \end{cases} \overset{\text{def}}{=} \mathcal{A}^\times_{(\tau(J))}$$

as claimed. □

Lemma-Definition 0.1.86: Let K be a field, $\mathcal{A} = \prod_{j=1}^n(\mathcal{A}_j, \mathfrak{m}_j, \kappa_j) \in \mathsf{FdCAlg}_K$, $M \in \mathsf{FdMod}(\mathcal{A})$ given by $\rho \colon \mathcal{A} \to \mathrm{End}_R(M)$, and $Z \in \mathcal{A}$.
(1) Units with respect to ρ:

$$\mathcal{A}^\times_\rho = \mathcal{A}^\times_{(I_\rho)} \overset{\text{def}}{=} \prod_{j=1}^{n} \begin{cases} \mathcal{A}_j^\times, \text{ if } j \in I_\rho \\ \mathcal{A}_j, \text{ otherwise} \end{cases} = \begin{cases} \mathfrak{m}_j^c, \text{ if } j \in I_\rho \\ \mathcal{A}_j, \text{ otherwise} \end{cases} = \bigcap_{j\in I_\rho} \mathfrak{M}_j^c. \tag{0.1.127}$$

(2) Singular cone with respect to ρ:

$$\mathrm{Sing}_\rho(\mathcal{A}) = \mathrm{Sing}_{I_\rho}(\mathcal{A}) = \bigcup_{j\in I_\rho} \mathfrak{M}_j \tag{0.1.128}$$

(3) Nil radical with respect to ρ:

$$\mathrm{nil}_\rho(\mathcal{A}) = \mathrm{nil}_{(I_\rho)}(\mathcal{A}) \overset{\text{def}}{=} \mathfrak{m}_\rho := \mathfrak{m}_{(I_\rho)} \overset{\text{def}}{=} \prod_{j=1}^{n} \begin{cases} \mathfrak{m}_j, \text{ if } j \in I_\rho \\ \mathcal{A}_j, \text{ otherwise} \end{cases} = \mathfrak{I}_{(I_\rho)}(\mathcal{A}) \tag{0.1.129}$$

(4) Singular “class” of Z with respect to ρ:

$$[Z]_\rho = [Z]_{(I_\rho)} \overset{\text{def}}{=} Z + \mathrm{Sing}_{(I_\rho)}(\mathcal{A}). \tag{0.1.130}$$

(5) Unital “class” of Z with respect to ρ:

$$[Z]^\times_\rho = [Z]^\times_{(I_\rho)} = \prod_{j=1}^{n} \begin{cases} z_j + \mathcal{A}_j^\times, \text{ if } j \in I \\ \mathcal{A}_j, \text{ otherwise,} \end{cases} \tag{0.1.131}$$

where $z_j \overset{\text{def}}{=} \chi_j(Z) \in \kappa_j$, $1 \leq j \leq n$.

Analogously for $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ and I_φ.

Proof: By Corollary 0.1.83 we can write $\rho = i_{\mathrm{cor}} \circ \rho^{\mathrm{cor}}$ with $\rho^{\mathrm{cor}} \overset{\text{def}}{=} (\rho_1, \ldots, \rho_m) = \bar{\rho} \circ \Pi_{(\tau)} \overset{\text{def}}{=} (\times_{k=1}^m \bar{\rho}_k) \circ (\times_{j\in I} \Delta_j) \circ \mathrm{pr}_I$ for $\tau \coloneqq \tau_\rho \colon [m] \to [n]$, $I \coloneqq I_\rho \overset{\text{def}}{=} \tau_\rho([m])$, where $\bar{\rho}_k \colon \mathcal{A}_{\tau(k)} \to \mathrm{End}_K(M_k)$ satisfy $\rho_k = \bar{\rho}_k \circ \mathrm{pr}^{\mathcal{A}}_{\tau(k)}$, $1 \leq k \leq m$. Now let $\rho^{\mathrm{mor}} \colon \mathcal{A} \to \mathcal{B} \overset{\text{def}}{=} \prod_{k=1}^m (\mathcal{B}_k, \mathfrak{n}_k, \lambda_k)$ be the corresponding *convenient* corestriction.

To (1): By Lemma 0.1.40 and Lemma 0.1.41 we have that $\mathcal{B}$ is unitally closed in $\mathrm{End}_R(M)$, hence $\mathcal{A}^\times_\rho \overset{\text{def}}{=} \rho^{-1}(\mathrm{GL}_R(M)) = \rho^{-1}(\mathcal{B}^\times) \overset{\text{def}}{=} (\rho^{\mathrm{mor}})^{-1}(\mathcal{B}^\times_{([m])})$, and the claim follows by taking $J \coloneqq [m]$ in Lemma 0.1.85 since $I_\rho = I_{\rho^{\mathrm{mor}}}$.

To (2): immediate from (1).

To (3): We have $\forall\, a = (a_1, \ldots, a_n) \in \mathcal{A}\ \forall\, k \in \mathbb{N} \colon \rho(a)^k = (\bar{\rho}_1(a_{\tau(1)})^k, \ldots, \bar{\rho}_m(a_{\tau(m)})^k) \in \mathcal{B}$. It follows that

$$\mathrm{nil}_\rho(\mathcal{A}) \overset{\text{def}}{=} \rho^{-1}(\mathrm{nil}(\mathrm{End}_K(M))) = \rho^{-1}\left(\prod_{k=1}^m \mathrm{nil}(\mathrm{End}_K(M_k))\right) = \rho^{-1}\left(\prod_{k=1}^m \mathrm{nil}(\mathcal{B}_k)\right) \overset{\text{def}}{=} (\rho^{\mathrm{mor}})^{-1}(\mathrm{nil}_{([m])}(\mathcal{B}))$$

The claim now follows from Lemma 0.1.85 with $J \coloneqq [m]$ since $I_\rho = I_{\rho^{\mathrm{mor}}}$.

To (4): immediate from (2).

To (5): The first equality is immediate from (1). The second equality follows from Lemma 0.1.47 (6) and $\mathcal{A}_j = \kappa_j \oplus_{\kappa_j} \mathfrak{m}_j$, $1 \leq j \leq n$. □

0.2. The Group of Units of $\mathcal{A}$

For any $\mathcal{A} \in \mathsf{FdCAlg}_K$, its group of units $\mathcal{A}^\times$ is an affine abelian algebraic group. If $K = \mathbb{K}$, then $\mathcal{A}^\times$ is also a Lie group, open and dense in $\mathcal{A}$ (with respect to the analytic topology). Since we are only interested in the commutative case, we can take an ad hoc approach in determining $\mathcal{A}^\times$. For $\mathcal{A} \in \mathsf{FdCAlg}_K$ we have by Lemma 0.1.57 that $\mathcal{A} \cong \prod_{j=1}^n (\mathcal{A}_j, \mathfrak{m}_j, \kappa_j)$, hence $\mathcal{A}^\times \cong \prod_{j=1}^n \mathcal{A}_j^\times$. Therefore the problem reduces to the local case $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \kappa)$. We will only consider the case when $\operatorname{char}(K) = 0$ and κ is a coefficient field for $\mathcal{A}$, in particular we have $\mathcal{A} \cong \kappa \oplus_\kappa \mathfrak{m}$ (see Section 0.1.11 for more details).

Proposition 0.2.1: Let $(\mathcal{A}, \mathfrak{m}, \kappa) \in \mathsf{LocArt}_\kappa$. Then we have an isomorphism of algebraic groups

$$\begin{aligned} \Phi_{\mathcal{A}} \colon \mathcal{A}^\times &\xrightarrow{\cong} \kappa^\times \times (\mathfrak{m}, +) \\ z \oplus X &\mapsto \left(z, \log\left(1 \oplus \frac{X}{z}\right)\right) \end{aligned} \tag{0.2.1}$$

with an inverse given by

$$\begin{aligned} \Psi_{\mathcal{A}} \colon \kappa^\times \times (\mathfrak{m}, +) &\xrightarrow{\cong} \mathcal{A}^\times \\ (w, Y) &\mapsto w \exp(Y) = w \oplus \big(w(\exp(Y) - 1)\big). \end{aligned} \tag{0.2.2}$$

Proof: Since $\mathcal{A} = \kappa \oplus_\kappa \mathfrak{m}$, every unit in $\mathcal{A}$ can be uniquely written as $z \oplus X = z\left(1 \oplus \frac{X}{z}\right)$ for some $z \in \kappa \setminus \{0\} = \kappa^\times$ and $X \in \mathfrak{m}$. As $\operatorname{char}(\kappa) = 0$, we have the formal power series

$$\exp(Z) \overset{\text{def}}{=} \sum_{k=0}^\infty \frac{Z^k}{k!} \in \kappa[[Z]] \text{ and } \log(1+Z) \overset{\text{def}}{=} \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k} Z^k \in \kappa[[Z]],$$

which are formally inverse to each other in the sense of Lemma 0.1.79. More precisely, we have that $\forall X \in \mathfrak{m} \colon \log(1 \oplus X) \in \mathfrak{m}$ and $\exp(X) \in 1 \oplus \mathfrak{m}$, giving well-defined polynomial maps since X is nilpotent. We thus get homomorphisms of abelian algebraic groups

$$\begin{aligned} \log \colon (1 \oplus \mathfrak{m}, \times_{\mathcal{A}}) &\to (\mathfrak{m}, +) \\ \exp \colon (\mathfrak{m}, +) &\to (1 \oplus \mathfrak{m}, \times_{\mathcal{A}}) \end{aligned}$$

that are inverse to each other since $\forall X \in \mathfrak{m} \colon \exp(\log(1 \oplus X)) = 1 \oplus X$ and $\forall Y \in \mathfrak{m} \colon \log(1 \oplus (\exp(Y) - 1)) = Y$. That is, $(1 \oplus \mathfrak{m}, \times) \cong (\mathfrak{m}, +)$. From here it is immediate to verify that the so defined maps $\Phi_{\mathcal{A}}$ and $\Psi_{\mathcal{A}}$ are group homomorphisms inverse to each other. Finally, these maps are clearly regular. $\square$

Remark: Alternatively, observe that the character $\chi_{\mathcal{A}} \colon \mathcal{A} \twoheadrightarrow \mathcal{A}/\mathfrak{m} = \kappa$ induces a SES

$$1 \to 1 \oplus \mathfrak{m} \hookrightarrow \mathcal{A}^\times \xrightarrow{\chi_{\mathcal{A}}^\times} \kappa^\times \to 1,$$

in Ab, which is right-splitted (possibly non-canonically in general) by the spectral inclusion $\iota_{\mathcal{A}}^\times := \iota_{\mathcal{A}}|\kappa^\times \hookrightarrow \mathcal{A}^\times$, hence indeed $\mathcal{A}^\times \cong \kappa^\times \times (1 \oplus \mathfrak{m}, \times_{\mathcal{A}})$ in Ab, from where one proceeds again as in the proof above. $\square$

Corollary-Definition 0.2.2: Let κ be an algebraically closed field of $\operatorname{char}(\kappa) = 0$ and $\mathcal{A} \in \mathsf{FdCAlg}_\kappa$. Then $\mathcal{A}^\times \cong \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{a}}^m$, where $n = \#\operatorname{Spec}\mathcal{A} = \#\operatorname{Spm}\mathcal{A} = \#\mathcal{X}(\mathcal{A})$ and $m = \dim_\kappa \mathcal{A} - n$. In this case $\mathcal{A}^\times$ is called for short *of type* (n, m).

Proof: We have $\mathcal{A} \cong \prod_{j=1}^n (\mathcal{A}_j, \mathfrak{m}_j, \kappa_j)$ with $(\mathcal{A}_j, \mathfrak{m}_j, \kappa_j) \in \mathsf{LocArt}_\kappa$, i.e. $\kappa_j = \kappa$, since κ_j/κ is finite and κ is algebraically closed, $1 \le j \le n$. Hence

$$\mathcal{A}^\times \cong \prod_{j=1}^n \mathcal{A}_j^\times \cong \prod_{j=1}^n (\kappa^\times \times (\mathfrak{m}_j, +)) = (\kappa^\times)^n \times \prod_{j=1}^n (\mathfrak{m}_j, +),$$

where $\dim_\kappa \prod_{j=1}^n (\mathfrak{m}_j, +) = \sum_{j=1}^n (\dim_\kappa \mathcal{A}_j - 1) = \dim_\kappa \mathcal{A} - n$ as desired. □

Example (due to Julian Rosen[26])**:** Conversely, any (n, m)-type occurs as the group of units of some $\mathcal{A} \in \mathsf{FdCAlg}_K$, namely:

$$\mathcal{A} := K^{n-1} \times K[x_1, \dots, x_m]/(x_1, \dots, x_m).$$

□

Proposition 0.2.3: Let $\mathcal{A} \in \mathsf{FdCAlg}_\mathbb{R}$. Then the following are equivalent:

(i) $\pi_0(\mathcal{A}^\times) = 1$;

(ii) $\mathcal{A}$ carries an (almost) complex structure;

(iii) $\mathcal{X}(\mathcal{A})$ consists only of *honestly* complex[27] characters.

Proof: Since $\mathbb{C}$ is the only non-trivial finite extension of $\mathbb{R}$ we have by Proposition 0.2.1 that $\mathcal{A}^\times \cong \prod_{j=1}^n \mathcal{A}_j^\times \cong \prod_{j=1}^n (\mathbb{K}_j^\times \times (\mathfrak{m}_j, +))$, where $\mathbb{K}_j \in \{\mathbb{R}, \mathbb{C}\}$ and these isomorphisms are clearly also as topological groups. Thus $\pi_0(\mathcal{A}^\times) = \prod_{j=1}^n \pi_0(\mathbb{K}_j^\times)$, whence $\pi_0(\mathcal{A}^\times) = 1 \Leftrightarrow \forall\, 1 \le j \le n \colon \mathbb{K}_j = \mathbb{C} \Leftrightarrow \forall\, 1 \le j \le n \colon \chi_j = \chi_{\mathcal{A}_j} \circ \mathrm{pr}_j^{\mathcal{A}} \in \mathcal{X}(\mathcal{A})$ is *honestly* $\mathbb{C}$-valued. □

Remarks:

(1) A different proof of the equivalence "(i) $\Leftrightarrow$ (ii)" in Proposition 0.2.3 using principal bundles is given in [GM96]. Incidentally, for some reason the authors stop short of stating / proving the explicit algebraic structure of $\mathcal{A}^\times$.

(2) The converse "(ii) $\Rightarrow$ (i)" remains true even if we drop commutativity of $\mathcal{A}$ because always $\operatorname{codim}_\mathbb{C} \operatorname{Sing}(\mathcal{A}) \ge 1$.

(3) Recall that in Lemma 0.1.62 we give a list of all possible (almost) complex structures on $\mathcal{A}$.

[26] see [MO01]

[27] i.e. their images are not contained in $\mathbb{R}$;

0.3. Normed Algebras & Some Matrix Analysis

All norms on finite-dimensional vector spaces are equivalent, but some norms are more equivalent than others.

By a *vector norm* on $\mathrm{M}_n(\mathbb{K})$ or, more generally, on any $\mathcal{A} \in \mathsf{FdAlg}_{\mathbb{K}}$ we shall mean a norm $\|\cdot\|$ that turns the underlying $\mathbb{K}$-vector space structure of $\mathcal{A}$ into a normed space, hence automatically into a $\mathbb{K}$-Banach space by finite-dimensionality of $\mathcal{A}$. By a *matrix norm* on $\mathrm{M}_n(\mathbb{K})$ or, more generally, an *algebra norm* on $\mathcal{A}$ we shall mean a vector norm $\|\cdot\|$ on $\mathcal{A}$ that is also *submultiplicative*, i.e. $\forall\, A, B \in \mathcal{A}$: $\|AB\| \leq \|A\|\,\|B\|$. Accordingly, such a norm turns $\mathcal{A}$ into a (finite-dimensional) Banach $\mathbb{K}$-algebra. Furthermore, a vector norm $\|\cdot\|$ on $\mathcal{A} \in \mathsf{FdAlg}_{\mathbb{K}}$ is called *normalized* (or *unital*) iff $\|1_{\mathcal{A}}\| = 1$. Note that in general we only have $\|1_{\mathcal{A}}\| \geq 1$, though.

While $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ itself comes equipped with a canonical (Archimedean) *absolute value* (in particular, *multiplicative*), this is manifestly not the case for a generic $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$. In trying to generalize complex analysis of a single variable to the broader setting of $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$, this is for the most part just a *cosmetic* issue. But every so often it also breaks some of the standard proofs, in which case it can present certain technical challenges that require a completely different line of argument. As is usually the case, this a feature, not a bug. Hence this section's goal is – among other things – to gauge one's expectations as to what is actually feasible in terms of norm properties in this more general context. To begin with, we remark that generally one cannot expect better than submultiplicativity, as the next two propositions amply show:

Proposition 0.3.1 (Gelfand-Mazur)**:** Let $\mathcal{A}$ be a normed division $\mathbb{C}$-algebra. Then $\mathcal{A} \cong \mathbb{C}$ isometrically. □

Corollary 0.3.2 ("Most $\mathbb{C}$-algebra norms are only submultiplicative")**:** Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) \in \mathsf{BanAlg}_{\mathbb{C}}$ such that $\|\cdot\|_{\mathcal{A}}$ is *multiplicative*. Then $\mathcal{A} \cong \mathbb{C}$ isometrically.

Proof: This is standard, so we only sketch the proof. One shows that, if $\forall\, x \in \mathcal{A}^\times :\ \left\|x^{-1}\right\| \leq \|x\|^{-1}$, then $\mathcal{A}^\times$ is closed in $\mathcal{A} \setminus \{0\}$. But $\mathcal{A}^\times$ is also open in $\mathcal{A} \setminus \{0\}$. However, since $\mathcal{A}$ is complex, $\mathcal{A} \setminus \{0\}$ is connected. Thus $\mathcal{A}^\times = \mathcal{A} \setminus \{0\}$, i.e. $\mathcal{A}$ is a division algebra, and Proposition 0.3.1 now implies the claim. □

Then there is also a real version of Proposition 0.3.1:

Proposition 0.3.3 (A "Frobenius Theorem" for normed algebras)**:** Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a normed division $\mathbb{R}$-algebra. Then $\mathcal{A} \cong \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ isometrically. □

However, the proof of Corollary 0.3.2 does not extend to $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) \in \mathsf{BanAlg}_{\mathbb{R}}$, since $\mathcal{A} \setminus \{0\}$ need no longer be connected, and so we cannot conclude that $\mathcal{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Indeed, there exist infinite-dimensional Banach $\mathbb{R}$-algebras with multiplicative norms (i.e. absolute values) [see UW60]. Nevertheless, a full list of all *finite-dimensional* real unital associative algebras admitting an absolute value is given by $\{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ as shown in [Alb47].

To reiterate, finite-dimensional (commutative) associative $\mathbb{K}$-algebras naturally fall into the category of (commutative) Banach algebras. But while Banach spaces and Banach algebras appear to be the most *convenient* setting that lends itself to our kind of analysis by virtue of their normedness and completeness, it is perhaps not the most *natural* one. This is because the Banach property is somewhat at odds with *algebraic niceness*: it is well-known that every commutative *Noetherian*

Banach algebra is in fact **finite-dimensional**. This even holds for the non-commutative case as well, see [ST74]. So, in a sense, the finite-dimensional setting is really forced upon us.

Suffices to say, norm theory on finite-dimensional $\mathbb{K}$-algebras can be easily considered a subject in its own right, especially when seen in the broader picture of *structured* Matrix Analysis. It should then come as no surprise that our main reference for this section will be [HJ13]. That said, we haven't actually established *existence* of algebra norms beyond absolute values yet. Let us quickly remedy that:

Lemma-Definition 0.3.4 (ℓ_1-norm)**:** For $A \in \mathrm{M}_n(\mathbb{K})$

$$\|A\|_1 := \sum_{i,j=1}^{n} |a_{ij}|$$

defines an algebra norm on $\mathrm{M}_n(\mathbb{K})$. □

In fact, a generic $\mathcal{A} \in \mathsf{FdAlg}_{\mathbb{K}}$ usually admits uncountably many (automatically equivalent) algebra norms, for instance as inherited from $\mathrm{M}_n(\mathbb{K})$ along some faithful representation $\rho\colon \mathcal{A} \hookrightarrow \mathrm{M}_n(\mathbb{K})$ (e.g. the regular representation $\rho_{\mathcal{A}}\colon \mathcal{A} \hookrightarrow \mathrm{M}_n(\mathbb{K})$ after fixing some $\mathbb{K}$-basis). We are suddenly spoilt for choice[28]. This necessitates the question as to what constitutes a good norm for some given application. While we can offer no general wisdom on the matter, we will try to gradually motivate our way through a few examples below.

A norm that is particularly close to geometric intuition is given by:

Lemma-Definition 0.3.5 (Frobenius[29] Norm)**:** For $A \in \mathrm{M}_n(\mathbb{K})$ the expression

$$\|A\|_{\mathrm{F}} := \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}}$$

defines a *matrix norm* on $\mathrm{M}_n(\mathbb{K})$. □

It arises from the *Frobenius inner product* $\langle A, B\rangle_{\mathrm{F}} := \mathrm{tr}(A^*B)$ and obviously coincides with the Euclidean norm on the *vector space* $\mathrm{M}_n(\mathbb{K})$ (with respect to the standard basis). Observe, however, that there is in general no a priori reason to expect that it would restrict to the Euclidean norm on some vector subspace $\mathcal{A} \cong \mathbb{K}^n$, e.g.

$$\left\|\begin{pmatrix} z & w \\ 0 & z \end{pmatrix}\right\|_{\mathrm{F}}^2 = 2\,|z|^2 + |w|^2 \neq |z|^2 + |w|^2$$

for $\mathcal{A} := \mathbb{K}[t]/(t^2) \hookrightarrow \mathrm{M}_2(\mathbb{K})$. This example illustrates the following interesting problem that appears intimately related to the possibility of *a finer structure theory* for $\mathcal{A} \in \mathsf{LocArt}_{\mathbb{K}}$:

Problem 3: Understand metric properties of $\mathcal{A} \xhookrightarrow{\rho_{\mathcal{A}}} \mathrm{M}_n(\mathbb{K})$ induced by some standard metrics and norms on $\mathrm{M}_n(\mathbb{K})$. □

Moreover, even though $\|\cdot\|_{\mathrm{F}}$ is indeed submultiplicative, it is unfortunately **not unital**, since $\|I_n\|_{\mathrm{F}} = \sqrt{n}$. This can be somewhat rectified – at the expense of a positive constant – via the following procedure:

Lemma 0.3.6 (Algebra norms out of vector norms)**:** Let $\mathcal{A} \in \mathsf{FdAlg}_{\mathbb{K}}$ and let $\|\cdot\|$ be a *vector norm* on $\mathcal{A}$.

[28] die Qual der Wahl;

[29] also called ℓ_2-norm, or a Schur norm, or a Hilbert-Schmidt norm;

(1) The rescaled vector norm $\alpha\|\cdot\|$ is an *algebra norm* if and only if $\alpha \geq \alpha_0$, where

$$\alpha_0 := \max\{\|AB\| : A, B \in \mathcal{A} \text{ with } \|A\| = \|B\| = 1\}.$$

(2) If $\|\cdot\|'$ is an *algebra norm* and the equivalence between $\|\cdot\|'$ and $\|\cdot\|$ is expressed via $c\|\cdot\|' \leq \|\cdot\| \leq C\|\cdot\|'$ for some $C, c > 0$, then $\alpha\|\cdot\|$ is also an algebra norm for $\alpha := C/c^2$.

Proof: This is [HJ13, Thm. 5.7.11], where it is stated for $\mathrm{M}_n(\mathbb{C})$, but the proof there carries through for any $\mathcal{A} \in \mathsf{FdAlg}_{\mathbb{K}}$ since the unit $\|\cdot\|$-sphere is always compact in the finite-dimensional setting. $\square$

Corollary 0.3.7: If $\|\cdot\|_2$ is the Euclidean norm on the vector space $\mathcal{A} \cong \mathbb{K}^n$, then there exists a constant $c = c_{\mathcal{A}} > 0$, depending solely on $\mathcal{A}$, such that $\forall\, A, B \in \mathcal{A}$: $\|AB\|_2 \leq c\|A\|_2\|B\|_2$. $\square$

In the same spirit:

Lemma-Definition 0.3.8 (ℓ_∞-norm)**:** Let $A \in \mathrm{M}_n(\mathbb{K})$, then

$$\|A\|_\infty := \max_{1 \leq i,j \leq n} |a_{ij}|$$

defines only a *vector norm* on $\mathrm{M}_n(\mathbb{K})$. Nevertheless, the rescaled ℓ_∞-norm

$$\|A\| := n\|A\|_\infty$$

is *submultiplicative.* $\square$

Notice that like with $\|\cdot\|_{\mathrm{F}}$ neither of these two algebra norms is *unital.* This suggests the following (extremely vague) intuition:

Meta-Lemma 3: *Geometric* algebra norms are not normalized and normalized algebra norms are perhaps not very intuitively geometric (with the judgement on that yet to pass). $\square$

So, let us now turn to unital algebra norms. There is a completely standard way to produce such:

Lemma-Definition 0.3.9 (Induced Norm)**:**

(1) Let $(V, \|\cdot\|_V), (W, \|\cdot\|_W) \in \mathsf{NrmVec}_{\mathbb{K}}$ and $\Phi \in \mathrm{Hom}_{\mathbb{K}}(V, W)$. Then

$$\|\Phi\|_{\mathrm{op}} := \|\Phi\|_{V,W} := \sup_{v \in V} \frac{\|\Phi(v)\|_W}{\|v\|_V} \stackrel{\text{def}}{=} \sup_{\|v\|_V = 1} \|\Phi(v)\|_W$$

will be called the *induced norm* or the *operator norm* on $\mathrm{Hom}_{\mathbb{K}}(V, W)$.

(2) If $(V, \|\cdot\|) \in \mathsf{NrmVec}_{\mathbb{K}}$, then the operator norm $\|\cdot\|_{\mathrm{End}} := \|\cdot\|_{\mathrm{End}_{\mathbb{K}}(V)} := \|\cdot\|_{\mathrm{op}} := \|\cdot\|_{V,V}$ is a *normalized algebra norm.*

(3) Let $p, q \in [1, \infty]$ and $(V, \|\cdot\|_p), (W, \|\cdot\|_q) \in \mathsf{FdVec}_{\mathbb{K}}$ be equipped with the p- and q-norm (with respect to some predefined bases), respectively. Then we put

$$\forall\, \Phi \in \mathrm{Hom}_{\mathbb{K}}(V, W) \colon\ \|\Phi\|_{p,q} := \|\Phi\|_{V,W} \stackrel{\text{def}}{=} \sup_{v \in V} \frac{\|\Phi(v)\|_q}{\|v\|_p}$$

for short. $\square$

Remarks:

(1) One can show that the notation of Lemma-Definition 0.3.9 is consistent with Lemma-Definition 0.3.4 and Lemma-Definition 0.3.8 in the sense that one indeed has $\forall\, A \in \mathrm{M}_n(\mathbb{K})$: $\|A\|_1 = \|A\|_{1,1}$ and $\|A\|_\infty = \|A\|_{\infty,\infty}$. Despite this, in all other cases one should not confuse the *induced* p-norm $\|\cdot\|_{p,p}$ with the *vector* p-norm on $\mathrm{M}_n(\mathbb{K})$, even though the shorter notation $\|\cdot\|_p := \|\cdot\|_{p,p}$ is also quite widespread in the literature. We will **not** be using the latter notation so as to avoid potential confusion. In particular, $\|\cdot\|_{2,2} \neq \|\cdot\|_{\mathrm{F}}$.

(2) Not all normalized algebra norms arise as *induced* norms. For example, one can show that the normalized algebra norm defined by

$$\|A\| := \max\{\|A\|_1, \text{max-row-sum-norm}(A)\}$$

on $\mathrm{M}_n(\mathbb{K})$ is **not** induced by any vector norm on $\mathbb{K}^n$. For the time being we are only interested in the *existence of normalized norms*, for which the above lemma suffices completely. □

Lemma-Definition 0.3.10 (Spectral Norm)**:** The operator norm

$$\|\cdot\|_{\mathrm{sp}} := \|\cdot\|_{2,2}$$

we be called *spectral norm* on $\mathrm{M}_n(\mathbb{K})$. We have

$$\forall\, A \in \mathrm{M}_n(\mathbb{K})\colon\ \|A\|_{\mathrm{sp}} = s_{\max}(A), \tag{0.3.1}$$

where $s_{\max}(A)$ denotes the largest *singular value*[30] of A. □

Remark: More abstractly, specifying a (Hermitian) inner product on $V \in \mathsf{FdVec}_{\mathbb{K}}$ gives rise to $V \xrightarrow{\cong} V^*$, which in turn induces an involution $(-)^*$ on $\mathrm{End}_{\mathbb{K}}(V) = V \otimes_{\mathbb{K}} V^*$. Then $\forall\, \Phi \in \mathrm{End}_{\mathbb{K}}(V)\colon \|\Phi\|_{\mathrm{sp}} := \sqrt{\lambda_{\max}(\Phi\Phi^*)} \overset{\text{def}}{=} s_{\max}(\Phi)$, where $\lambda_{\max}$ denotes the largest eigenvalue. In other words, $\|\cdot\|_{\mathrm{sp}}$ (equivalently, $s_{\max}$) actually depends on the choice of an inner product structure on V rather than a $\mathbb{K}$-basis. □

Lemma 0.3.11 (Standard Algebra Norm Relations)**:** Let $A, B \in \mathrm{M}_n(\mathbb{K})$ be arbitrary. We have the following inequality relations between algebra norms:

(1) $\|A\|_{\mathrm{sp}} \le \|A\|_{\mathrm{F}} \le \sqrt{n}\,\|A\|_{\mathrm{sp}}$;
(2) $\|AB\|_{\mathrm{F}} \le \|A\|_{\mathrm{sp}}\,\|B\|_{\mathrm{F}}$ (stronger submultiplicativity of the Frobenius norm);
(3) $\|A\|_{\mathrm{sp}}^2 \le \|A\|_{1,1}\,\|A\|_{\infty,\infty}$. □

The abundance of *necessarily equivalent* algebra norms on $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$ can be leveraged as follows. Since many analytic statements are essentially independent of the choice of an *equivalent* norm, or rather, are valid for all (equivalent) norms *simultaneously*, this suggests – aided by the next proposition – that in many situations a more appropriate notion to consider instead is that of the *spectral radius*, even though it is itself not a norm.

Definition 0.3.12 (Spectral Radius)**:** Let $A \in \mathrm{M}_n(\mathbb{K})$ and let $\sigma(A)$ denote its spectrum. Then

$$\varrho(A) := \max\{|z| : z \in \sigma(A)\}$$

is called the *spectral radius* of A. Given $\mathcal{A} \hookrightarrow \mathrm{M}_n(\mathbb{K})$, we will also denote $\varrho_{\mathcal{A}} := \varrho|_{\mathcal{A}}$.

Proposition 0.3.13: We have:

(1) $\forall\, A \in \mathrm{M}_n(\mathbb{C})\, \forall$ matrix norm $\|\cdot\|:\ \|A\| \ge \varrho(A)$;
(2) $\forall\, A \in \mathrm{GL}_n(\mathbb{C})\, \forall$ matrix norm $\|\cdot\|:\ \left\|A^{-1}\right\| \ge \varrho(A)^{-1}$;
(3) $\forall\, A \in \mathrm{M}_n(\mathbb{C})\ \forall \varepsilon > 0\ \exists$ matrix norm $\|\cdot\| : \varrho(A) \le \|A\| \le \varrho(A) + \varepsilon$;
(4) $\forall\, A \in \mathrm{M}_n(\mathbb{C})\colon \varrho(A) = \inf\{\|A\| : \|\cdot\| \text{ matrix norm}\} = \inf\{\|A\|_{\mathrm{op}} : \|\cdot\|_{\mathrm{op}} \text{ operator norm}\}$.

Proof: These are [HJ13, Thm. 5.6.9] and [HJ13, Lemma 5.6.10] . □

Remark: Notice that the algebra norm $\|\cdot\|$ in (3) depends not only on ε but also on A.

The following is an easy but very useful consequence of Proposition 0.3.13 (3), that allows us to move back and forth between spectral radius and algebra norms:

[30] the more standard notation for this is $\sigma_{\max}(A)$ instead, but we will not be using it due to notational overload from $\sigma(A)$ denoting the spectrum of A or $\sigma \in \mathrm{Aut}_{\mathbb{K}}(\mathcal{A})$ denoting a $\mathbb{K}$-algebra automorphism of $\mathcal{A}$;

Corollary 0.3.14 (Principle of spectral comparison)**:** Let $A, B \in \mathrm{M}_n(\mathbb{C})$.
(1) If $\forall$ *matrix norm* $\|\cdot\|$: $\|A\| \leq \|B\|$, then $\varrho(A) \leq \varrho(B)$.
(2) Conversely, if $\varrho(A) < \varrho(B)$, then $\exists$ *matrix norm* $\|\cdot\|$: $\|A\| < \|B\|$.

Proof: To (1): take $\inf_{\|\cdot\|}$ on both sides and apply Proposition 0.3.13 (4).

To (2): Let $\varepsilon > 0$ such that $\varrho(A) + \varepsilon < \varrho(B)$. Then by Proposition 0.3.13 there exists an algebra norm $\|\cdot\|$ on $\mathrm{M}_n(\mathbb{K})$, depending on both A and ε, such that $\|A\| \leq \varrho(A) + \varepsilon < \varrho(B) \leq \|B\|$. ☐

Furthermore, we have:

Proposition 0.3.15 (Gelfand)**:** Let $A \in \mathrm{M}_n(\mathbb{C})$ and let $\|\cdot\|$ be an arbitrary matrix norm. Then:
(1) $\lim\limits_{k\to\infty} A^k = 0$ if and only if $\varrho(A) < 1$;
(2) $\varrho(A) = \lim\limits_{k\to\infty} \|A^k\|^{\frac{1}{k}}$.

Proof: This is [HJ13, Thm. 5.6.12] and [HJ13, Cor. 5.6.14]. ☐

Lemma 0.3.16: We have:
(1) $\varrho \geq 0$;
(2) $\varrho(I_n) = 1$;
(3) $\forall\, z \in \mathbb{K}\ \forall\, A$: $\varrho(zA) = |z|\,\varrho(A)$;
(4) $\forall\, [A, B] = 0$: $\varrho(A + B) \leq \varrho(A) + \varrho(B)$;
(5) $\forall\, [A, B] = 0$: $\varrho(AB) \leq \varrho(A)\varrho(B)$;

Proof: (4) is a standard exercise, and (5) follows for example from Proposition 0.3.15 (2). ☐

Remarks:
(1) The property $[A, B] = 0$ for $A, B \in \mathrm{M}_n(\mathbb{K})$ is crucial for (4) and (5) to hold.
(2) $\varrho(A) = 0$ need not imply $A = 0$, e.g. take $A \coloneqq \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. ☐

Lemma 0.3.17 (Spectral radius in $\mathsf{FdCAlg}_{\mathbb{K}}$)**:** Let $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$ and let $\|\cdot\|_{\mathcal{A}}$ be an arbitrary algebra norm on $\mathcal{A}$.
(1) The spectral radius $\varrho_{\mathcal{A}}$ is a *normalized algebra seminorm* on $\mathcal{A}$, that is:
  (i) $\varrho_{\mathcal{A}} \geq 0$;
  (ii) $\varrho_{\mathcal{A}}(1_{\mathcal{A}}) = 1$;
  (iii) $\forall\, z \in \mathbb{K}\ \forall\, A \in \mathcal{A}$: $\varrho_{\mathcal{A}}(zA) = |z|\,\varrho(A)$;
  (iv) $\forall\, A, B \in \mathcal{A}$: $\varrho_{\mathcal{A}}(A + B) \leq \varrho_{\mathcal{A}}(A) + \varrho_{\mathcal{A}}(B)$;
  (v) $\forall\, A, B \in \mathcal{A}$: $\varrho_{\mathcal{A}}(AB) \leq \varrho_{\mathcal{A}}(A)\varrho_{\mathcal{A}}(B)$;
(2) If $\mathcal{A} = \prod_{j=1}^{n}(\mathcal{A}_j, \mathfrak{m}_j, \mathbb{K}_j)$, $\mathbb{K}_j \in \{\mathbb{R}, \mathbb{C}\}$, then writing $Z = (z_1 \oplus X_1, \ldots, z_n \oplus X_n) \in \mathcal{A}$ for $z_j \in \mathbb{K}_j$ and $X_j \in \mathfrak{m}_j$, $1 \leq j \leq n$, we have:
$$\forall\, Z \in \mathcal{A}:\ \varrho_{\mathcal{A}}(Z) = \max_{1\leq j\leq n} |z_j|\,. \tag{0.3.2}$$
Hence $\forall\, Z \in \mathcal{A}$:
$$\begin{aligned} \|Z\|_{\mathcal{A}} &\geq \max_{1\leq j\leq n} |z_j|\,, \\ \left\|Z^{-1}\right\|_{\mathcal{A}} &\geq \Big(\min_{1\leq j\leq n} |z_j|\Big)^{-1}. \end{aligned} \tag{0.3.3}$$
(3) If $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{K})$, then:
  (i) $\forall\, Z \in \mathcal{A}$: $\varrho_{\mathcal{A}}(Z) \geq 0$;
  (ii) $\varrho_{\mathcal{A}}(1_{\mathcal{A}}) = 1$;
  (iii) $\forall\, \lambda \in \mathbb{K}\ \forall\, Z \in \mathcal{A}$: $\varrho_{\mathcal{A}}(\lambda A) = |\lambda|\,\varrho(Z)$;
  (iv) $\forall\, Z_1, Z_2 \in \mathcal{A}$: $\varrho_{\mathcal{A}}(Z_1 + Z_2) \leq \varrho_{\mathcal{A}}(Z_1) + \varrho_{\mathcal{A}}(Z_2)$;

(v) $\varrho_{\mathcal{A}} = |\cdot| \circ \chi_{\mathcal{A}}$, hence $\forall\, Z_1, Z_2 \in \mathcal{A}\colon \varrho_{\mathcal{A}}(Z_1 Z_2) = \varrho_{\mathcal{A}}(Z_1)\varrho_{\mathcal{A}}(Z_2)$;

(vi) $\mathfrak{m} = \{Z \in \mathcal{A} : \varrho_{\mathcal{A}}(Z) = 0\}$ and $\mathcal{A}^\times = \{Z \in \mathcal{A} : \varrho_{\mathcal{A}}(A) > 0\}$;

(vii) $\forall\, Z \in \mathcal{A}^\times\colon \left\|Z^{-1}\right\|_{\mathcal{A}} \geq 1/\,|z|$.

It follows that in the special case of a local $\mathcal{A}$, $\varrho_{\mathcal{A}}$ is in fact a *"semi"*[31] *absolute value* on $\mathcal{A}$. Conversely, if $\varrho_{\mathcal{A}}$ is *multiplicative*, then $\mathcal{A}$ must be local.

Proof: To (1): immediate from Lemma 0.3.16.

To (2): immediate from the triangular representation of $\mathcal{A}$ (see also Corollary 0.1.73) and Proposition 0.3.13.

To (3): follows from (2). □

Due to the only-submultiplicativity property of algebra norms it is generally difficult to tightly estimate inverses *from above*. We collect next some results that can help with that.

Lemma 0.3.18: Let $\|\cdot\|$ be a unital matrix norm on $\mathrm{M}_n(\mathbb{C})$ and let $A \in \mathrm{M}_n(\mathbb{C})$ with $\|A\| < 1$. Then:

$$\frac{1}{1 + \|A\|} \leq \left\|\frac{1}{1 - A}\right\| \leq \frac{1}{1 - \|A\|} \tag{0.3.4}$$

Proof: Standard exercise. □

But when $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{K})$, we can do in fact better:

Lemma 0.3.19 (Standard Upper Bound of the Inverse in $(\mathcal{A}, \mathfrak{m}, \mathbb{K})$)**:** Let $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{K})$ be equipped with an unital algebra norm $\|\cdot\|_{\mathcal{A}}$. Write $Z := z \oplus X \in \mathcal{A}^\times$ for $z \in \mathbb{C}^\times$ and $X \in \mathfrak{m}$, and let $\nu := \nu(X) := \min\{k \in \mathbb{N} : X^k = 0\}$. Then:

$$\left\|Z^{-1}\right\|_{\mathcal{A}} \leq \frac{1}{|z|}\sum_{k=0}^{\nu-1}\left(\frac{\|X\|_{\mathcal{A}}}{|z|}\right)^k = \begin{cases} \frac{1 - \|X\|_{\mathcal{A}}^\nu/|z|^\nu}{|z| - \|X\|_{\mathcal{A}}}, & \text{if } |z| \neq \|X\|_{\mathcal{A}} \\ \frac{\nu}{|z|}, & \text{if } |z| = \|X\|_{\mathcal{A}}\,. \end{cases} \tag{0.3.5}$$

In particular, if $\|X\|_{\mathcal{A}} \leq |z|$, then we obtain the crude but independent of $\|X\|_{\mathcal{A}}$ estimate:

$$\left\|Z^{-1}\right\|_{\mathcal{A}} \leq \frac{\nu}{|z|}. \tag{0.3.6}$$

More generally:

$$\forall\, n \in \mathbb{N}_0\colon \left\|Z^{-n}\right\|_{\mathcal{A}} \leq \frac{1}{|z|^n}\sum_{k=0}^{\nu-1}\binom{n+k-1}{k}\frac{\|X\|_{\mathcal{A}}^k}{|z|^k} \tag{0.3.7}$$

Proof: Since $X \in \mathfrak{m} = \mathrm{nil}(\mathcal{A})$ with $X^\nu = 0$, we have the finite geometric series

$$Z^{-1} = \frac{1}{z \oplus X} = \frac{1}{z(1 - (-X/z))} = \frac{1}{z}\sum_{k=0}^{\nu-1}(-1)^k\frac{X^k}{z^k},$$

which implies the first inequality. The second inequality is then just a special case of the first one. The third inequality follows similarly from the above geometric series. □

Remarks:

(1) Note that $\nu \in \mathbb{N}$ can be instead chosen uniformly for the whole $\mathfrak{m}$ as $\mathfrak{m}$ itself is nilpotent. This is of course less optimal than picking ν for an individual X.

(2) Indeed, the estimate in Eq. (0.3.5) is not only valid for $\|X\|_{\mathcal{A}} \geq |z|$ unlike Eq. (0.3.4), but it is also sharper than Eq. (0.3.4) for the case $\|X\|_{\mathcal{A}} < |z|$: since $\|X\|_{\mathcal{A}} < |z|$, we have

[31] for a lack of a better term;

$1 - (\|X\|_{\mathcal{A}} / |z|)^{\nu} < 1$, hence

$$\left\|Z^{-1}\right\|_{\mathcal{A}} \leq \frac{1 - \|X\|_{\mathcal{A}}^{\nu} / |z|^{\nu}}{|z| - \|X\|_{\mathcal{A}}} < \frac{1}{|z| - \|X\|_{\mathcal{A}}}.$$

Proposition 0.3.20 (Yun–Gu–Piazza–Politi–Güngör)**:** Let $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{K})$ with $\dim_{\mathbb{K}} \mathcal{A} = n$. Then:

$$\forall\, Z \in \mathcal{A}^{\times}:\ \left\|Z^{-1}\right\|_{\mathrm{sp}} = \frac{1}{s_{\min}(Z)} \leq \frac{1}{(n-1)^{\frac{n-1}{2}}} \frac{\|Z\|_{\mathrm{F}}^{n-1}}{|z|^{n}} \leq \left(\frac{n}{n-1}\right)^{\frac{n-1}{2}} \frac{\|Z\|_{\mathrm{sp}}^{n-1}}{|z|^{n}} \tag{0.3.8}$$

where $s_{\min}(Z)$ denotes the smallest singular value of Z.

Proof: Since $\dim_{\mathbb{K}} \mathcal{A} = n$, there exists by Section 0.1.12 a lower-triangular faithful representation $\rho\colon \mathcal{A} \hookrightarrow \mathrm{M}_n(\mathbb{K})$, in particular $\det(Z) = z^n$ for $Z \in \mathcal{A}$. The statement then follows from an inequality obtained by [YG97] and later independently by [PP02] with correction by [Gün10] (see also the accounts in [Zou12] and [LX21]). □

Example: Let $\mathcal{A} := \mathbb{K}[t]/(t^2) = \mathbb{K}[\epsilon]$ for $\epsilon \neq 0$, $\epsilon^2 = 0$. Then

$$\mathcal{A} \cong \left\{ Z = \begin{pmatrix} z & 0 \\ w & z \end{pmatrix} : z, w \in \mathbb{K} \right\},$$

whence one can calculate

$$\|Z\|_{\mathcal{A}} := \|Z\|_{\mathrm{sp}} = s_{\max}(Z) = \frac{\sqrt{4\,|z|^2 + |w|^2} + |w|}{2}, \tag{0.3.9}$$

which is smooth away from $\mathcal{A} \setminus \mathbb{K}$. Beyond that, since $\nu = n = 2$ and $s_{\max}(X) = |w|$, Eq. (0.3.5) yields the estimate

$$\left\|Z^{-1}\right\|_{\mathrm{sp}} \leq \frac{1}{|z|}\left(1 + \frac{|w|}{|z|}\right) = \frac{|z| + |w|}{|z|^2},$$

whereas Eq. (0.3.8) gives

$$\left\|Z^{-1}\right\|_{\mathrm{sp}} \leq \frac{\sqrt{2}}{|z|^2} \frac{\sqrt{4\,|z|^2 + |w|^2} + |w|}{2} = \frac{\sqrt{4\,|z|^2 + |w|^2} + |w|}{\sqrt{2}\,|z|^2} > \frac{|z| + |w|}{|z|^2}$$

away from 0. □

Remark: Even though the above example shows that Eq. (0.3.8) generically yields a looser bound than Eq. (0.3.5), it relates $\left\|Z^{-1}\right\|_{\mathrm{sp}}$ to $\|Z\|_{\mathrm{sp}}$ in a rather nice closed-form expression. This has potential interpretation advantages.

Problem 4 (Structured Smoothness of $\|\cdot\|_{\mathrm{sp}}$)**:** It is well-known that $\|\cdot\|_{\mathrm{sp}}$ generally fails to be differentiable away from 0 at points (= matrices) that have eigenvalues of multiplicity > 1. On the other hand, observe that $Z \in \mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{K})$ has exactly one eigenvalue z, but of maximal multiplicity $n = \dim_{\mathbb{K}}(\mathcal{A})$. Despite that, low-dimensional examples as the above suggest the surprising behavior that $\|\cdot\|_{\mathrm{sp}}$ is differentiable (really, real-analytic) away from $\mathbb{K} \hookrightarrow \mathcal{A}$. Prove or disprove this observation. □

Lemma 0.3.21: Let $V \in \mathsf{FdVec}_{\mathbb{K}}$ and $v_1, v_2 \in V$. Then $v_1 = \lambda v_2$ for some $|\lambda| = 1$ if and only if $\forall\, \|\cdot\|_V :\ \|v_1\|_V = \|v_2\|_V$.

Proof: If v_1 and v_2 are collinear, this is clear. Suppose that they are not collinear and define $W := \mathrm{span}_{\mathbb{K}}\{v_1, v_2\} \subseteq V$ a 2-dimensional $\mathbb{K}$-vector subspace. Further define $\|x_1 v_1 + x_2 v_2\|_W := \sqrt{x_1^2 + \alpha x_2^2}$ for some $\alpha > 0$, $\alpha \neq 1$. It is standard that $\|\cdot\|_W$ is indeed a norm on $W \subseteq V$ and that it can be extended to a norm on V (e.g. utilize a direct complement). But $\|v_1\|_W = 1$, $\|v_2\|_W = \alpha \neq 1$, a contradiction. □

Definition 0.3.22: Let $(V, \|\cdot\|_V) \in \mathsf{NrmVec}_{\mathbb{K}}$, $v \in V$, and $S \subseteq V$. We put:

(1) Distance of v from S with respect to $\|\cdot\|_V$: $\operatorname{dist}_V(v, S) \coloneqq \inf_{w \in S} \|v - w\|_V$;

(2) Diameter of S with respect to $\|\cdot\|_V$: $\operatorname{diam}_V(S) \coloneqq \sup_{v,w \in S} \|v - w\|_V$.

Lemma 0.3.23 (Quotient Norm)**:** Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) \in \mathsf{NrmAlg}_{\mathbb{K}}$ and $I \lhd \mathcal{A}$ *closed*. Then

$$\|\bar{a}\|_{\mathcal{A}/I} \coloneqq \inf_{\alpha \in \bar{a}} \|\alpha\|_{\mathcal{A}} \overset{\text{def}}{=} \operatorname{dist}_{\mathcal{A}}(a, I)$$

defines again an algebra norm on $\mathcal{A}/I$. If $\|\cdot\|_{\mathcal{A}}$ is unital, then so is $\|\cdot\|_{\mathcal{A}/I}$.

Proof: It is standard that $\|\cdot\|_{\mathcal{A}/I}$ is a well-defined *vector* norm on $\mathcal{A}/I$. For the submultiplicativity we have $\forall\, a_1, a_2 \in \mathcal{A}$:

$$\begin{aligned}\|\bar{a}_1\|_{\mathcal{A}/I} \|\bar{a}_2\|_{\mathcal{A}/I} &= \inf_{x_1,x_2 \in I} \|a_1 - x_1\|_{\mathcal{A}} \|a_2 - x_2\|_{\mathcal{A}} \geq \inf_{x_1,x_2 \in I} \|(a_1 - x_1)(a_2 - x_2)\|_{\mathcal{A}} = \\ &= \inf_{x_1,x_2 \in I} \|a_1a_2 - a_1x_2 - a_2x_1 + x_1x_2\|_{\mathcal{A}} \geq \inf_{x \in I} \|a_1a_2 - x\| \overset{\text{def}}{=} \|\overline{a_1a_2}\|_{\mathcal{A}/I} = \\ &= \|\bar{a}_1\bar{a}_2\|_{\mathcal{A}/I}\end{aligned}$$

since $a_1x_2 + a_2x_1 - x_1x_2 \in I \lhd \mathcal{A}$. Finally, if $\|\cdot\|_{\mathcal{A}}$ is unital, then

$$\left\|1_{\mathcal{A}/I}\right\|_{\mathcal{A}/I} \overset{\text{def}}{=} \operatorname{dist}_{\mathcal{A}}(1_{\mathcal{A}}, I) \leq \operatorname{dist}_{\mathcal{A}}(1_{\mathcal{A}}, 0_{\mathcal{A}}) = \|1_{\mathcal{A}}\|_{\mathcal{A}} = 1,$$

hence $\left\|1_{\mathcal{A}/I}\right\|_{\mathcal{A}/I} = 1$ since we have in any case $\left\|1_{\mathcal{A}/I}\right\|_{\mathcal{A}/I} \geq 1$ by submultiplicativity. □

Recall that given $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) \in \mathsf{NrmAlg}_{\mathbb{K}}$, we have by definition $(M, \|\cdot\|_M) \in \mathsf{NrmLMod}(\mathcal{A})$ if $M \in \mathsf{LMod}(\mathcal{A})$ and the following compatability relation between the norms holds:

$$\forall\, a \in \mathcal{A}\; \forall\, x \in M\colon\; \|a \cdot x\|_M \leq \|a\|_{\mathcal{A}} \|x\|_M\,.$$

Analogously for $\mathsf{NrmRMod}(\mathcal{A})$ and $\mathsf{NrmBMod}(\mathcal{A})$. In the same vein, given now $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) \in \mathsf{NrmCAlg}_{\mathbb{K}}$, we have $(\mathcal{B}, \|\cdot\|_{\mathcal{B}}) \in \mathsf{NrmAlg}(\mathcal{A})$ if $\mathcal{B} \in \mathsf{Alg}(\mathcal{A})$ and

$$\forall\, a \in \mathcal{A}\; \forall\, b \in \mathcal{B}\colon\; \|a \cdot b\|_{\mathcal{B}} \leq \|a\|_{\mathcal{A}} \|b\|_{\mathcal{B}}\,.$$

Lemma 0.3.24 (Tautological Norm Compatibility)**:** Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) \in \mathsf{NrmAlg}_{\mathbb{K}}$ with $\|\cdot\|_{\mathcal{A}}$ being unital and $(M, \|\cdot\|_M) \in \mathsf{LMod}(\mathcal{A})$ via some (continuous) $\rho\colon \mathcal{A} \to \operatorname{End}_{\mathbb{K}}(M)$.

(1) $\|\rho\|_{\text{op}} \overset{\text{def}}{=} \|\rho\|_{\mathcal{A},\operatorname{End}_{\mathbb{K}}(M)} = 1 \Leftrightarrow (M, \|\cdot\|_M) \in \mathsf{NrmLMod}(\mathcal{A})$.

(2) If $(M, \|\cdot\|_M) \in \mathsf{NrmLMod}(\mathcal{A})$, then $\left(\operatorname{End}_{\mathbb{K}}(M), \|\cdot\|_{\operatorname{End}_{\mathbb{K}}(M)} \overset{\text{def}}{=} \|\cdot\|_{M,M}\right) \in \mathsf{NrmBMod}(\mathcal{A})$.

(3) If $(M, \|\cdot\|_M) \in \mathsf{NrmLMod}(\mathcal{A})$, then only $\|\cdot\|_{\operatorname{End}_{\mathbb{K}}(M)}|_{\rho(\mathcal{A})} \leq \|\cdot\|_{\mathcal{A}/\operatorname{Ker}\rho}$ via $\rho(\mathcal{A}) \cong \mathcal{A}/\operatorname{Ker}\rho$.

Proof: To (1): "$\Leftarrow$": We have:

$$\begin{aligned}\|\rho\|_{\text{op}} &\overset{\text{def}}{=} \|\rho\|_{\mathcal{A},\operatorname{End}_{\mathbb{K}}(M)} \overset{\text{def}}{=} \sup_{\|a\|_{\mathcal{A}}=1} \|\rho(a)\|_{\operatorname{End}_{\mathbb{K}}(M)} \overset{\text{def}}{=} \sup_{\|a\|_{\mathcal{A}}=1} \sup_{\|x\|_M=1} \|\rho(a)x\|_M \overset{\text{def}}{=} \sup_{\|a\|_{\mathcal{A}}=1} \sup_{\|x\|_M=1} \|a \cdot x\|_M \leq \\ &\leq \sup_{\|a\|_{\mathcal{A}}=1} \sup_{\|x\|_M=1} \|a\|_{\mathcal{A}} \|x\|_M = 1.\end{aligned}$$

On the other hand, since both $\|\cdot\|_{\operatorname{End}_{\mathbb{K}}(M)}$ and $\|\cdot\|_{\mathcal{A}}$ are unital, we get

$$\|\rho\|_{\text{op}} \overset{\text{def}}{=} \sup_{\|a\|_{\mathcal{A}}=1} \|\rho(a)\|_{\operatorname{End}_{\mathbb{K}}(M)} \geq \|\rho(1_{\mathcal{A}})\|_{\operatorname{End}_{\mathbb{K}}(M)} = 1.$$

It follows that $\|\rho\|_{\text{op}} = 1$.

"$\Rightarrow$": We have indeed $\forall\, a \in \mathcal{A}\; \forall\, x \in M$:

$$\|a \cdot x\|_M \overset{\text{def}}{=} \|\rho(a)x\|_M \leq \|\rho(a)\|_{\operatorname{End}_{\mathbb{K}}(M)} \|x\|_M \leq \|\rho\|_{\mathcal{A},\operatorname{End}_{\mathbb{K}}(M)} \|a\|_{\mathcal{A}} \|x\|_M = \|a\|_{\mathcal{A}} \|x\|_M$$

as claimed.

To (2): For the $\mathsf{NrmLMod}(\mathcal{A})$-part we have $\forall\, a \in \mathcal{A}\ \forall\, \Phi \in \mathrm{End}_{\mathbb{K}}(M)$:

$$\|a \cdot \Phi\|_{\mathrm{End}_{\mathbb{K}}(M)} \overset{\text{def}}{=} \|\rho(a)\Phi\|_{\mathrm{End}_{\mathbb{K}}(M)} \overset{\text{def}}{=} \sup_{\|x\|_M=1} \|(\rho(a)\Phi)x\|_M \overset{\text{def}}{=} \sup_{\|x\|_M=1} \|a \cdot (\Phi x)\|_M \leq$$
$$\leq \sup_{\|x\|_M=1} \|a\|_{\mathcal{A}} \|\Phi x\|_M = \|a\|_{\mathcal{A}} \sup_{\|x\|_M=1} \|\Phi x\|_M \overset{\text{def}}{=} \|a\|_{\mathcal{A}} \|\Phi\|_{\mathrm{End}_{\mathbb{K}}(M)} .$$

For the $\mathsf{NrmRMod}(\mathcal{A})$-part we have $\forall\, a \in \mathcal{A}\ \forall\, \Phi \in \mathrm{End}_{\mathbb{K}}(M)$:

$$\|\Phi \cdot a\|_{\mathrm{End}_{\mathbb{K}}(M)} \overset{\text{def}}{=} \sup_{\|x\|_M=1} \|(\Phi\rho(a))x\|_M \leq \sup_{\|x\|_M=1} \|\Phi\rho(a)\|_{\mathrm{End}_{\mathbb{K}}(M)} \|x\|_M = \|\Phi\rho(a)\|_{\mathrm{End}_{\mathbb{K}}(M)} \leq$$
$$\leq \|\Phi\|_{\mathrm{End}_{\mathbb{K}}(M)} \|\rho(a)\|_{\mathrm{End}_{\mathbb{K}}(M)} \leq \|\Phi\|_{\mathrm{End}_{\mathbb{K}}(M)} \|\rho\|_{\mathrm{op}} \|a\|_{\mathcal{A}} \overset{(1)}{=} \|\Phi\|_{\mathrm{End}_{\mathbb{K}}(M)} \|a\|_{\mathcal{A}}$$

as desired.

To (3): We have indeed $\forall\, a \in \mathcal{A}\ \forall\, x \in \operatorname{Ker}\rho$: $\|\rho(a)\|_{\mathrm{End}_{\mathbb{K}}(M)} = \|\rho(a+x)\|_{\mathrm{End}_{\mathbb{K}}(M)} \leq \|a+x\|_{\mathcal{A}}$ by (1), hence $\forall\, a \in \mathcal{A}$: $\|\rho(a)\|_{\mathrm{End}_{\mathbb{K}}(M)} \leq \inf_{x \in \operatorname{Ker}\rho} \|a+x\|_{\mathcal{A}} \overset{\text{def}}{=} \|\bar{a}\|_{\mathcal{A}/\operatorname{Ker}\rho}$ as claimed. □

Remark: Observe that, unlike the $\mathsf{NrmLMod}(\mathcal{A})$-case, the proof for the $\mathsf{NrmRMod}(\mathcal{A})$-case in (2) relies on (1) because despite commutativity of $\mathcal{A}$ the representation ρ need not map the latter into $\mathcal{Z}(\mathrm{End}_{\mathbb{K}}(M))$. □

Lemma-Definition 0.3.25 (Norms for Finite (Co)Products)**:**

(1) Finite Products in $\mathsf{NrmAlg}_{\mathbb{K}}$: if $(\mathcal{A}_j, \|\cdot\|_{\mathcal{A}_j})_{1\leq j\leq n} \subset \mathsf{NrmAlg}_{\mathbb{K}}$, then

$$\left\| \times_{j=1}^{n} a_j \right\|_{\times} := \max_{1\leq j\leq n} \|a_j\|_{\mathcal{A}_j} \tag{0.3.10}$$

defines a *submultiplicative* norm on $\mathcal{A} := \prod_{j=1}^{n} \mathcal{A}_j$. If additionally $\|\cdot\|_{\mathcal{A}_j}$, $1 \leq j \leq n$, are unital, then so is $\|\cdot\|_{\mathcal{A}} := \|\cdot\|_{\times}$.

(2) Finite Coproducts in $\mathsf{NrmLMod}(\mathcal{A})$: Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) \in \mathsf{NrmAlg}_{\mathbb{K}}$ and let $(M_k, \|\cdot\|_{M_k})_{1\leq k\leq m} \subset \mathsf{NrmLMod}(\mathcal{A})$. If $\|\cdot\|_{\mathbb{R}^m}$ is any *monotonic* norm on $\mathbb{R}^m$, then

$$\|\oplus_{k=1}^{m} x_k\|_M := \left\|(\|x_1\|_{M_1}, \ldots, \|x_m\|_{M_m})\right\|_{\mathbb{R}^m}$$

defines an $\mathcal{A}$-module norm on $M := \bigoplus_{k=1}^{m} M_k$. In particular, we set

$$\forall (x_1, \ldots, x_m) \in M:\ \|\oplus_{k=1}^{m} x_k\|_{\oplus} := \left\|(\|x_1\|_{M_1}, \ldots, \|x_m\|_{M_m})\right\|_2 .$$

Proof: To (1): This is immediate from the fact that $\|\cdot\|_\infty$ is itself a unital *algebra* norm on $\mathbb{R}^n$.

To (2): Write $x = \oplus_{k=1}^{m} x_k$. Since $\|\cdot\|_{\mathbb{R}^m}$ is monotonic, we have

$$\forall\, a \in \mathcal{A}\ \forall\, x \in M:\ \|a \cdot x\|_M \overset{\text{def}}{=} \left\|(\|a \cdot x_1\|_{M_1}, \ldots, \|a \cdot x_m\|_{M_m})\right\|_{\mathbb{R}^m} \leq$$
$$\leq \left\|(\|a\|_{\mathcal{A}} \|x_1\|_{M_1}, \ldots, \|a\|_{\mathcal{A}} \|x_m\|_{M_m})\right\|_{\mathbb{R}^m} =$$
$$= \|a\|_{\mathcal{A}} \left\|(\|x_1\|_{M_1}, \ldots, \|x_m\|_{M_m})\right\|_{\mathbb{R}^m} \overset{\text{def}}{=} \|a\|_{\mathcal{A}} \|x\|_M$$

as desired. □

Remark: It is well-known that a norm in $\mathbb{R}^n$ is *monotonic* iff it is *absolute*, that is, depends only on the absolute values of the vector coordinates (see [BSW61] for the original treatment or the account in [JN91]). □

To summarize, we will have to deal with the following **norm package**:

- $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$ equipped with some (hopefully unital) *algebra* norm $\|\cdot\|_{\mathcal{A}}$;
- $\mathrm{End}_{\mathbb{K}}(\mathcal{A})$ naturally equipped with $\|\cdot\|_{\mathrm{End}_{\mathbb{K}}(\mathcal{A})} \overset{\text{def}}{=} \|\cdot\|_{\mathcal{A},\mathcal{A}}$;
- $\rho_{\mathcal{A}} \colon \mathcal{A} \hookrightarrow \mathrm{End}_{\mathbb{K}}(\mathcal{A})$ equipped with $\|\rho_{\mathcal{A}}\|_{\mathrm{op}} \overset{\text{def}}{=} \|\rho\|_{\mathcal{A},\mathrm{End}_{\mathbb{K}}(\mathcal{A})}$;

and more generally:

- $M \in \mathsf{FdMod}(\mathcal{A})$ equipped with some *vector* norm $\|\cdot\|_M$;
- $\mathrm{End}_{\mathbb{K}}(M)$ naturally equipped with $\|\cdot\|_{\mathrm{End}_{\mathbb{K}}(M)} \stackrel{\text{def}}{=} \|\cdot\|_{M,M}$;
- $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$ equipped itself with $\|\rho\|_{\mathrm{op}} := \|\rho\|_{\mathcal{A},\mathrm{End}_{\mathbb{K}}(M)}$ and $\|\cdot\|_{\rho(\mathcal{A})} \stackrel{\text{def}}{=} \|\cdot\|_{\mathrm{End}_{\mathbb{K}}(M)}|_{\rho(\mathcal{A})}$;
- $\varphi\colon \mathcal{A} \to \mathcal{B}$ equipped with $\|\varphi\|_{\mathrm{op}} \stackrel{\text{def}}{=} \|\varphi\|_{\mathcal{A},\mathcal{B}}$ vs. $\varphi^{\mathrm{rep}} \stackrel{\text{def}}{=} \rho_\varphi \stackrel{\text{def}}{=} \rho_{\mathcal{B}} \circ \varphi\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(\mathcal{B})$ equipped with $\|\varphi^{\mathrm{rep}}\|_{\mathrm{op}} \stackrel{\text{def}}{=} \|\varphi^{\mathrm{rep}}\|_{\mathcal{A},\mathrm{End}_{\mathbb{K}}(\mathcal{B})}$;
- $\rho^{\mathrm{mor}}\colon \mathcal{A} \to \mathcal{B} \hookrightarrow \mathrm{End}_{\mathbb{K}}(M)$ equipped with $\|\rho^{\mathrm{mor}}\|_{\mathrm{op}} \stackrel{\text{def}}{=} \|\rho^{\mathrm{mor}}\|_{\mathcal{A},\mathcal{B}}$ as well as the other corestrictions ρ^{im}, ρ^{cor}.

This begs the question as to whether one can make some quality-of-life arrangements so that all these norms play nice together. To this end, the following lemma falls into the verify-once-and-forget category:

Lemma 0.3.26 (Consistent Choice of Norms)**:** Let $\mathcal{A}, \mathcal{B} \in \mathsf{FdCAlg}_{\mathbb{K}}$ and $M \in \mathsf{FdMod}_{\mathcal{A}}$ via $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$.

(1) If $\|\cdot\|_{\mathcal{A}}$ is any unital algebra norm on $\mathcal{A}$, then

$$\|\cdot\|_{\mathrm{End}_{\mathbb{K}}(\mathcal{A})}|_{\rho_{\mathcal{A}}(\mathcal{A})} = \|\cdot\|_{\mathcal{A}},$$

where $\|\cdot\|_{\mathrm{End}_{\mathbb{K}}(\mathcal{A})} \stackrel{\text{def}}{=} \|\cdot\|_{\mathcal{A},\mathcal{A}}$. That is, the inherited along the regular representation $\rho_{\mathcal{A}}$ induced norm agrees with the original algebra norm. In other words, the identification $\mathcal{A} \cong \rho_{\mathcal{A}}(\mathcal{A})$ is *norm-consistent.*

(2) Given $\|\cdot\|_M$, there exists a unital algebra norm $\|\cdot\|_{\mathcal{A}}$ (depending on $\|\cdot\|_M$), with respect to which $\|\cdot\|_M$ is an $\mathcal{A}$-module norm. In particular, if we choose $\|\cdot\|_M := \|\cdot\|_2$, then $\|\cdot\|_{\mathrm{End}_{\mathbb{K}}(M)} = \|\cdot\|_{\mathrm{sp}}$ and hence $\|\cdot\|_{\rho(\mathcal{A})} = \|\cdot\|_{\mathrm{sp}}|_{\rho(\mathcal{A})}$.

(3) Given $\|\cdot\|_{\mathcal{A}}$, there exists a compatible $\mathcal{A}$-module norm $\|\cdot\|_M$ (depending on $\|\cdot\|_{\mathcal{A}}$). In particular, picking $\|\cdot\|_{\mathcal{A}} := \|\cdot\|_{\mathrm{sp}}$ via $\mathcal{A} \xhookrightarrow{\rho_{\mathcal{A}}} \mathrm{End}_{\mathbb{K}}(\mathcal{A})$ for some choice of an inner product on the underlying $\mathbb{K}$-vector space $\mathcal{A}$, we can find a compatible $\|\cdot\|_M$.

(4) If $M = \mathcal{B}$ in (2) and $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$, then there exist submultiplicative unital norms $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{B}}$ such that

$$\forall\, a \in \mathcal{A}\ \forall\, b \in \mathcal{B}\colon\ \|a \cdot b\|_{\mathcal{B}} \le \|a\|_{\mathcal{A}}\,\|b\|_{\mathcal{B}}$$

and

$$\|\varphi\|_{\mathrm{op}} = \|\varphi^{\mathrm{rep}}\|_{\mathrm{op}} \stackrel{\text{def}}{=} \|\rho_\varphi\|_{\mathcal{A},\mathrm{End}_{\mathbb{K}}(\mathcal{B})} = 1.$$

(5) As a continuation of (2), let $(M_k, \|\cdot\|_{M_k} = \|\cdot\|_2)_{1 \le k \le m} \subset \mathsf{NrmFdMod}(\mathcal{A})$, $M = \bigoplus_{k=1}^m M_k$, and consider $\prod_{k=1}^m \mathrm{End}_{\mathbb{K}}(M_k) \hookrightarrow \mathrm{End}_{\mathbb{K}}(M)$, where accordingly $\|\cdot\|_{\mathrm{End}_{\mathbb{K}}(M_k)} = \|\cdot\|_{\mathrm{sp}}$, $1 \le k \le m$. Then

$$\|\cdot\|_M \stackrel{\text{def}}{=} \|\cdot\|_2 = \|\cdot\|_{\oplus}$$

(with respect to the concatenated basis), hence

$$\|\cdot\|_{M,M} = \|\cdot\|_{\mathrm{sp}}$$

and we have

$$\|\cdot\|_{\mathrm{sp}}|_{\prod_{k=1}^m \mathrm{End}_{\mathbb{K}}(M_k)} = \|\cdot\|_{\times} \stackrel{\text{def}}{=} \|\cdot\|_{\mathrm{Codom}(\rho^{\mathrm{cor}})}.$$

In other words, the module norm $\|\cdot\|_2$ is consistent with respect to direct sums and the algebra norm $\|\cdot\|_{\mathrm{sp}}$ is consistent with respect to the *corresponding* products of corner rings.

(6) Continuing (5), we have

$$\|\cdot\|_{\mathrm{Codom}(\rho^{\mathrm{mor}})} = \|\cdot\|_{\mathrm{sp}}|_{\mathrm{Codom}(\rho^{\mathrm{mor}})}.$$

Moreover:

$$\left\|\rho^{\mathrm{im}}\right\|_{\mathrm{op}} = \left\|\rho^{\mathrm{mor}}\right\|_{\mathrm{op}} = \left\|\rho^{\mathrm{cor}}\right\|_{\mathrm{op}} = \|\rho\|_{\mathrm{op}} = 1,$$

where the operator norms are taken with respect to the corresponding codomains as defined in Section 0.1.4.

(7) "Functoriality": if $\mathcal{A} \xrightarrow{\varphi} \mathcal{B} \xrightarrow{\psi} \mathcal{C}$ in $\mathsf{FdCAlg}_{\mathbb{K}}$, then there exist $\|\cdot\|_{\mathcal{A}}$, $\|\cdot\|_{\mathcal{B}}$, and $\|\cdot\|_{\mathcal{C}}$ such that

$$\|\varphi\|_{\mathcal{A},\mathcal{B}} = \|\psi\|_{\mathcal{B},\mathcal{C}} = \|\psi \circ \varphi\|_{\mathcal{A},\mathcal{C}} = 1.$$

Proof: To (1): Indeed, identifying $\mathcal{A} \cong \rho_{\mathcal{A}}(\mathcal{A})$, we have $\forall\, a \in \mathcal{A}$:

$$\|a\|_{\mathcal{A}} = \|a \cdot 1_{\mathcal{A}}\|_{\mathcal{A}} \leq \|a\|_{\mathrm{End}_{\mathbb{K}}(\mathcal{A})} \|1_{\mathcal{A}}\|_{\mathcal{A}} = \|a\|_{\mathrm{End}_{\mathbb{K}}(\mathcal{A})} \overset{\mathrm{def}}{=} \sup_{\|x\|_{\mathcal{A}}=1} \|a \cdot x\|_{\mathcal{A}} \leq \sup_{\|x\|_{\mathcal{A}}=1} \|a\|_{\mathcal{A}} \|x\|_{\mathcal{A}} = \|a\|_{\mathcal{A}}\,,$$

that is, $\forall\, a \in \mathcal{A}\colon\ \|a\|_{\mathrm{End}_{\mathbb{K}}(\mathcal{A})} = \|a\|_{\mathcal{A}}$ as desired.

To (2): Let $\|\cdot\|'_{\mathcal{A}}$ be *any* submultiplicative unital norm on $\mathcal{A}$ and fix some $\|\cdot\|_M$ (e.g. $\|\cdot\|_M \coloneqq \|\cdot\|_2$ for some choice of $\mathbb{K}$-basis for M), which in turn induces some $\|\cdot\|_{\mathrm{End}_{\mathbb{K}}(M)}$. Put $\|\cdot\|'_{\mathrm{op}} \coloneqq \|\cdot\|'_{\mathcal{A},\mathrm{End}_{\mathbb{K}}(M)}$ for the corresponding operator norm with respect to $\|\cdot\|'_{\mathcal{A}}$ and $\|\cdot\|_{\mathrm{End}_{\mathbb{K}}(M)}$. Now define $\forall\, a \in \mathcal{A}\colon\ \|a\|_{\mathcal{A}} \coloneqq \max\{\|a\|'_{\mathcal{A}}\,, \|\rho(a)\|_{\mathrm{End}_{\mathbb{K}}(M)}\}$. Then $\|\cdot\|_{\mathcal{A}}$ is a unital submultiplicative norm on $\mathcal{A}$ equivalent to $\|\cdot\|'_{\mathcal{A}}$ (regardless of finite-dimensionality). Indeed, unitality, absolute homogeneity, and positive definiteness of $\|\cdot\|_{\mathcal{A}}$ are clear, while subadditivity and submultiplicativity are immediate from $\|\cdot\|_{\times}$ being an algebra norm on $\mathbb{R} \times \mathbb{R}$. As for the norm equivalence, we have $\forall\, a \in \mathcal{A}$:

$$\|a\|'_{\mathcal{A}} \leq \|a\|_{\mathcal{A}} \overset{\mathrm{def}}{=} \max\{\|a\|'_{\mathcal{A}}\,, \|\rho(a)\|_{\mathrm{End}_{\mathbb{K}}(M)}\} \leq \max\{\|a\|'_{\mathcal{A}}\,, \|\rho\|'_{\mathrm{op}} \|a\|'_{\mathcal{A}}\} = \max\{1, \|\rho\|'_{\mathrm{op}}\} \|a\|'_{\mathcal{A}}\,.$$

Next put $\|\cdot\|_{\mathrm{op}} \coloneqq \|\cdot\|_{\mathcal{A},\mathrm{End}_{\mathbb{K}}(M)}$ with respect to $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_{\mathrm{End}_{\mathbb{K}}(M)}$. By Lemma 0.3.24 it is enough to show that $\|\rho\|_{\mathrm{op}} = 1$. To this end, decompose $\mathbb{B}_1^{\|\cdot\|_{\mathcal{A}}}(0) = X \cup Y$ with $X \coloneqq \{a \in \mathcal{A} : \|a\|'_{\mathcal{A}} \leq \|\rho(a)\|_{\mathrm{End}_{\mathbb{K}}(M)} = 1\}$ and $Y \coloneqq \{a \in \mathcal{A} : \|\rho(a)\|_{\mathrm{End}_{\mathbb{K}}(M)} < \|a\|'_{\mathcal{A}} = 1\}$. Then $\|\rho\|_{\mathrm{op}} \overset{\mathrm{def}}{=} \max\left\{\sup_{a\in X} \|\rho(a)\|_{\mathrm{End}_{\mathbb{K}}(M)}\,, \sup_{a \in Y} \|\rho(a)\|_{\mathrm{End}_{\mathbb{K}}(M)}\right\}$. We clearly have $\sup_{a\in X} \|\rho(a)\|_{\mathrm{End}_{\mathbb{K}}(M)} = 1$, provided $X \neq \varnothing$, and $\sup_{a\in Y} \|\rho(a)\|_{\mathrm{End}_{\mathbb{K}}(M)} \leq \|a\|'_{\mathcal{A}} = 1$, so it suffices to show that $X \neq \varnothing$. But this is also clear because $X \ni 1_{\mathcal{A}}$.

To (3): Let $\|\cdot\|'_M$ be an arbitrary norm on M. Then define

$$\forall\, x \in M\colon\ \|x\|_M \coloneqq \sup_{\|a\|_{\mathcal{A}}=1} \|a \cdot x\|'_M\,.$$

Observe that $\forall\, x \in M\colon\ \|x\|_M < \infty$ by (automatic) continuity of the action[32]. Taking $u \coloneqq 1_{\mathcal{A}} / \|1_{\mathcal{A}}\|_{\mathcal{A}}$, in particular $\|u\|_{\mathcal{A}} = 1$, we have

$$\forall\, x \in M\colon\ \|x\|_M \geq \|u \cdot x\|'_M = \frac{\|x\|'_M}{\|1_{\mathcal{A}}\|_{\mathcal{A}}},$$

hence $\|\cdot\|_M \geq 0$ and $\|x\|_M = 0 \Leftrightarrow x = 0$. Moreover, $\|\cdot\|_M$ is clearly absolutely homogeneous. Furthermore:

$$\begin{aligned}\forall\, x, y \in M\colon\ \|x + y\|_M &\overset{\mathrm{def}}{=} \sup_{\|a\|_{\mathcal{A}}=1} \|a \cdot (x + y)\|'_M \leq \sup_{\|a\|_{\mathcal{A}}=1} (\|a \cdot x\|'_M + \|a \cdot y\|'_M) \leq \\ &\leq \sup_{\|a\|_{\mathcal{A}}=1} \|a \cdot x\|'_M + \sup_{\|a\|_{\mathcal{A}}=1} \|a \cdot y\|'_M \overset{\mathrm{def}}{=} \|x\|_M + \|y\|_M\,,\end{aligned}$$

i.e. $\|\cdot\|_M$ is subadditive. Next we show that

$$\forall\, x \in M\colon\ \sup_{\|a\|_{\mathcal{A}}\leq 1} \|a \cdot x\|'_M = \sup_{\|a\|_{\mathcal{A}}=1} \|a \cdot x\|'_M\,.$$

[32] more generally, the construction works for topological $\mathcal{A}$-modules M, which is automatic in the finite-dimensional case;

It suffices to show that $\forall x \in M\colon \sup_{\|a\|_{\mathcal{A}}\le 1} \|a\cdot x\|'_M \le \sup_{\|a\|_{\mathcal{A}}=1} \|a\cdot x\|'_M$. Indeed we have

$$\forall x \in M\colon \sup_{\|a\|_{\mathcal{A}}\le 1} \|a\cdot x\|'_M = \sup_{\substack{\|a\|_{\mathcal{A}}\le 1\\ a\neq 0}} \|a\|_{\mathcal{A}} \left\|\frac{a}{\|a\|_{\mathcal{A}}}\cdot x\right\|'_M \le \sup_{\substack{\|a\|_{\mathcal{A}}\le 1\\ a\neq 0}} \left\|\frac{a}{\|a\|_{\mathcal{A}}}\cdot x\right\|'_M = \sup_{\|b\|_{\mathcal{A}}=1} \|b\cdot x\|'_M$$

as desired. Finally, we obtain $\forall a \in \mathcal{A}\setminus\{0\}\ \forall x \in M$:

$$\|a\cdot x\|_{\mathcal{A}} \overset{\text{def}}{=} \sup_{\|b\|_{\mathcal{A}}=1} \|b\cdot a\cdot x\|'_M \le \sup_{\|c\|_{\mathcal{A}}\le\|a\|_{\mathcal{A}}} \|c\cdot x\|'_M = \|a\|_{\mathcal{A}} \sup_{\|c\|_{\mathcal{A}}\le\|a\|_{\mathcal{A}}} \left\|\frac{c}{\|a\|_{\mathcal{A}}}\cdot x\right\|'_M \le$$
$$\le \|a\|_{\mathcal{A}} \sup_{\|d\|_{\mathcal{A}}\le 1} \|d\cdot x\| = \|a\|_{\mathcal{A}} \sup_{\|d\|_{\mathcal{A}}=1} \|d\cdot x\| \overset{\text{def}}{=} \|a\|_{\mathcal{A}} \|x\|_M$$

by the preceding equality and the fact that for $c \coloneqq ba$ we have $\|c\|_{\mathcal{A}} = \|ba\|_{\mathcal{A}} \le \|b\|_{\mathcal{A}} \|a\|_{\mathcal{A}} = \|a\|_{\mathcal{A}}$. Clearly, the construction puts no restrictions on $\|\cdot\|_{\mathcal{A}}$.

To (4): Pick any unital submultiplicative norms $\|\cdot\|'_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{B}}$. We have $\forall a \in \mathcal{A}\colon \|\rho_\varphi(a)\|_{\mathrm{End}_{\mathbb{K}}(\mathcal{B})} \overset{\text{def}}{=} \|\rho_{\mathcal{B}}(\varphi(a))\|_{\mathrm{End}_{\mathbb{K}}(\mathcal{B})} = \|\varphi(a)\|_{\mathcal{B}}$ by (1). Thus the same procedure $\|a\|_{\mathcal{A}} \coloneqq \max\{\|a\|'_{\mathcal{A}}, \|\varphi(a)\|_{\mathcal{B}}\} = \max\{\|a\|'_{\mathcal{A}}, \|\rho_\varphi(a)\|_{\mathrm{End}_{\mathbb{K}}(\mathcal{B})}\}$, $a \in \mathcal{A}$, as in (2) yields $\|\varphi\|_{\mathrm{op}} = \|\rho_\varphi\|_{\mathrm{op}} = 1$ and $\forall a \in \mathcal{A}\ \forall b \in \mathcal{B}\colon \|a\cdot b\|_{\mathcal{B}} \le \|a\|_{\mathcal{A}} \|b\|_{\mathcal{B}}$.

To (5): Write $A \coloneqq \times_{k=1}^m A_k \in \prod_{k=1}^m \mathrm{End}_{\mathbb{K}}(M_k) \hookrightarrow \mathrm{End}_{\mathbb{K}}(M)$ and $x \coloneqq \oplus_{k=1}^m x_k \in M = \bigoplus_{k=1}^m M_k$. Then $\forall 1 \le k \le m$:

$$\|A\|_{\mathrm{sp}} \overset{\text{def}}{=} \sup_{\|x\|_2=1} \|Ax\|_2 \overset{\text{def}}{=} \sup_{\|x\|_2=1} \|\oplus_{\ell=1}^m A_\ell x_\ell\|_2 \ge \sup_{\substack{\|x_k\|_2=1\\ x_{\neq k}=0}} \|\oplus_{\ell=1}^m A_\ell x_\ell\|_2 = \sup_{\|x_k\|_2=1} \|A_k x_k\| \overset{\text{def}}{=} \|A_k\|_{\mathrm{sp}},$$

hence $\|A\|_{\mathrm{sp}} \ge \max_{1\le k\le m} \|A_k\|_{\mathrm{sp}} \overset{\text{def}}{=} \|A\|_\times$. On the other hand,

$$\|A\|_{\mathrm{sp}}^2 \overset{\text{def}}{=} \sup_{\sum_{k=1}^m \|x_k\|_2^2=1} \|\oplus_{k=1}^m A_k x_k\|_2^2 \overset{\text{def}}{=} \sup_{\sum_{k=1}^m \|x_k\|_2^2=1} \sum_{k=1}^m \|A_k x_k\|_2^2 \le \sup_{\sum_{k=1}^m \|x_k\|_2^2=1} \sum_{k=1}^m \|A_k\|_{\mathrm{sp}}^2 \|x_k\|_2^2 =$$
$$= \sup\left\{\sum_{k=1}^m t_k \|A_k\|_{\mathrm{sp}}^2 : \sum_{k=1}^m t_k = 1 \text{ and } \forall 1 \le k \le m\colon t_k \ge 0\right\} = \max_{1\le k\le m} \|A_k\|_{\mathrm{sp}}^2 \overset{\text{def}}{=} \|A\|_\times^2.$$

It follows that $\|A\|_{\mathrm{sp}} = \|A\|_\times$ as desired.

To (6): Following (5), write ${}_{\mathcal{A}}M = \bigoplus_{k=1}^m {}_{\mathcal{A}}M_k$ with $\|\cdot\|_{M_k} \coloneqq \|\cdot\|_2$ and corresponding representations $\rho_k\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M_k)$, $1 \le k \le m$. By Lemma-Definition 0.1.15 we have $\rho^{\mathrm{mor}} \overset{\text{def}}{=} (\rho_1^{\mathrm{im}}, \dots, \rho_m^{\mathrm{im}})\colon \mathcal{A} \to \prod_{k=1}^m \mathcal{B}_k$, where $\mathcal{B}_k \coloneqq \rho_k(\mathcal{A})$ with $\|\cdot\|_{\mathcal{B}_k} \overset{\text{def}}{=} \|\cdot\|_{\mathrm{End}_{\mathbb{K}}(M_k)}|_{\mathcal{B}_k} = \|\cdot\|_{\mathrm{sp}}|_{\mathcal{B}_k}$, $1 \le k \le m$. Hence $\forall b = \times_{k=1}^m b_k \in \prod_{k=1}^m \mathcal{B}_k$:

$$\|\times_{k=1}^m b_k\|_{\prod_{k=1}^m \mathcal{B}_k} \overset{\text{def}}{=} \|\times_{k=1}^m b_k\|_\times \overset{\text{def}}{=} \max_{1\le k\le m} \|b_k\|_{\mathcal{B}_k} \overset{\text{def}}{=} \max_{1\le k\le m} (\|b_k\|_{\mathrm{sp}}|_{\mathcal{B}_k}) =$$
$$= \max_{1\le k\le m} \|b_k\|_{\mathrm{sp}} \overset{\text{def}}{=} \|b\|_\times \overset{(5)}{=} \|b\|_{\mathrm{sp}}|_{\prod_{k=1}^m \mathrm{End}_{\mathbb{K}}(M_k)} = \|b\|_{\mathrm{sp}} \overset{\text{def}}{=} \|b\|_{\mathrm{End}_{\mathbb{K}}(M)}.$$

Or more succinctly:

$$\|\cdot\|_{\prod_{k=1}^m \mathcal{B}_k} \overset{\text{def}}{=} (\|\cdot\|_{\mathrm{End}_{\mathbb{K}}(M_k)}|_{\mathcal{B}_k})_{\times_{k=1}^m} = \|\cdot\|_{\prod_{k=1}^m \mathrm{End}_{\mathbb{K}}(M_k)}|_{\prod_{k=1}^m \mathcal{B}_k} \overset{\text{def}}{=} \|\cdot\|_\times|_{\prod_{k=1}^m \mathcal{B}_k} = \|\cdot\|_{\mathrm{sp}}|_{\prod_{k=1}^m \mathcal{B}_k},$$

which proves the first claim. In particular, in choosing the spectral norm as a default one, product norms commute with corner norm restrictions. Finally, since $\|\cdot\|_{\rho(\mathcal{A})}$, $\|\cdot\|_{\mathrm{Codom}(\rho^{\mathrm{mor}})}$, and $\|\cdot\|_{\mathrm{Codom}(\rho^{\mathrm{cor}})}$ are restrictions of $\|\cdot\|_{\mathrm{End}_{\mathbb{K}}(M)} = \|\cdot\|_{\mathrm{sp}}$, the second claim follows from (2).

To (7): Let $\|\cdot\|'_{\mathcal{A}}$, $\|\cdot\|'_{\mathcal{B}}$, and $\|\cdot\|_{\mathcal{C}}$ be any unital submultiplicative norms on $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, respectively. As in (4) define successively $\forall b \in \mathcal{B}\colon \|b\|_{\mathcal{B}} \coloneqq \max\{\|b\|'_{\mathcal{B}}, \|\psi(b)\|_{\mathcal{C}}\}$ and

$$\forall a \in \mathcal{A}\colon \|a\|_{\mathcal{A}} \coloneqq \max\{\|a\|'_{\mathcal{A}}, \|\varphi(a)\|_{\mathcal{B}}\} \overset{\text{def}}{=} \max\{\|a\|'_{\mathcal{A}}, \|\varphi(a)\|'_{\mathcal{B}}, \|\psi(\varphi(a))\|_{\mathcal{C}}\}.$$

By (4) we have $\|\varphi\|_{\mathcal{A},\mathcal{B}} = \|\psi\|_{\mathcal{B},\mathcal{C}} = 1$. Finally, as in (2), put

$$X := \left\{a \in \mathcal{A} : \max\{\|a\|'_{\mathcal{A}}, \|\varphi(a)\|'_{\mathcal{B}}\} \leq \|\psi(\varphi(a))\|_{\mathcal{C}} = 1\right\} \subseteq \mathbb{B}_1^{\|\cdot\|_{\mathcal{A}}}(0)$$

and

$$Y := \mathbb{B}_1^{\|\cdot\|_{\mathcal{A}}}(0) \setminus X \overset{\text{def}}{=} \left\{a \in \mathcal{A} : \|\psi(\varphi(a))\|_{\mathcal{C}} < \max\{\|a\|'_{\mathcal{A}}, \|\varphi(a)\|'_{\mathcal{B}}\} = 1\right\}.$$

Then proceed as in (2). □

Remarks:

(1) Observe that in (3) we could have equivalently taken the in retrospect technically simpler definition $\|x\|_M := \sup_{\|a\|_{\mathcal{A}} \leq 1} \|a \cdot x\|'_M$ in place of the stricter $\sup_{\|a\|_{\mathcal{A}}=1} \|a \cdot x\|'_M$. Moreover, one can show that $\|\cdot\|_M \sim \|\cdot\|'_M$ regardless of finite-dimensionality.

(2) Clearly, (5) and consequently (6) works for any $\|\cdot\|_{p,p}$ in place of $p = 2$, but among them $\|\cdot\|_{\text{sp}} \overset{\text{def}}{=} \|\cdot\|_{2,2}$ is *special*. Other convenient (p,p)-operator norms include $\|\cdot\|_1$ and $\|\cdot\|_\infty$, though.

(3) The $\mathcal{A}$-module norm compatibility condition is often given in the literature as

$$\exists\, C > 0\ \forall\, a \in \mathcal{A}\ \forall\, x \in M\colon\ \|a \cdot x\|_M \leq C\, \|a\|_{\mathcal{A}}\, \|x\|_M$$

instead. By Lemma 0.3.24 we have that $C = 1$ is equivalent to $\|\rho\|_{\text{op}} = 1$ (in fact, we can clearly take $C := \|\rho\|_{\text{op}}$). But by Lemma 0.3.26, starting with either $\|\cdot\|_{\mathcal{A}}$ or $\|\cdot\|_M$, we can always arrange for an equivalent norm so that ρ becomes weakly contractive. Getting rid of $C > 0$ is not only more aesthetically pleasing, but it also behaves more naturally with respect to change of scalars.

(4) Clearly, (7) can be extended by induction to any finite chain of morphisms in $\mathsf{FdCAlg}_{\mathbb{K}}$ so that any intermediate compositions are also normalized.

(5) Working with ρ and φ^{rep} is thus more flexible than working with φ and ρ^{mor} due to the norm consistency of the regular representation, whereas the latter would require something like fixing $\|\cdot\|_{\text{End}} := \|\cdot\|_{\text{sp}}$, i.e. $\|\cdot\|_M := \|\cdot\|_2$ in order for the (convenient) corestrictions of ρ to be consistent with $\|\cdot\|_\times$. □

Definition 0.3.27 (Spectral Balls)**:** Let $\mathcal{A} \in \mathsf{BanAlg}_{\mathbb{K}}$, $Z_0 \in \mathcal{A}$, and $R > 0$. Then

$$\mathbb{B}^{\text{sp}}_{\mathcal{A}}(Z_0, R) := \{Z \in \mathcal{A} : \varrho_{\mathcal{A}}(Z - Z_0) < R\}$$

$$\overline{\mathbb{B}}^{\text{sp}}_{\mathcal{A}}(Z_0, R) := \{Z \in \mathcal{A} : \varrho_{\mathcal{A}}(Z - Z_0) \leq R\}$$

will denote the open and the closed spectral R-balls of $\mathcal{A}$, respectively.

Remark: If $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{K}) \in \mathsf{LocArt}_{\mathbb{K}}$, then by Corollary 0.1.73 we have explicitly:

$$\mathbb{B}^{\text{sp}}_{\mathcal{A}}(0, R) = \begin{cases} (-R, R) \times \mathfrak{m}, \text{ if } \mathbb{K} = \mathbb{R} \\ \mathbb{D}_R(0) \times \mathfrak{m}, \text{ if } \mathbb{K} = \mathbb{C} \end{cases}$$

and

$$\overline{\mathbb{B}}^{\text{sp}}_{\mathcal{A}}(0, R) = \begin{cases} [-R, R] \times \mathfrak{m}, \text{ if } \mathbb{K} = \mathbb{R} \\ \overline{\mathbb{D}}_R(0) \times \mathfrak{m}, \text{ if } \mathbb{K} = \mathbb{C} \end{cases}$$

Hence for finite-dimensional commutative local $\mathbb{K}$-algebras the spectral balls are just *infinite boxes* or *infinite cylinders*, depending on $\mathbb{K}$, rather than actual balls.

Definition 0.3.28 (Spectral Polyballs/Polycylinders)**:** Let $\mathcal{A} = \prod_{j=1}^n (\mathcal{A}_j, \mathfrak{m}_j, \mathbb{C})$ and let $Z_0 \in \mathcal{A}$.

Let $\underline{r} \coloneqq (r_1, \ldots, r_n) \in [0,\infty]^n$ and $\underline{R} \coloneqq (R_1, \ldots, R_n) \in [0,\infty]^n$. Then we define:

$$\mathbb{B}^{\mathrm{sp}}_{\mathcal{A}}(Z_0, \underline{R}) \coloneqq \prod_{j=1}^{n} \mathbb{B}^{\mathrm{sp}}_{\mathcal{A}_j}(\mathrm{pr}_j(Z_0), R_j)$$

and

$$\overline{\mathbb{B}}^{\mathrm{sp}}_{\mathcal{A}}(Z_0, \underline{R}) \coloneqq \prod_{j=1}^{n} \overline{\mathbb{B}}^{\mathrm{sp}}_{\mathcal{A}_j}(\mathrm{pr}_j(Z_0), R_j)$$

Remarks:

(1) Definition 0.3.27 and Definition 0.3.28 agree only when $R_1 = \cdots = R_n = R$.

(2) Unlike Definition 0.3.27, however, Definition 0.3.28 generally *would have depended* on the choice of product decomposition for $\mathcal{A}$ among all isomorphic ones if we didn't *internalize*.

Definition 0.3.29 (Scalar Disks)**:** Let $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{C}}$ and denote by $\iota_{\mathcal{A}} \colon \mathbb{C} \hookrightarrow \mathcal{A}$ the canonical inclusion of scalars. Furthermore, let $R > 0$ and $Z_0 \in \mathcal{A}$. Then

$$\mathbb{D}_R(Z_0) \coloneqq \mathbb{D}_{\mathcal{A}}(Z_0, R) \coloneqq Z_0 + \iota_{\mathcal{A}}(\mathbb{D}_R(0))$$

and

$$\overline{\mathbb{D}}_R(Z_0) \coloneqq \overline{\mathbb{D}}_{\mathcal{A}}(Z_0, R) \coloneqq Z_0 + \iota_{\mathcal{A}}(\overline{\mathbb{D}}_R(0))$$

will be called respectively the open[33] and the closed *scalar* discs of radius R around Z_0.

[33]obviously not an open subset of $\mathcal{A}$, but only in the relative topology of the accordingly shifted complex plane in $\mathcal{A}$;

1. Differentiation over $\mathcal{A}$

In Section 1.1 we introduce our main working definitions. To this end, given $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, it will be convenient to make a superficial distinction between the equivalent notions of a (finite-dimensional) representation $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$ as a special kind of an algebra morphism (representation POV) and the object ${}_{\mathcal{A}}M \in \mathsf{FdMod}(\mathcal{A})$ as a (finite-dimensional) $\mathbb{K}$-vector space equipped with an $\mathcal{A}$-action (module POV). Given the first type of data, we will speak of ρ-differentiability, whereas given the second type of data, we will speak of $\mathcal{A}$-differentiability, but the reader should not feel confounded as they will mean one and the same. Of special interest will be representations that arise from morphisms $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ in $\mathsf{FdCAlg}_{\mathbb{K}}$ since these then enable "functorial" behavior on the analytic side. Now, given such a representation ρ, we will associate to it three different types of sheaves, $\mathscr{L}_\rho$, $\mathscr{F}^{\mathrm{str}}_\rho$, and $\mathscr{F}_\rho$, that encode compatibility of the derivative with the $\mathcal{A}$-structure (i.e. $\mathcal{A}$-equivariance of the derivative as a linear map) and that represent three main classes of *basic regularity* for ρ-differentiability. One should note, though, that in all three cases, given a ρ-differentiable function $f\colon U \to {}_{\mathcal{A}}M$, the ($\rho$-)derivative f' is again a function $f'\colon U \to M$. This is the first similarity to single-variable complex analysis. The first type, $\mathscr{L}$, is an adaptation of the *classical* definition of differentiability via a *limit of a difference quotient* (hence the notation). The third one is just Frechét differentiability with an $\mathcal{A}$-equivariant condition on the differential. The middle one $\mathscr{F}^{\mathrm{str}}$ is an auxiliary notion, a *stricter* version of $\mathscr{F}$ that imposes an additional condition on the o-behavior of the remainder term. The reader should rightly wonder why all the fuss with three distinct definitions (and we haven't even included the Gateaux version of $\mathcal{A}$-equivariant differentiability yet!). Regardless of which definition one deems to be the conceptually right one, one of the main objectives of Section 1.1 is to study $\mathcal{A}$-differentiability in its natural, categorical setting. That is to say, to study the interplay between the analysis definition on the one hand and the elementary categorical properties, objects, constructions in both $\mathsf{FdCAlg}_{\mathbb{K}}$ and $\mathsf{Mod}(\mathcal{A})$ such as monomorphisms, epimorphisms, products etc. on the other. The regularity class $\mathscr{L}_\rho$ behaves well in that regard, but unfortunately it has just one "minor" drawback: it is the conceptually wrong definition in the Banach setting as it is limited to the finite-dimensional setting since infinite-dimensional commutative Banach $\mathbb{K}$-algebras need not have dense groups of units, which renders said definition completely unviable in that situation. On the other hand, $\mathscr{F}_\rho$ is the natural definition in the Banach setting, but it does not *appear* to behave well enough with respect to some essential categorical constructions, and it turns out that $\mathscr{F}^{\mathrm{str}}$ is the correct choice for this key categorical behavior. Even though we are going to show that the distinction between the three definitions disappears quickly once one imposes *continuous* differentiability, it is instructive to understand how far one can take the theory without such an assumption – recall namely that requiring continuity of the derivative for holomorphic functions is a completely superfluous assumption. As the reader would have probably already guessed at this point, with some additional care much of the theory in this section carries through more generally for $\mathcal{A} \in \mathsf{CBanAlg}_{\mathbb{K}}$ and $E \in \mathsf{BanMod}(\mathcal{A})$ instead (this will be a subject of different work though), which we decided to reflect in parts of the proof language on purpose.

As a first order of business after introducing the main definitions in Section 1.1 we revisit some key results of [Wat92a]. In our notation Waterhouse shows that $\mathscr{L}_{\mathcal{A}} \subseteq \mathscr{F}_{\mathcal{A}}$ by first showing that $\mathscr{L}_{\mathcal{A}}$-differentiable functions decompose in accordance with the decomposition of $\mathcal{A}$ as a product of local Artin algebras and then he invokes the explicit algebraic structure of local $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{K})$ to show that then $\mathscr{L}_{\mathcal{A}} \subseteq \mathscr{F}_{\mathcal{A}}$, thus concluding its validity for all $\mathcal{A}$. We give a straightfoward, *purely topological* proof of the more general inclusion $\mathscr{L}_\rho \subseteq \mathscr{F}^{\mathrm{str}}_\rho \subseteq \mathscr{F}_\rho$, dispensing with any *algebraic* reasoning altogether. Since our argument relies on $\mathcal{A}^\times$ being dense in $\mathcal{A}$, the reader might still argue that it is algebraic in nature – however, we also show that *any* dense set of invertible elements does the job, which is what enables the definition of $\mathscr{L}$ to begin with! This also turns out to be crucial for the "functorial" behavior of $\mathscr{L}$. Later we show the aforementioned decomposition as a very special case of the interplay between $\mathscr{L}$ and $\mathscr{F}^{\mathrm{str}}$ on the one hand and algebra epimorphisms (including projections out of products) on the other. We also give a substantial strenghtening in the one-$\mathcal{A}$-variable case of an important foundational theorem in [GM96]: they show that a function is $\mathcal{A}$-analytic if and only if it is analytic and *infinitely* $\mathscr{L}_{\mathcal{A}}$-differentiable, whereas we show that once-$\mathscr{L}_{\mathcal{A}}$-differentiable actually suffices. We then move on to show miscellaneous ρ-differentiation rules, including a "mixed" Leibniz rule for the product of a φ-differentiable function with a ρ-differentiable function, given certain compatibility between the morphism φ and the representation ρ, as well as various chain rules that exhibit the previously mentioned "functorial" behavior with respect to composition of morphisms in $\mathsf{FdCAlg}_{\mathbb{K}}$. We remark that not all of them behave as straightforward as one would expect.

In Section 1.2 we study the interaction between the three regularity classes $\mathscr{L}$, $\mathscr{F}^{\mathrm{str}}$, and $\mathscr{F}$ and some elementary categorical constructions. As previously alluded, we show in particular that if ρ factorizes through an epimorphism π, then so does any f of class $\mathscr{L}$ or $\mathscr{F}^{\mathrm{str}}$. This is the basis for two types of results: various *automatic* ρ-differentiable extensions and *naturality with respect to internal and external products*, which in turn eventually leads to the full factorization of ρ-differentiable functions in accordance with the standard factorization of ρ and the reduction to the case of indecomposable faithful modules over local $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{K})$.

In Section 1.3 we will state the generalized Cauchy–Riemann(–Scheffers) equations associated to ρ-differentiable functions. We also derive corresponding generalized Wirtinger operators in two different ways: one that later on shows them to be precisely the "coefficients" of $\mathrm{d}\omega$, where $\omega := \rho(\mathrm{d}Z)f(Z)$ is the canonical ${}_{\mathcal{A}}M$-valued 1-form associated to the function $f\colon U \to {}_{\mathcal{A}}M$, $U \subseteq \mathcal{A}$ open, (see below) and a second one as a result of a convenient (co)frame change in "Artinified" (co)tangent space that is universal for all $\mathcal{A}$.

In Section 1.4 we study the canonical differential 1-form $\omega := \rho(\mathrm{d}Z)f(Z) \overset{\mathrm{def}}{=} \mathrm{d}Z \cdot f(Z) \in \Omega^1(U, {}_{\mathcal{A}}M)$ associated to a (possibly ρ-differentiable) function $f\colon U \to M$. A key property is that ω is d-closed if and only if $f(Z)$ is ρ-differentiable. Closely related to that is the question of ρ-integrability and ρ-primitives, i.e. the existence of a ρ-differentiable function $F\colon U \to {}_{\mathcal{A}}M$ such that $F' = f$. To that end, we prove a ρ-analogue of Cauchy–Goursat's Integral Theorem (after Pringsheim), which then implies infinite integrability of ρ-differentiable functions $f(Z)$ on sufficiently nice domains. In other words, this class of functions shares the infinite "reverse regularity" property of usual holomorphic functions. Observe that we don't assume $f(Z)$ to be continuously ρ-differentiable or for $\mathcal{A}$ to carry a complex structure.

A final remark on notation. Even though for the most part we will not need to assume an underlying complex structure (except at the few places stated otherwise), we will use the notation $Z \in \mathcal{A}$ and $Z = z \oplus X$ as a remainder that our end-goal is the complex case and that we are actually working over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ in all cases, not merely over $\mathbb{R}$, even though of course $\mathbb{C}$ is an $\mathbb{R}$-algebra. This is stronger.

1.1. Differentiability with respect to a representation

Even though we are mostly interested in the complex case, it is instructive to try and develop the basic theory more generally over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Instead of working within a fixed object $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$, our starting point will rather be a non-trivial finite-dimensional representation $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$ for some $M \in \mathsf{FdVec}_{\mathbb{K}}$ (equivalently, $M \in \mathsf{FgMod}(\mathcal{A})$). Observe that in the special case $\dim_{\mathbb{K}} M = \dim_{\mathbb{K}} \mathcal{A}$ and ρ faithful this is the same as a choice of an $\mathcal{A}$-structure on $M \in \mathsf{FdVec}_{\mathbb{K}}$. Moreover, following Grothendieck's relative point of view, a special case of interest will be those representations arising from morphisms $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ in $\mathsf{FdCAlg}_{\mathbb{K}}$, where recall $\rho_\varphi \overset{\text{def}}{=} \varphi^{\text{rep}} \overset{\text{def}}{=} \rho_{\mathcal{B}} \circ \varphi$. For much of the theory, the representation case subsumes the morphism case, however the morphism case ultimately gives rise to qualitatively different theory, so at times we will need to treat them in parallel. We remark that there is an abundance of such morphisms:

Example: Let $\mathcal{A} \coloneqq \mathbb{K}[X,Y]/(X^2+Y^3, XY^2, Y^4)$. Then $\mathcal{A}$ is a local $\mathbb{K}$-algebra of $\dim_{\mathbb{K}} \mathcal{A} = 6$, and the quotient map $\pi\colon \mathcal{A} \twoheadrightarrow \mathcal{A}/(\overline{Y}^3) \cong \mathbb{K}[X,Y]/(X^2, XY^2, Y^3) =: \mathcal{B}$, $\dim_{\mathbb{K}} \mathcal{B} = 5$, is a non-trivial (necessarily local) morphism. □

Recall that $\mathcal{A}_\rho^\times \overset{\text{def}}{=} \rho^{-1}(\mathrm{GL}(M))$ is an open dense monoid in $\mathcal{A}$ as $\mathcal{A}_\rho^\times \supseteq \mathcal{A}^\times$. Likewise for $\mathcal{A}_\varphi^\times$.

Definition 1.1.1 (Classical ρ/φ-Differentiability)**:** Let $U \subseteq \mathcal{A}$ be open.

(1) Classical ρ/φ-differentiability at a point: let $Z_0 \in U$.

(i) Representation Setting: a function $f\colon U \to {}_{\mathcal{A}}M$ is called *classically or "limit-wise" ρ-differentiable* at Z_0 if the limit of the following *difference quotient limit* exists:

$$f'(Z_0) \coloneqq \lim_{\substack{H \to 0 \\ H \in \mathcal{A}_\rho^\times \cap (U - Z_0)}} \rho(H)^{-1}(f(Z_0+H) - f(Z_0)). \tag{1.1.1}$$

(ii) Morphism Setting: a function $f\colon U \to \mathcal{B}$ is called *classically or "limit-wise" φ-differentiable* at Z_0 if the limit of the following *difference quotient limit* exists:

$$f'(Z_0) \coloneqq \lim_{\substack{H \to 0 \\ H \in \mathcal{A}_\varphi^\times \cap (U - Z_0)}} \frac{f(Z_0+H) - f(Z_0)}{\varphi(H)}. \tag{1.1.2}$$

(2) f is called *classically ρ/φ-differentiable* if it is classically ρ/φ-differentiable everywhere in U.

(3) Spaces of classically ρ/φ-differentiable functions on U:

$$\begin{aligned}
\mathscr{L}_\rho(U) &\coloneqq \mathscr{L}_{\mathcal{A}}(U, {}_{\mathcal{A}}M) \coloneqq \{f\colon U \to M \mid f \text{ is classically } \rho\text{-differentiable}\}\\
\mathscr{L}_\varphi(U) &\coloneqq \mathscr{L}_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathcal{B}) \overset{\text{def}}{=} \{f\colon U \to \mathcal{B} \mid f \text{ is classically } \varphi\text{-differentiable}\}\\
\mathscr{L}_{\mathcal{A}}(U) &\coloneqq \mathscr{L}_{\mathrm{id}_{\mathcal{A}}}(U) \overset{\text{def}}{=} \mathscr{L}_{\mathcal{A}}(U, \mathcal{A}).
\end{aligned}$$

(4) Spaces of k-times classically ρ/φ-differentiable functions: for $k \in \mathbb{N} \cup \{\infty\}$ we also put

$$\begin{aligned}
\mathscr{L}_\rho^k(U) &\coloneqq \mathscr{L}_{\mathcal{A}}^k(U, {}_{\mathcal{A}}M) \coloneqq \{f\colon U \to M \mid f \text{ is } k\text{-times classically } \rho\text{-differentiable}\}\\
\mathscr{L}_\varphi^k(U) &\coloneqq \mathscr{L}_{\mathcal{A}}^k(U, {}_{\mathcal{A}}\mathcal{B}) \overset{\text{def}}{=} \{f\colon U \to \mathcal{B} \mid f \text{ is } k\text{-times classically } \varphi\text{-differentiable}\}.
\end{aligned}$$

Given $v \in M$, it defines an $\mathcal{A}$-linear map $\hat{v}\colon {}_{\mathcal{A}}\mathcal{A} \to {}_{\mathcal{A}}M$, $a \mapsto a \cdot v \overset{\text{def}}{=} \rho(a)(v)$, which gives rise to the "regular module representation" $\Upsilon \coloneqq \Upsilon_M\colon M \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, M) \hookrightarrow \mathrm{Hom}_{\mathbb{R}}(\mathcal{A}, M)$, $v \mapsto \hat{v}$, with the inverse being given by $\Lambda \coloneqq \Lambda_M \coloneqq \Upsilon_M^{-1}\colon \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, M) \to M$, $\Phi \mapsto v \coloneqq \Phi(1_{\mathcal{A}})$. Completely analagously, $B \in \mathcal{B}$ induces a tautologically $\mathcal{A}$-linear map $\hat{B}\colon \mathcal{A} \to \mathcal{B}$, $A \mapsto \varphi(A)B$.

Definition 1.1.2 (Strict Fréchet ρ/φ-Differentiability)**:** Let $U \subseteq \mathcal{A}$ be open.

(1) Strict Fréchet ρ/φ-differentiability at a point: let $Z_0 \in U$.

(i) Representation Setting: a function $f\colon U \to {}_{\mathcal{A}}M$ is called *Fréchet strictly ρ-differentiable* at Z_0 if there exists $v \in M$ such that

$$\begin{aligned}
f(Z_0+H) &= f(Z_0) + H \cdot v + r(H) \overset{\text{def}}{=} f(Z_0) + \rho(H)v + r(H) \overset{\text{def}}{=}\\
&\overset{\text{def}}{=} f(Z_0) + \hat{v}(H) + r(H),
\end{aligned} \tag{1.1.3}$$

where $\|r(H)\|_M = o(\|\rho(H)\|_{\mathrm{End}})$ as $H \to 0$. In this case $f'(Z_0) := v$, *canonically identified* as $f'(Z_0) := \hat{v} \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, M) \subseteq \mathrm{Hom}_{\mathbb{K}}(\mathcal{A}, M)$, is called the ρ-derivative of f at Z_0.

(ii) Morphism Setting: a function $f\colon U \to \mathcal{B}$ is called *Fréchet strictly φ-differentiable* at Z_0 if there exists $B \in \mathcal{B}$ such that

$$\begin{aligned} f(Z_0+H) &= f(Z_0) + H\cdot B + r(H) \stackrel{\text{def}}{=} f(Z_0) + B\varphi(H) + r(H) \stackrel{\text{def}}{=} \\ &\stackrel{\text{def}}{=} f(Z_0) + \hat{B}(H) + r(H), \end{aligned} \tag{1.1.4}$$

where $\|r(H)\|_{\mathcal{B}} = o(\|\varphi(H)\|_{\mathcal{B}})$ as $H \to 0$. In this case $f'(Z_0) := B$, *canonically identified* as $f'(Z_0) := \hat{B} \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{B}) \subseteq \mathrm{Hom}_{\mathbb{K}}(\mathcal{A}, \mathcal{B})$, is called the φ-derivative of f at Z_0.

(2) f is called *Fréchet strictly ρ/φ-differentiable* if it is Fréchet strictly ρ/φ-differentiable everywhere in U.

(3) Spaces of Fréchet strictly ρ/φ-differentiable functions:

$$\begin{aligned} &\mathscr{F}^{\mathrm{str}}_{\rho}(U) := \mathscr{F}^{\mathrm{str}}_{\mathcal{A}}(U, {}_{\mathcal{A}}M) := \{f\colon U \to M \mid f \text{ is Fréchet strictly } \rho\text{-differentiable}\} \\ &\mathscr{F}^{\mathrm{str}}_{\varphi}(U) := \mathscr{F}^{\mathrm{str}}_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathcal{B}) \stackrel{\text{def}}{=} \{f\colon U \to \mathcal{B} \mid f \text{ is Fréchet strictly } \varphi\text{-differentiable}\} \\ &\mathscr{F}^{\mathrm{str}}_{\mathcal{A}}(U) := \mathscr{F}^{\mathrm{str}}_{\mathrm{id}_{\mathcal{A}}}(U) \stackrel{\text{def}}{=} \mathscr{F}^{\mathrm{str}}_{\mathcal{A}}(U, \mathcal{A}). \end{aligned}$$

(4) Spaces of k-times Fréchet strictly ρ/φ-differentiable functions: for $k \in \mathbb{N} \cup \{\infty\}$ we also put

$$\begin{aligned} &\mathscr{F}^{\mathrm{str},k}_{\rho}(U) := \mathscr{F}^{\mathrm{str},k}_{\mathcal{A}}(U, {}_{\mathcal{A}}M) := \{f\colon U \to M \mid f \text{ is } k\text{-times Fréchet strictly } \rho\text{-differentiable}\} \\ &\mathscr{F}^{\mathrm{str},k}_{\varphi}(U) := \mathscr{F}^{\mathrm{str},k}_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathcal{B}) \stackrel{\text{def}}{=} \{f\colon U \to \mathcal{B} \mid f \text{ is } k\text{-times Fréchet strictly } \varphi\text{-differentiable}\}. \end{aligned}$$

Definition 1.1.3 (Fréchet ρ-Differentiability)**:** Let $U \subseteq \mathcal{A}$ be open.

(1) Fréchet ρ/φ-differentiability at a point: let $Z_0 \in U$.

(i) Representation Setting: a function $f\colon U \to {}_{\mathcal{A}}M$ is called *Fréchet ρ-differentiable* at Z_0 if there exists $v \in M$ such that

$$\begin{aligned} f(Z_0+H) &= f(Z_0) + H\cdot v + r(H) \stackrel{\text{def}}{=} f(Z_0) + \rho(H)v + r(H) \stackrel{\text{def}}{=} \\ &\stackrel{\text{def}}{=} f(Z_0) + \hat{v}(H) + r(H), \end{aligned} \tag{1.1.5}$$

where $\|r(H)\|_M = o(\|H\|_{\mathcal{A}})$ as $H \to 0$. In this case $f'(Z_0) := v$, *canonically identified* as $f'(Z_0) := \hat{v} \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, M) \subseteq \mathrm{Hom}_{\mathbb{K}}(\mathcal{A}, M)$, is called the ρ-derivative of f at Z_0.

(ii) Morphism Setting: a function $f\colon U \to \mathcal{B}$ is called *Fréchet φ-differentiable* at Z_0 if there exists $B \in \mathcal{B}$ such that

$$\begin{aligned} f(Z_0+H) &= f(Z_0) + H\cdot B + r(H) \stackrel{\text{def}}{=} f(Z_0) + B\varphi(H) + r(H) \stackrel{\text{def}}{=} \\ &\stackrel{\text{def}}{=} f(Z_0) + \hat{B}(H) + r(H), \end{aligned} \tag{1.1.6}$$

where $\|r(H)\|_{\mathcal{B}} = o(\|H\|_{\mathcal{A}})$ as $H \to 0$. In this case $f'(Z_0) := B$, *canonically identified* as $f'(Z_0) := \hat{B} \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{B}) \subseteq \mathrm{Hom}_{\mathbb{K}}(\mathcal{A}, \mathcal{B})$, is called the φ-derivative of f at Z_0.

(2) f is called *Fréchet ρ/φ-differentiable* if it is Fréchet ρ/φ-differentiable everywhere in U.

(3) Spaces of Fréchet ρ/φ-differentiable functions:

$$\begin{aligned} &\mathscr{F}_{\rho}(U) := \mathscr{F}_{\mathcal{A}}(U, {}_{\mathcal{A}}M) := \{f\colon U \to M \mid f \text{ is Fréchet } \rho\text{-differentiable}\} \\ &\mathscr{F}_{\varphi}(U) := \mathscr{F}_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathcal{B}) \stackrel{\text{def}}{=} \{f\colon U \to \mathcal{B} \mid f \text{ is Fréchet } \varphi\text{-differentiable}\} \\ &\mathscr{F}_{\mathcal{A}}(U) := \mathscr{F}_{\mathrm{id}_{\mathcal{A}}}(U) \stackrel{\text{def}}{=} \mathscr{F}_{\mathcal{A}}(U, \mathcal{A}). \end{aligned}$$

(4) Spaces of k-times Fréchet ρ/φ-differentiable functions: for $k \in \mathbb{N} \cup \{\infty\}$ we also put

$$\begin{aligned} &\mathscr{F}^{k}_{\rho}(U) := \mathscr{F}^{k}_{\mathcal{A}}(U, {}_{\mathcal{A}}M) := \{f\colon U \to M \mid f \text{ is } k\text{-times Fréchet } \rho\text{-differentiable}\} \\ &\mathscr{F}^{k}_{\varphi}(U) := \mathscr{F}^{k}_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathcal{B}) \stackrel{\text{def}}{=} \{f\colon U \to \mathcal{B} \mid f \text{ is } k\text{-times Fréchet } \varphi\text{-differentiable}\}. \end{aligned}$$

Remarks:

(1) $\mathscr{L}$ stands for the *limit-wise definition*, $\mathscr{F}$ stands for *Fréchet*, $\mathscr{G}$ *will* stand for *Gateaux*, and $\mathscr{C}^k_\rho$ (and similar) will stand for (k-times) *continuous* differentiability. Also, note that $\mathscr{L}^1_\rho \overset{\text{def}}{=} \mathscr{L}_\rho$, $\mathscr{F}^{\text{str},1}_\rho \overset{\text{def}}{=} \mathscr{F}^{\text{str}}_\rho$, and $\mathscr{F}^1_\rho \overset{\text{def}}{=} \mathscr{F}_\rho$ (and likewise for φ). We will also say that a function f or a differential form ω is of (regularity) class $\mathscr{L}^k_\rho$, $\mathscr{F}^{\text{str},k}_\rho$, $\mathscr{F}^k_\rho$, or $\mathscr{G}^k_\rho$, respectively. Moreover, observe in the definitions that k-times ρ/φ-differentiability is **not** assumed to be *continuous*, unlike the standard definition for $\mathscr{C}^k$, $k \in \mathbb{N}$. This is on purpose: recall that continuity of the derivative of a function complex-differentiable in a neighbourhood turns out to be a superfluous assumption (cf. the Cauchy–Goursat Integral Theorem). Finally, to be completely rigorous, we have

$$\mathscr{L}^\infty_\rho(U) \overset{\text{def}}{=} \bigcap_{k\in\mathbb{N}} \mathscr{L}^k_\rho(U)$$

and likewise for $\mathscr{F}^{\text{str},\infty}_\rho(U)$, $\mathscr{F}^\infty_\rho(U)$, and a morphism $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$.

(2) Both notations $\mathscr{L}_\rho(U)$ and $\mathscr{L}_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$ have its advantages and disadvantages depending on context, so we will continue using them interchangeably. For example, for two different $\mathcal{A}$-modules ${}_{\mathcal{A}}M$ and ${}_{\mathcal{A}}N$ it is more clear to give them explicitly, i.e. $\mathscr{L}_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$ and $\mathscr{L}_{\mathcal{A}}(U, {}_{\mathcal{A}}N)$, whereas it's entirely possible to have two different representations $\rho_1, \rho_2 \colon \mathcal{A} \to \operatorname{End}_{\mathbb{K}}(M)$ (for example, differing by $\sigma \in \operatorname{Aut}_{\mathbb{K}}(\mathcal{A})$), i.e. two different $\mathcal{A}$-module structures on M, in which case the notation $\mathscr{L}_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$ would be ambiguous. Likewise for $\mathscr{F}$ and $\mathscr{F}^{\text{str}}$. *In any case, the second notations are meant to suggest that instead of ρ-differentiability one could also simply speak of $\mathcal{A}$-differentiability since in essence these are differentiability conditions that respect the $\mathcal{A}$-structure on the level of modules, i.e. representation point of view (as a map) vs. module point of view.* However, for such usage it is crucial to specify the $\mathcal{A}$-module (structure) (see also (15) below), much more so than in the case of $\mathbb{K}$-vector spaces. Nevertheless, when the module and the ρ-regularity class are understood, speaking plainly of $\mathcal{A}$-differentiability should not cause any confusion.

(3) Since $\operatorname{Ker}(\rho)$ may in general be non-trivial, the condition $\|r(H)\|_M = o(\|\rho(H)\|_{\text{End}})$ as $H \to 0$ in Eq. (1.1.3) should be understood as $\forall\, \varepsilon > 0\ \exists\, \delta = \delta(\varepsilon) > 0\ \forall\, \|H\|_{\mathcal{A}} < \delta\colon\ \|r(H)\|_M \le \varepsilon\, \|\rho(H)\|_{\text{End}}$. Likewise for Eq. (1.1.4).

(4) The only difference between Definition 1.1.2 and Definition 1.1.3 is in the o-condition for the remainder term $r(H)$, namely $\|r(H)\|_M = o(\|\rho(H)\|_{\text{End}})$ in $\mathscr{F}^{\text{str}}_\rho$ vs. $\|r(H)\|_M = o(\|H\|_{\mathcal{A}})$ in $\mathscr{F}_\rho$ (as $H \to 0$). Note that $\mathscr{F}^{\text{str}}_{\mathcal{A}} \overset{\text{def}}{=} \mathscr{F}_{\mathcal{A}}$. The condition $\mathscr{F}_\rho$ can be briefly summarized as *$\mathcal{A}$-equivariant (total or Fréchet) differentiability*, in particular we clearly have the implication $f \in \mathscr{F}_\rho(U) \Rightarrow f$ Fréchet (or totally) differentiable in the standard sense. Some authors describe $\mathscr{F}_{\mathcal{A}}$ in a rather clumsy way as $\mathcal{A}$-differentiability *in the sense of Lorch* (in reference to [Lor43]), but Lorch's definition is just Fréchet differentiability with the derivative being $\mathcal{A}$-linear… On the other hand, $\mathscr{F}^{\text{str}}_\rho$ is more of an auxiliary definition, motivated by the analytic setting $\sum_{k=1}^{\infty} B_k \varphi(Z)^k$, but is also more in alignment with Definition 1.1.1. Even though the distinction between $\mathscr{F}^{\text{str}}_\rho$ and $\mathscr{F}_\rho$ will eventually disappear once we impose $\mathscr{C}^1$-regularity, without it $\mathscr{F}^{\text{str}}_\rho$ actually satisfies some convenient properties that $\mathscr{F}_\rho$ does not. In fact, we have

$$\mathscr{F}^{\text{str}}_\rho(U) \subseteq \mathscr{F}_\rho(U)$$

in any case since $\forall\, H \in \mathcal{A}\colon\ \|\rho(H)\|_{\text{End}} \le \|\rho\|_{\text{op}}\, \|H\|_{\mathcal{A}} = \|H\|_{\mathcal{A}}$, which only justifies the term further. We remark that similarly *classical* ρ-differentiability will turn out to be a special case of Fréchet ρ-differentiability (as it should), see Proposition 1.1.11 below.

(5) Continuing the previous point, observe that the expression in Eq. (1.1.1) makes sense because $\mathcal{A}^\times \subset \mathcal{A}$ is dense. However, if $\dim_{\mathbb{K}} \mathcal{A} = \infty$, then $\mathcal{A}^\times$ need no longer remain dense in $\mathcal{A}$ (see for instance [DF03]), which precludes something like Definition 1.1.1, and in that case something like Eq. (1.1.5) is generically the only sensible definition (also used in [Lor43] for the case $\mathbb{K} = \mathbb{C}$ and $\rho = \mathrm{id}_{\mathcal{A}}$).

(6) More generally, given $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$, $M, N \in \mathsf{FdMod}(\mathcal{A})$, $U \subseteq M$ open, and $k \in \mathbb{N} \cup \{\infty\}$, one can define $\mathscr{F}^k_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$ to be the $\mathcal{A}$-module of k-times Fréchet (=totally) differentiable maps $f \colon U \to M$ with $\mathcal{A}$-linear differential, i.e. $\forall\, Z \in U \colon (Df)(Z) \in \mathrm{Hom}_{\mathcal{A}}({}_{\mathcal{A}}M, {}_{\mathcal{A}}N) \subseteq \mathrm{Hom}_{\mathbb{R}}(M, N)$. In that sense, if $U \subseteq \mathcal{A}$ open, then

$$\mathscr{F}^k_{\mathbb{K}}(U, M) \overset{\text{def}}{=} \{f \colon U \to M \mid f \text{ is } k\text{-times totally } \mathbb{K}\text{-differentiable}\}. \tag{1.1.7}$$

Observe that again $\forall\, k \in \mathbb{N} \colon \mathscr{F}^k_{\mathbb{R}}(U, M) \subsetneqq \mathscr{C}^k(U, M)$. This will be occasionally useful notation.

(7) (ρ, λ)-Differentiability: one can combine Definition 1.1.2 and Definition 1.1.3 as follows. Let $\lambda \colon \mathcal{A} \to \mathfrak{B}$ is a morphism into a *not necessarily commutative* normed $\mathbb{K}$-algebra $\mathfrak{B}$. A function $f \colon U \to M$ will be called (ρ, λ)-differentiable in U, written $f \in \mathscr{F}_{\rho,\lambda}(U)$ if for every $Z_0 \in U$ there exists $f'(Z_0) \in M$ such that

$$f(Z_0 + H) = f(Z_0) + \rho(H) f'(Z_0) + r(H),$$

where $\|r(H)\|_M = o(\|\lambda(H)\|_{\mathfrak{B}})$ as $H \to 0$. Then we recover $\mathscr{F}^{\mathrm{str}}_\rho = \mathscr{F}_{\rho,\rho}$ and $\mathscr{F}_\rho = \mathscr{F}_{\rho,\mathrm{id}_{\mathcal{A}}}$.

(8) The abstract operator $\Phi \in \mathrm{Hom}_{\mathbb{K}}(\mathcal{A}, M)$, $H \mapsto \Phi(H)$, that should express the first-order linear approximation of f in Definition 1.1.2 and Definition 1.1.3, is represented by some $v \in M$ as an *inner*[1] linear map $\Phi(H) = \hat{v}(H) := \rho(H) v \overset{\text{def}}{=} H \cdot v$, and the differential $Df = \Phi$ is then $\mathcal{A}$-linear with respect to the $\mathcal{A}$-module structure on M. Conversely, since $\mathcal{A}$ is unital, any $\mathcal{A}$-linear $\Phi \in \mathrm{Hom}_{\mathbb{K}}(\mathcal{A}, M)$ is inner because $\forall\, H \in \mathcal{A} \colon \Phi(H) = H \cdot \Phi(1_{\mathcal{A}}) \overset{\text{def}}{=} \rho(H) v$ for $v := \Phi(1_{\mathcal{A}})$. In other words, we have an *equivalence between $\mathcal{A}$-linearity and inner representability of Df for unital $\mathcal{A}$*. Furthermore, due to unitality we have not only uniqueness of Φ (from Fréchet differentiability) but also uniqueness of its inner representative: if $\Phi(H) = \rho(H) v = \rho(H) w$, then taking any $H \in \mathcal{A}^\times$, e.g. any $H \in \mathbb{K}^\times \subset \mathcal{A}^\times$, implies $v = w$. We remark that uniqueness of the inner representative of Φ may fail if $\mathcal{A}$ were non-unital, for example if $\mathcal{A} = \mathfrak{N}$ were a nilpotent finite-dimensional commutative associative $\mathbb{K}$-algebra: in that case M may admit a non-trivial submodule $M \supseteq M' \neq 0$ such that $\mathfrak{N} \cdot M' = 0$, i.e. the homomorphism $\Upsilon_M \colon M \to \mathrm{Hom}_{\mathbb{K}}(\mathfrak{N}, M)$, $v \mapsto \hat{v}$, may happen to have a non-trivial kernel. For instance, if we were to define Fréchet $\mathfrak{N}$-differentiability in the same manner, then $f'(Z_0)$ would be well-defined only up to $(\mathrm{mod}\ \mathfrak{N}^{\nu-1})$, where ν is the order of nilpotence of $\mathfrak{N}$.

(9) Higher (continuous) ρ/φ-differentiability: observe that for all three differentiability classes, $\mathscr{L}_\rho$, $\mathscr{F}^{\mathrm{str}}_\rho$, and $\mathscr{F}_\rho$, the ρ-derivative is naturally again a function with the same domain and codomain $f' \colon U \to {}_{\mathcal{A}}M$ (or $f' \colon U \to \mathcal{B}$, respectively). It thus makes indeed sense to speak of *higher* (continuous) ρ/φ-differentiability and higher ρ/φ-derivatives. On the other hand, writing $\mathcal{A}^\vee := \mathrm{Hom}_{\mathbb{R}}(\mathcal{A}, \mathbb{R})$ for the *real* dual of $\mathcal{A}$ and $M_0 := M$, $M_{k+1} := \mathrm{Hom}_{\mathbb{R}}(\mathcal{A}, M_k) \cong M \otimes_{\mathbb{R}} (\mathcal{A}^\vee)^{\otimes(k+1)}$, $k \in \mathbb{N}_0$, we have that $\{M_k\}_{k \in \mathbb{N}_0}$ are naturally $\mathcal{A}$-modules via the action of $\mathcal{A}$ on the M-"factor", or equivalently, recursively via

$$\forall\, A \in \mathcal{A}\ \forall\, \Phi_{k+1} \in \mathrm{Hom}_{\mathbb{R}}(\mathcal{A}, M_k)\ \forall\, a \in \mathcal{A} \colon (A \cdot \Phi_{k+1})(a) := A \cdot (\Phi_{k+1}(a)).$$

Suppose now that $f \in \mathscr{C}^n(U, M)$ only. Then *generically* we only have $D^k f \colon U \to M_k \cong M \otimes_{\mathbb{R}} (\mathcal{A}^\vee)^{\otimes k}$ as a *real*-differentiable function, $1 \leq k \leq n$. If additionally $(D^k f)(Z)$ is $\mathcal{A}$-linear, $Z \in U$, then in fact $D^k f \colon U \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, {}_{\mathcal{A}}M_{k-1}) \cong M_{k-1} \hookrightarrow M_k$, since clearly $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, {}_{\mathcal{A}}M_{k-1}) \subseteq \mathrm{Hom}_{\mathbb{K}}(\mathcal{A}, M_{k-1})$, hence actually $D^{k+1} f \colon U \to \mathrm{Hom}_{\mathbb{R}}(\mathcal{A}, M_{k-1})$, $1 \leq$

[1] the term is loosely borrowed from group theory, cf. inner vs. outer automorphisms;

$k \le n$. If moreover $(D^k f)(Z)$, $Z \in U$, is $\mathcal{A}$-linear for all $1 \le k \le n$, then by replacing all $\mathrm{Hom}_{\mathbb{R}}(\mathcal{A}, M_k)$ by $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, M_k)$ in this procedure it follows inductively that in fact $\forall\, 1 \le k \le n\colon D^k f\colon U \to M$, with the $(\mathcal{A}^\vee)^{\otimes k}$-factors being killed off. Thus, as expected, both pictures are consistent with each other. To make this even more explicit, we can extend the natural isomorphism $\Lambda\colon \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, {}_{\mathcal{A}}M) \xrightarrow{\cong} {}_{\mathcal{A}}M$ of $\mathcal{A}$-modules from earlier to

$$\begin{aligned} \Lambda_k\colon \mathcal{L}^k_{\mathcal{A}}(\mathcal{A}, {}_{\mathcal{A}}M) &\xrightarrow{\cong} {}_{\mathcal{A}}M \\ \Phi_k &\mapsto \Phi_k(\underbrace{1_{\mathcal{A}}, \dots, 1_{\mathcal{A}}}_{k \text{ times}}) \\ \Upsilon_k \coloneqq \Lambda_k^{-1}\colon {}_{\mathcal{A}}M &\xrightarrow{\cong} \mathcal{L}^k_{\mathcal{A}}(\mathcal{A}, {}_{\mathcal{A}}M) \\ v &\mapsto \big(\hat{v}_k\colon (Z_1, \dots, Z_k) \mapsto Z_1 \cdots Z_k \cdot v\big) \end{aligned}$$

In particular, we have the following correspondence between the higher real total derivatives and the higher ρ-derivatives: if $f \in \mathscr{F}^n_\rho(U)$, then

$$\forall\, 1 \le k \le n\; \forall\, Z \in U\colon f^{(k)}(Z) = \big((D^k f)(Z)\big)(\underbrace{1_{\mathcal{A}}, \dots, 1_{\mathcal{A}}}_{k \text{ times}}) \tag{1.1.8}$$

We will revisit this in Proposition 1.1.16.

(10) Continuing the previous point, unlike the case $\mathscr{F}^{\mathrm{str},k}_\rho(U) \subseteq \mathscr{F}^k_\rho(U)$, where in particular Fréchet differentiability of f implies continuity of f, we haven't established (yet) the implication $f \in \mathscr{L}_\rho(U) \Rightarrow f$ continuous. This will happen in Proposition 1.1.9.

(11) We *could have defined* more succinctly[2] $\mathscr{L}_\varphi \coloneqq \mathscr{L}_{\rho_\varphi}$ and $\mathscr{F}_\varphi \coloneqq \mathscr{F}_{\rho_\varphi}$ since these are tautologically true by the current definitions anyway. Moreover, $\|\rho_\varphi(H)\|_{\mathrm{End}} \overset{\mathrm{def}}{=} \|\rho_{\mathcal{B}}(\varphi(H))\|_{\mathrm{End}} = \|\varphi(H)\|_{\mathcal{B}}$ by Lemma 0.3.24. It follows that we also have $\mathscr{F}^{\mathrm{str}}_\varphi \overset{\mathrm{def}}{=} \mathscr{F}^{\mathrm{str}}_{\rho_\varphi}$. On the other hand, observe however that a ρ-differentiable function f is M-valued, whereas a φ_ρ-differentiable function is $\mathcal{B}$-valued with $\mathcal{B}$ only *acting* on M (cf. Lemma-Definition 0.1.15). In other words, $\mathscr{L}_\rho \ne \mathscr{L}_{\rho^{\mathrm{mor}}}$, $\mathscr{F}^{\mathrm{str}}_\rho \ne \mathscr{F}^{\mathrm{str}}_{\rho^{\mathrm{mor}}}$, and $\mathscr{F}_\rho \ne \mathscr{F}_{\rho^{\mathrm{mor}}}$ by definition. Likewise for the higher ρ-derivatives. In summary, we only have

$$\mathscr{L}^k_\varphi = \mathscr{L}^k_{\varphi^{\mathrm{rep}}},\ \mathscr{F}^{\mathrm{str},k}_\varphi = \mathscr{F}^{\mathrm{str},k}_{\varphi^{\mathrm{rep}}},\ \mathscr{F}^k_\varphi = \mathscr{F}^k_{\varphi^{\mathrm{rep}}} \tag{1.1.9}$$

for all $k \in \mathbb{N} \cup \{\infty\}$ (and also for $k = \omega$ after Definition 1.1.15).

(12) Independence of o-conditions from the choice of equivalent norms: let $X \in \mathsf{Top}$, $(V, \|\cdot\|_V)$, $(W, \|\cdot\|_W) \in \mathsf{NrmVec}_{\mathbb{K}}$, and $f\colon X \to V$ and $g\colon X \to W$ be functions with $\|f(x)\|_V = o(\|g(x)\|_W)$. Let $\|\cdot\|'_V$ and $\|\cdot\|'_W$ be some further norms on V and W, respectively, and let $C_V, c_W > 0$ be such that $\|\cdot\|'_V \le C_V \|\cdot\|_V$ and $c_W \|\cdot\|_W \le \|\cdot\|'_W$. Now let $\varepsilon' > 0$ be arbitrary. If $\forall\, \varepsilon > 0\; \exists\, \mathcal{N}_\varepsilon \ni p\; \forall\, x \in \mathcal{N}_\varepsilon\colon \|f(x)\|_V \le \varepsilon \|g(x)\|_V$, then taking $\varepsilon \coloneqq \frac{c_W}{C_V}\varepsilon'$, we have $\forall\, x \in \mathcal{N}_\varepsilon\colon \|f(x)\|'_V \le C_V \|f(x)\|_V \le C_V \varepsilon \|g(x)\|_W \le C_V \varepsilon \frac{1}{c_W} \|g(x)\|'_W = \varepsilon' \|g(x)\|'_W$ as required. This actually shows that only a weaker (one-sided) relation than norm equivalence is needed for o-conditions to be independent of the choice of norms.

(13) In Eq. (1.1.1) the usage of $\rho(H)^{-1}$ instead of $\rho(H^{-1})$ allows for $H \in \mathcal{A}_{\mathrm{sing}} \overset{\mathrm{def}}{=} \mathcal{A} \setminus \mathcal{A}^\times$ with $H \to 0$, for which we still get $\rho(H) \in \mathrm{GL}(M)$ with $\rho(H) \to 0$. In combination with Lemma 1.1.10 below this allows for a certain amount of flexibility in proving certain "functorial-like" statements later.

(14) In the present work we take a **firm stand** that *in principle* one should distinguish between *holomorphy* (complex- or other differentiability in a *neighbourhood* of a point) and *analyticity* (convergent power series expansions) and in particular that it is a *theorem* (rather than a definition) of Complex Analysis that these two notions agree in the complex setting. Distin-

[2] but we didn't!

guishing between both notions even in the setting of standard complex analysis is a matter of good taste! However, it is debatable to what extent the term *holomorphy* is justified in the absence of a complex structure. On the one hand, $\mathcal{A}$-linearity of Df imposes a generalized Cauchy-Riemann PDE system condition on f (see Section 1.3) regardless of the ground field (for example, this point of view on holomorphy is taken in [Sal23] as indicated by its title). Such PDE conditions in turn only accentuate further the need to distinguish between ρ-differentiability *at a point* and ρ-differentiability *in a neighbourhood* of said point. On the other hand, in the absence of a complex structure, these generalized Cauchy–Riemann PDEs do not lead in general to analyticity (see for example [GM96]), so they are not as strong a condition as the original. To avoid terminological ambiguity we propose that one should instead speak of ***real* holomorphy**. This is somewhat justified by the fact that under certain mild assumptions ρ-differentiability implies slightly higher automatic regularity (though in general not as strong as analyticity as in the complex case) even without an almost complex structure and also shares some other regularity properties of holomorphy such as infinite integrability (see Proposition 1.4.12).

(15) Equation (1.1.1) naturally suggests to try and extend *classical* ρ-differentiability to other *limit* points in $\mathcal{A}$ in the following sense: if $W_0 \in \mathcal{A}$, then we can consider the expression

$$f'_{(W_0)}(Z_0) := \lim_{\substack{H \to W_0 \\ H \in \mathcal{A}^\times_\rho \cap (U - Z_0)}} \rho(H)^{-1}(f(Z_0 + H) - f(Z_0)).$$

If $W_0 \in \mathcal{A}^\times$, then by continuity of f (Proposition 1.1.9) this is just $f'_{(W_0)}(Z_0) = \rho(W_0)^{-1}(f(Z_0 + W_0) - f(Z_0))$, so the concept is more interesting for $W_0 \in \mathcal{A} \backslash \mathcal{A}^\times$. For example, if $W_0 \in \mathrm{nil}(\mathcal{A})$, it will turn out that such limits exist when $\mathbb{K} = \mathbb{C}$ and f is ρ-holomorphic at Z_0 and moreover can be used to give an alternative proof[3] of the generalized homological version of Cauchy's Integral Formula (Theorem 2.3.6).

(16) Comparison with other notions of differentiability: somewhat differently than the usual notions of differentiability, ρ/φ-differentiability of f is entwined in both the domain and the *codomain* of f for it depends on a choice of a representation ρ. Moreover, observe that M need not be a free $\mathcal{A}$-module. For example, this relationship becomes apparent once we state the generalized Cauchy-Riemann equations for a morphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ (see Proposition 1.3.1). In contrast, the Cauchy-Riemann equations for holomorphic functions taking values in, say, any $V \in \mathsf{FdVec}_{\mathbb{C}}$, $f\colon \mathbb{C} \to V$, are always the same.

(17) In essence, the definition of φ-differentiability appears to be using only the fact that $\mathcal{B}$ is a normed $\mathcal{A}$-module. So, how does the algebra structure of $\mathcal{B}$ come into play? An important feature of the ρ/φ-definition is that if $f\colon U \to M$ is a ρ-holomorphic function, then the derivative f' is again a function $U \to M$ and likewise for φ-differentiable $f\colon U \to \mathcal{B}$. This is in stark contrast to the situation when $\mathcal{A}$ and $\mathcal{B}$ are considered only as $\mathbb{K}$-vector spaces. One might thus suggest that a better analogy would be to take holomorphic functions $f\colon \mathbb{C} \to V$, $V \in \mathsf{FdVec}_{\mathbb{C}}$, as above since their derivatives retain the same domain and codomain $f'\colon \mathbb{C} \to V$. Unfortunately, in general these cannot be composed with each other without further ado, unlike in the morphism setting (it should come as no surprise that if f is φ-differentiable and g is ψ-differentiable, then $g \circ f$ will be $\psi \circ \varphi$-differentiable, provided compatible domains and codomains). One can overcome this defi(ni)ciency by introducing holomorphic functions between vector spaces $f\colon V_1 \to V_2$, which then can be composed as $V_1 \to V_2 \to V_3$, but their derivatives have again codomains different than the original ones, e.g. $f'\colon V_1 \to \mathrm{Hom}_{\mathbb{C}}(E_1, E_2)$ etc. We have come a full circle.

[3] a modification of Dixon's proof for the $\mathcal{A}$-setting;

(18) It is a natural question to what extent building a representation ρ or a morphism φ into Definition 1.1.3 might be trivial. There are some possibilities for this: for example $\mathscr{F}_{\mathcal{A}}(U, {}_{\mathcal{A}}M) \cong \mathscr{F}_{\mathcal{A}}(U) \otimes_{\mathcal{A}} M$, or for $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ perhaps a φ-differentiable function $f\colon U \to \mathcal{B}$ is always given as a composition $\varphi \circ g$ for some $g \in \mathcal{O}_{\mathcal{A}}(U)$. The answer to the latter is in the negative: clearly, $f(Z) := \varphi(Z)b$ with $b \notin \operatorname{Im}(\varphi)$ cannot be written in this manner. Another possibility is perhaps that $f\colon U \to \mathcal{B}$ lifts to a $\mathcal{B}$-differentiable function $g\colon V \to \mathcal{B}$, $V \supseteq \varphi(U)$, i.e. $f = g \circ \varphi$. Notice that unless φ is surjective, which requires $\dim_{\mathbb{K}} \mathcal{B} \leq \dim_{\mathbb{K}} \mathcal{A}$ in any case, $\varphi(U)$ won't be open (in $\mathcal{B}$).

Example: Let $Z := z_1 + \epsilon z_2 \in \mathbb{K}[\epsilon]$, $\epsilon^2 = 0$, and define $f(Z) := f(z_1, z_2) := f_1(z_1, z_2) + \epsilon f_2(z_1, z_2)$, where $f_1(z_1, z_2) := z_1^3 - z_1^2 + 1$ and $f_2(z_1, z_2) := 2z_1^3 + 3z_1^2 z_2 + z_1^2 - 2z_1 z_2 + 3$. Then f is $\mathbb{K}[\epsilon]$-differentiable because $f(Z) = (1+2\epsilon)Z^3 + (-1+\epsilon)Z^2 + (1+3\epsilon) \in \mathbb{K}[\epsilon][Z]$ with $f'(Z) = (3+6\epsilon)Z^2 + (-2+2\epsilon)Z$. □

For categorical purposes and interactions it is important to establish that Definition 1.1.1 admits certain amount of flexibility when it comes to the choice of a *dense neighbourhood* $\mathcal{A}_{\varphi}^{\times}$ of 0 and that it can be replaced by other similar subsets, for example by $\mathcal{A}^{\times}$ itself, but also not only by it, since $\mathcal{A}^{\times}$ itself does not behave categorically in the desired way: it does not respect preimages since in general we only have $\varphi^{-1}(\mathcal{B}^{\times}) \supseteq \mathcal{A}^{\times}$. While it is technically possible to indeed fix $\mathcal{A}^{\times}$ everywhere, this would require non-stop reference to the algebraic product structure of $\mathcal{A}$ and the corresponding idempotents, thus invoking algebraic arguments where purely topological ones suffice, as we shall see. We will shortly give a first example of that.

Definition 1.1.4: Let $X \in \mathsf{Top}$ and $x_0 \in X$. Then $D \subseteq X$ is called a *dense(ly defined) neighbourhood* of x_0 if $\bar{D}$ is a neighbourhood of x_0. Note that we do not require $D \ni x_0$.

Definition 1.1.5: Let (X, d_X) and (Y, d_Y) be two metric spaces, $x_0 \in X$ a point, and $f\colon X \to Y$ a map. Then f is called *densely (sequentially) continuous* at x_0 if there exists a dense neighbourhood D of x_0 such that

$$\lim_{\substack{x \to x_0 \\ x \in D}} f(x) = f(x_0).$$

In that case, f will be called *(sequentially) densely continuous at x_0 with respect to D* or *(sequentially) D-continuous at x_0* for short.

Definition 1.1.6 (D-Restricted Classical ρ-Differentiability)**:** Let $U \subseteq \mathcal{A}$ be open, $f\colon U \to {}_{\mathcal{A}}M$ a map, $Z_0 \in U$ a point, and fix a dense neighbourhood $D \subseteq \mathcal{A}_{\rho}^{\times}$ of 0.

(1) f is called *classically D-ρ-differentiable* at Z_0 if the limit of the following difference quotient exists:

$$f_D'(Z_0) := \lim_{\substack{H \to 0 \\ H \in D \cap (U - Z_0)}} \rho(H)^{-1}(f(Z_0 + H) - f(Z_0)). \tag{1.1.10}$$

(2) f is called *classically D-ρ-differentiable* if it is classically D-ρ-differentiable everywhere in U.

(3) Spaces of classically D-ρ-differentiable functions:

$$\mathscr{L}_{D,\rho}(U) := \mathscr{L}_{D,\mathcal{A}}(U, {}_{\mathcal{A}}M) := \{f\colon U \to M \mid f \text{ is classically } D\text{-}\rho\text{-differentiable}\}.$$

Lemma 1.1.7 (Densely Sequential Continuity at a Point)**:** Let $U \subseteq \mathcal{A}$ be open, $f\colon U \to M$ a map, $Z_0 \in U$ a point, and $D \subseteq \mathcal{A}_{\rho}^{\times}$ a dense neighbourhood of 0. If f is $\mathscr{L}_{D,\rho}$-differentiable at Z_0, then f is also sequentially D-continuous at Z_0.

Proof: Put $D(Z_0) := D \cap (U - Z_0)$. As $D \subseteq \mathcal{A}_{\rho}^{\times}$, we get indeed

$$\lim_{\substack{H \to 0 \\ H \in D(Z_0)}} (f(Z_0 + H) - f(Z_0)) = \lim_{\substack{H \to 0 \\ H \in D(Z_0)}} \rho(H) \lim_{\substack{H \to 0 \\ H \in D(Z_0)}} \rho(H)^{-1}(f(Z_0 + H) - f(Z_0)) = 0 \cdot f_D'(Z_0) = 0$$

since ρ is (automatically) continuous and hence both limits exist. $\square$

As an organic part of this section we are now going to revisit several results of [Wat92a]. We remark that Waterhouse calls $\mathcal{A}$-differentiable (p. 185) what we here call *classically* $\mathcal{A}$-differentiable or of class $\mathscr{L}_{\mathcal{A}}$ for short. In Section 2 of that paper he proves an important "separation theorem" (decomposition of f with respect to the product decomposition of $\mathcal{A}$) for the class $\mathscr{L}_{\mathcal{A}}$, which we will prove a little later anyway as a consequence of the interplay between the $\mathcal{A}$-linearity of the derivative and elementary categorical considerations in $\mathsf{FdCAlg}_{\mathbb{K}}$ (see Proposition 1.2.9 (2) and onwards). He writes on p. 185 in the proof of the theorem that "Our writing of $\mathcal{A}$ as a product would have made this theorem almost obvious *if we knew that f is differentiable as a function of its real variable.*" In Section 5 of said paper he then proves that if $\mathcal{A}$ is local, i.e. $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{K})$, and f is of class $\mathscr{L}_{\mathcal{A}}$, then it is of class $\mathscr{F}_{\mathcal{A}}$ (hence totally differentiable), *by invoking the basic algebraic structure of* $\mathcal{A} = \mathbb{K} \oplus_{\mathbb{K}} \mathfrak{m}$. After that, Waterhouse continues on p. 189: *"Observe that some such proof is needed even to show that f is continuous"*. We are now going to show that none of this is needed. More precisely, we are going to show that both continuity and total differentiability of *classically* $\mathcal{A}$-differentiable functions in fact follow *for purely topological reasons* instead of algebraic ones (as they should!), *circumventing the structure theory of local $\mathcal{A}$-s altogether.* To begin with, being "densely" sequentially continuous *everywhere* indeed suffices to establish full continuity *everywhere*[4]:

Lemma 1.1.8: Let (X, d_X) and (Y, d_Y) be two metric spaces, $S \subseteq X$ a dense subset, and $f \colon X \to Y$ a map such that $\forall\, x \in X \colon \lim\limits_{\substack{s \to x \\ s \in S}} f(s) = f(x)$. Then f is continuous.

Proof: By hypothesis, we have that $\forall\, x \in X\ \forall\, \varepsilon > 0\ \exists\, \delta = \delta(x, \varepsilon) > 0\, \forall\, s \in S \colon d_X(s, x) < \delta \Rightarrow d_Y(f(s), f(x)) < \varepsilon$. Now let $(x_n)_{n \in \mathbb{N}} \subset X$ be an arbitrary sequence with $x_n \to x$ and put $\delta_n := \delta(x_n, \frac{1}{n})$, $n \in \mathbb{N}$. Since $S \subseteq X$ is dense, we have that $\forall\, n \in \mathbb{N}\ \exists\, s_n \in S \colon d_X(s_n, x_n) < \min\{\delta_n, \frac{1}{n}\}$. In particular, $s_n \to x$ since $d_X(s_n, x) \leq d_X(s_n, x_n) + d_X(x_n, x) < \frac{1}{n} + d_X(x, x_n) \to 0$ as $n \to \infty$, hence $\lim\limits_{n \to \infty} f(s_n) = f(x)$. Furthermore, by construction $d_Y(f(s_n), f(x_n)) < \frac{1}{n}$ since $d_X(s_n, x_n) < \delta_n$, $n \in \mathbb{N}$. Thus, $d_Y(f(x), f(x_n)) \leq d_Y(f(x), f(s_n)) + d_Y(f(s_n), f(x_n)) \to 0$ as $n \to \infty$, as desired. $\square$

Proposition 1.1.9 (Continuity of $\mathscr{L}_{D,\rho}$-Differentiable Functions)**:** Let $U \subseteq \mathcal{A}$ be open and $D \subseteq \mathcal{A}_\rho^\times$ a densely defined neighbourhood of 0. Then $\mathscr{L}_{D,\mathcal{A}}(U, {}_{\mathcal{A}}M) \subseteq \mathscr{C}(U, {}_{\mathcal{A}}M)$.

Proof: By Lemma 1.1.7 we have $\forall\, Z_0 \in U \colon \lim\limits_{\substack{H \to 0 \\ H \in D(Z_0)}} f(Z_0 + H) = f(Z_0)$. Taking $S := U \cap \bigcup_{Z_0 \in U} (Z_0 + D)$ gives a dense subset of U, and the claim now follows from Lemma 1.1.8. $\square$

Lemma 1.1.10 (Dense Replacement Principle)**:** Let $U \subseteq \mathcal{A}$ be open and $D \subseteq \mathcal{A}_\rho^\times$ an arbitrary densely defined neighbourhood of 0. Then $\mathscr{L}_\rho(U) = \mathscr{L}_{D,\rho}(U)$ and $\forall\, f \in \mathscr{L}_\rho(U) \colon f' = f'_D$.

Proof: Clearly, $\mathscr{L}_\rho(U) \subseteq \mathscr{L}_{D,\rho}(U)$. We need to show that $\mathscr{L}_{D,\rho}(U) \subseteq \mathscr{L}_\rho(U)$. To this end, let $f \in \mathscr{L}_{D,\rho}(U)$, fix $Z_0 \in U$, and define the function $r_{Z_0} \colon \mathcal{N} \to {}_{\mathcal{A}}M$, $r_{Z_0}(H) := f(Z_0 + H) - f(Z_0) - \rho(H) f'_D(Z_0)$, on some sufficiently small open neighbourhood $\mathcal{N} \ni 0$ such that $Z_0 + \mathcal{N} \subseteq U$. We consider the function $R_{Z_0} \colon (\mathcal{N} \cap \mathcal{A}_\rho^\times) \cup \{0\} \to {}_{\mathcal{A}}M$ defined by

$$R_{Z_0}(H) := \begin{cases} \rho(H)^{-1} r_{Z_0}(H), \text{ if } H \in \mathcal{N} \cap \mathcal{A}_\rho^\times \\ 0, \text{ if } H = 0. \end{cases}$$

It suffices to show that $R_{Z_0}(H)$ is (sequentially) continuous. Since f is $\mathscr{L}_{D,\rho}$-differentiable at Z_0, we have by definition that $R_{Z_0}(H)$ is sequentially D-continuous at 0. Moreover, f is $\mathscr{L}_{D,\rho}$-differentiable at every $Z_0 + H$, hence f is sequentially D-continuous at every $Z_0 + H$ by Lemma 1.1.7. Since ρ is

[4] this is clearly false if we replace *everywhere* by *a single point*;

(automatically) continuous, it now follows that $R_{Z_0}(H)$ is also sequentially D-continuous everywhere on $\mathcal{N} \cap \mathcal{A}_\rho^\times$. That is, $R_{Z_0}(H)$ is everywhere D-continuous, hence by Lemma 1.1.8 it is everywhere continuous by taking $S := \mathcal{N} \cap \mathcal{A}_\rho^\times \cap \bigcup_{H \in \mathcal{N} \cap \mathcal{A}_\rho^\times}(H + D)$. $\square$

Remarks:

(1) Observe that it has been essential so far to use *neighbourhood conditions* in order to establish *properties at a point.* Notably, $\mathscr{L}_\rho$-"holomorphy" of f at Z_0, i.e. $\mathscr{L}_\rho$-differentiability of f in a neighbourhood of Z_0, is required in order to establish continuity at Z_0 (hence at every point) as we needed an everywhere dense subset S of said neighbourhood.

(2) Note that Lemma 1.1.10 is not just about *calculating* $\lim\limits_{H \to 0}$ by using a different dense neighbourhood, *provided* the limit exists, but rather about guaranteeing that said limit exists independently of the choice of a dense neighbourhood.

(3) Lemma 1.1.10 will be the basis of the "functorial" behavior of $\mathscr{L}_\rho$, respectively $\mathscr{L}_\varphi$, with respect to various pre- and post-compositions as these will inadvertently change the dense neighbourhood, over which the limit is taken. $\square$

Proposition 1.1.11: Let $U \subseteq \mathcal{A}$ open and $k \in \mathbb{N} \cup \{\infty\}$. Then $\mathscr{L}_\rho^k(U) \subseteq \mathscr{F}_\rho^{\mathrm{str},k}(U) \subseteq \mathscr{F}_\rho^k(U) \subseteq \mathscr{F}_\mathbb{R}(U, M)$. Moreover, if ρ is faithful, then in fact $\mathscr{F}_\rho^{\mathrm{str},k}(U) = \mathscr{F}_\rho^k(U)$.

Proof: It suffices to consider only the case $k = 1$.

"$\mathscr{L}_\rho(U) \subseteq \mathscr{F}_\rho^{\mathrm{str}}(U)$:" Fix $Z_0 \in U \subseteq \mathcal{A}$ and put $r(H) := f(Z_0 + H) - f(Z_0) - \rho(H)f'(Z_0)$ taking values in ${}_{\mathcal{A}}M$. Since $f \in \mathscr{L}_\rho(U)$, we have by definition that $\forall\, \varepsilon > 0\ \exists\, \delta = \delta(\varepsilon) > 0\ \forall\, H \in \mathcal{A}_\rho^\times\colon\ \|H\|_\mathcal{A} < \delta \Rightarrow \left\|\rho(H)^{-1} r(H)\right\|_M < \varepsilon$. Hence $\forall\, H \in \mathcal{A}_\rho^\times$:

$$\|H\|_\mathcal{A} < \delta \Rightarrow \|r(H)\|_M = \left\|\rho(H)\rho(H)^{-1} r(H)\right\|_M \le \|\rho(H)\|_{\mathrm{End}} \left\|\rho(H)^{-1} r(H)\right\|_M < \varepsilon \|\rho(H)\|_{\mathrm{End}}.$$

Now let $K \in \mathcal{A}$ be arbitrary with $\|K\|_\mathcal{A} < \delta(\varepsilon)$. Since $\mathcal{A}_\rho^\times \subset \mathcal{A}$ is dense, there exists now a sequence $(H_n)_{n\in\mathbb{N}} \subset \mathcal{A}_\rho^\times$ with $\forall\, n \in \mathbb{N}\colon\ \|H_n\|_\mathcal{A} < \delta(\varepsilon)$ such that $H_n \to K$ as $n \to \infty$. As $f \in \mathscr{L}_\rho(U)$, f is in particular also $\mathscr{L}_\rho$-differentiable at $Z_0 + K$, hence $f(Z_0 + K) = \lim\limits_{n\to\infty} f(Z_0 + H_n)$ by Lemma 1.1.7. It follows that $r(K) = \lim\limits_{n\to\infty} r(H_n)$ as ρ is (automatically) continuous. Thus, $\|r(K)\|_M = \lim\limits_{n\to\infty} \|r(H_n)\|_M \le \lim\limits_{n\to\infty} \varepsilon \|\rho(H_n)\|_{\mathrm{End}} = \varepsilon \|\rho(K)\|_{\mathrm{End}}$, again by continuity of ρ. In other words, $\|r(H)\|_M = o(\|\rho(H)\|_{\mathrm{End}})$ for *all* H with $H \to 0$, which proves that $f \in \mathscr{F}_\rho^{\mathrm{str}}(U)$.

"$\mathscr{F}_\rho^{\mathrm{str}}(U) \subseteq \mathscr{F}_\rho(U)$:" If $f \in \mathscr{F}_\rho(U)$, then $\forall\, \varepsilon > 0\ \forall\, \|H\|_\mathcal{A} < \delta(\varepsilon)\colon\ \|r(H)\|_M \le \varepsilon \|\rho(H)\|_{\mathrm{End}} \le \varepsilon \|\rho\|_{\mathrm{op}} \|H\|_\mathcal{A} = \varepsilon \|H\|_\mathcal{A}$. We thus have $\|r(H)\|_M = o(\|H\|_\mathcal{A})$ as $H \to 0$.

Finally, if ρ is faithful, then ρ is also bounded from below as it automatically has a closed range in the finite-dimensional case. Thus, we also have $\|H\|_\mathcal{A} \le c_\rho \|\rho(H)\|_{\mathrm{End}}$ for some $c_\rho > 0$, implying the claim. $\square$

Problem 5: Prove or disprove $\mathscr{L}_\rho(U) \subsetneq \mathscr{F}_\rho^{\mathrm{str}}(U) \subsetneq \mathscr{F}_\rho(U)$. Observe that to show a hypothetical $\mathscr{F}_\rho(U) \subseteq \mathscr{L}_\rho(U)$ is by the *dense replacement principle* the same as showing

$$\lim_{\substack{H \to 0 \\ H \in \mathcal{A}^\times}} \left\|H^{-1} \cdot r(H)\right\|_M = 0,$$

where we are given that $\forall\, \varepsilon > 0\ \exists\, \delta = \delta(\varepsilon) > 0\ \forall\, \|H\|_\mathcal{A} < \varepsilon\colon\ \|r(H)\|_M \le \delta \|H\|_\mathcal{A}$. However, a *naive* estimation only gives

$$\left\|H^{-1} \cdot r(H)\right\|_M \le \left\|H^{-1}\right\|_\mathcal{A} \|r(H)\|_M \le \left\|H^{-1}\right\|_\mathcal{A} \delta \|H\|_\mathcal{A} = \delta \left\|H^{-1}\right\|_\mathcal{A} \|H\|_\mathcal{A} \ge \delta \left\|H^{-1} H\right\| = \delta.$$

So, if $\mathscr{F}_\rho(U) \subseteq \mathscr{L}_\rho(U)$ is in fact true, then an additional, less trivial argument would be needed. $\square$

But one can do only so much without higher regularity and a more precise form of the remainder term $r(H) := r_{Z_0}(H) := f(Z_0 + H) - f(Z_0) - \rho(H)f'(Z_0)$.

Lemma 1.1.12 (Integral MVT)**:** Let $U \subseteq \mathcal{A}$ be open, $Z_0 \in U$, and $f \in \mathscr{F}_\rho(U) \cap \mathscr{C}^1(U, M)$. If $[Z_0, Z_0 + H] \subseteq U$, then:

$$f(Z_0 + H) - f(Z_0) = \rho(H) \int_0^1 f'(Z_0 + tH)\mathrm{d}t \overset{\text{def}}{=} H \cdot \int_0^1 f'(Z_0 + tH)\mathrm{d}t. \tag{1.1.11}$$

Consequently,

$$r(H) = \rho(H) \int_0^1 (f'(Z_0 + tH) - f'(Z_0))\mathrm{d}t \overset{\text{def}}{=} H \cdot \int_0^1 (f'(Z_0 + tH) - f'(Z_0))\mathrm{d}t. \tag{1.1.12}$$

Proof: Consider $g\colon [0, 1] \to M$, $g(t) := f(Z_0 + tH)$. Since $f \in \mathscr{F}_\rho(U) \cap \mathscr{C}^1(U, M)$, we have $g \in \mathscr{C}^1([0, 1], M)$ and $g'(t) = (Df)(Z_0 + tH)(H) = H \cdot f'(Z_0 + tH)$. Moreover,

$$g' \in \mathscr{C}([0, 1], \mathrm{Hom}_{\mathbb{K}}(\mathrm{End}_{\mathbb{K}}(M), M))$$

via the identification with $\Upsilon \circ g'$. The claim now follows from [Lan93, Thm. 4.2, p. 341]. $\square$

Corollary 1.1.13 (Infinitesimal Conservation of Ideals)**:** Let $U \subseteq \mathcal{A}$ be open, $Z_0 \in U$, and $f \in \mathscr{F}_\rho(U) \cap \mathscr{C}^1(U, M)$. If $H \in \mathfrak{a} \lhd \mathcal{A}$ such that $[Z_0, Z_0 + H] \subseteq U$, then $f(Z_0 + H) - f(Z_0) \in \rho(\mathfrak{a})M$. In particular, if $f \in \mathscr{F}_{\mathcal{A}}(U) \cap \mathscr{C}^1(U, \mathcal{A})$, we have more succinctly $\Delta Z \in \mathfrak{a} \Rightarrow \Delta f \in \mathfrak{a}$.

Proof: Immediate from Lemma 1.1.12. $\square$

Proposition 1.1.14: Let $U \subseteq \mathcal{A}$ be open. Then $\forall\, k \in \mathbb{N}$:

$$\mathscr{L}^k_\rho(U) \cap \mathscr{C}^k(U, M) = \mathscr{F}^{\mathrm{str},k}_\rho(U) \cap \mathscr{C}^k(U, M) = \mathscr{F}^k_\rho(U) \cap \mathscr{C}^k(U, M).$$

Proof: It is enough to show the case $k = 1$. To this end, by Proposition 1.1.11 it suffices to show that $\mathscr{F}_\rho(U) \cap \mathscr{C}^1(U, M) \subseteq \mathscr{L}_\rho(U) \cap \mathscr{C}^1(U, M)$. Thus let $f \in \mathscr{F}_\rho(U) \cap \mathscr{C}^1(U, M)$. Since $\mathcal{A}$ is in particular locally convex, we have by Lemma 1.1.12 that

$$\lim_{\substack{H \to 0 \\ H \in \mathcal{A}^\times_\rho}} \rho(H)^{-1}\big(f(Z_0 + H) - f(Z_0)\big) = \lim_{\substack{H \to 0 \\ H \in \mathcal{A}^\times_\rho}} \int_0^1 f'(Z_0 + tH)\mathrm{d}t = \int_0^1 \lim_{\substack{H \to 0 \\ H \in \mathcal{A}^\times_\rho}} f'(Z_0 + tH)\mathrm{d}t = f'(Z_0)$$

since $f'(Z_0 + tH) \xrightarrow{H \to 0} f'(Z_0)$ uniformly in t by continuity of f' and compactness of $[0, 1]$. It follows that $f \in \mathscr{L}_\rho(U)$, with the $\mathscr{F}_\rho$- and $\mathscr{L}_\rho$-derivatives agreeing with each other, in particular the $\mathscr{L}_\rho$-derivative also being continuous. $\square$

Remark: Observe that Lemma 1.1.12 allows to drop the faithfulness assumption on ρ due to the additional regularity. $\square$

Thus, for *continuous* ρ/φ-differentiability the distinction between the three regularity classes disappears in the finite-dimensional setting and we can use them interchangeably. We now fix some *working definitions* accordingly:

Definition 1.1.15: Let $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$, $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$ a representation, $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ a morphism, and $U \subseteq \mathcal{A}$ open.

(1) For $\mathbb{K} = \mathbb{R}$ and $k \in \mathbb{N} \cup \{\infty\}$ we put:

$$\mathscr{C}^k_\rho(U) := \mathscr{C}^k_{\mathcal{A}}(U, {}_{\mathcal{A}}M) := \mathscr{L}^k_\rho(U) \cap \mathscr{C}^k(U, M) = \mathscr{F}^{\mathrm{str},k}_\rho(U) \cap \mathscr{C}^k(U, M) = \mathscr{F}^k_\rho(U) \cap \mathscr{C}^k(U, M)$$
$$\mathscr{C}^k_\varphi(U) := \mathscr{C}^k_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathcal{B}) := \mathscr{L}^k_\varphi(U) \cap \mathscr{C}^k(U, \mathcal{B}) = \mathscr{F}^{\mathrm{str},k}_\varphi(U) \cap \mathscr{C}^k(U, \mathcal{B}) = \mathscr{F}^k_\varphi(U) \cap \mathscr{C}^k(U, \mathcal{B})$$
$$\mathscr{C}^k_{\mathcal{A}}(U) := \mathscr{C}^k_{\mathrm{id}_{\mathcal{A}}}(U) \overset{\text{def}}{=} \mathscr{L}^k_{\mathcal{A}}(U) \cap \mathscr{C}^k(U, \mathcal{A}) = \mathscr{F}^{\mathrm{str},k}_{\mathcal{A}}(U) \cap \mathscr{C}^k(U, \mathcal{A}) = \mathscr{F}^k_{\mathcal{A}}(U) \cap \mathscr{C}^k(U, \mathcal{A}).$$

If f is an element of some of these spaces, then f will be called *k-times* continuously $\rho/\varphi/\mathcal{A}$-*differentiable*, respectively.

(2) $\mathcal{A}$-Analyticity:

(i) Representation Setting: a function $f\colon U \to M$ is called *ρ-analytic* or *$\mathcal{A}$-analytic*[5] (on U) if every $Z_0 \in U$ admits a neighbourhood $\mathcal{N} \ni Z_0$, for which there exist $\{v_k\}_{k\in\mathbb{N}_0} \subset M$ such that $f|_{\mathcal{N}}$ is given by a formal power series

$$f|_{\mathcal{N}}(Z) = \sum_{k=0}^{\infty} (Z - Z_0)^k \cdot v_k \in {}_{\mathcal{A}}M[[Z - Z_0]] \tag{1.1.13}$$

that is point-wise convergent with respect to $\|\cdot\|_M$ for all $Z \in \mathcal{N}$. We will denote by ${}_{\mathcal{A}}M\{Z\}$ the subset of ${}_{\mathcal{A}}M[[Z]]$ of all power series that are point-wise convergent in some neighbourhood of $0 \in \mathcal{A}$

(ii) Morphism Setting: a function $f\colon U \to \mathcal{B}$ is called *φ-analytic* or *$\mathcal{A}$-analytic*[6] (on U) if every $Z_0 \in U$ admits a neighbourhood $\mathcal{N} \ni Z_0$, for which there exist $\{B_k\}_{k\in\mathbb{N}_0} \subset \mathcal{B}$ such that $f|_{\mathcal{N}}$ is given by a formal power series

$$f|_{\mathcal{N}}(Z) = \sum_{k=0}^{\infty} B_k \varphi(Z - Z_0)^k \tag{1.1.14}$$

that is point-wise convergent with respect to $\|\cdot\|_{\mathcal{B}}$ for all $Z \in \mathcal{N}$. We will denote by $\mathcal{B}_\varphi\{Z\} \coloneqq \mathcal{B}\{\varphi(Z)\}$ the subset of $\mathcal{B}_\varphi[[Z]] \coloneqq \mathcal{B}[[\varphi(Z)]]$ of all power series that are point-wise convergent in some neighbourhood of $0_{\mathcal{A}}$.

(iii) Spaces of $\mathcal{A}$-analytic functions:

$$\begin{aligned}
\mathscr{C}^\omega_\rho(U) &\coloneqq \mathscr{C}^\omega_{\mathcal{A}}(U, {}_{\mathcal{A}}M) \coloneqq \{f : U \to {}_{\mathcal{A}}M \mid f \text{ is } \rho\text{-analytic}\}\\
\mathscr{C}^\omega_\varphi(U) &\coloneqq \mathscr{C}^\omega_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathcal{B}) \overset{\text{def}}{=} \{f : U \to {}_{\mathcal{A}}\mathcal{B} \mid f \text{ is } \varphi\text{-analytic}\}\\
\mathscr{C}^\omega_{\mathcal{A}}(U) &\coloneqq \mathscr{C}^\omega_{\mathrm{id}_{\mathcal{A}}}(U)
\end{aligned}$$

(3) In the special case $\mathbb{K} = \mathbb{C}$ we also put

$$\begin{aligned}
\mathcal{O}_\rho(U) &\coloneqq \mathcal{O}_{\mathcal{A}}(U, {}_{\mathcal{A}}M) \coloneqq \mathscr{L}_\rho(U) = \mathscr{F}^{\text{str}}_\rho(U) = \mathscr{F}_\rho(U)\\
\mathcal{O}_\varphi(U) &\coloneqq \mathcal{O}_\varphi(U, {}_{\mathcal{A}}M) \coloneqq \mathscr{L}_\varphi(U) = \mathscr{F}^{\text{str}}_\varphi(U) = \mathscr{F}_\varphi(U)\\
\mathcal{O}_{\mathcal{A}}(U) &\coloneqq \mathcal{O}_{\mathrm{id}_{\mathcal{A}}}(U) \overset{\text{def}}{=} \mathscr{L}_{\mathcal{A}}(U) = \mathscr{F}^{\text{str}}_{\mathcal{A}}(U) = \mathscr{F}_{\mathcal{A}}(U).
\end{aligned}$$

If f is an element of these spaces, then f will be called *$\rho/\varphi/\mathcal{A}$-holomorphic*, respectively.

Examples:

(1) If $\sigma \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{C})$ is the complex conjugation, then $\mathcal{O}_\sigma$ is precisely the sheaf of anti-holomorphic functions on $\mathbb{C}$.

(2) If $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{C}}$, then $\mathcal{O}_{\iota_{\mathcal{A}}}$ is the sheaf of $\mathbb{C}$-holomorphic functions with values in the (finite-dimensional) commutative Banach algebra $\mathcal{A}$.

(3) If $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C})$, then $\mathcal{O}_{\chi_{\mathcal{A}}}$ is the sheaf of all $\mathcal{A}$-holomorphic functions with values in $\mathbb{C}$ that are translation-invariant under $\mathfrak{m}$, i.e. *trivial lifts* of $\mathbb{C}$-holomorphic functions to $\mathcal{A}$.

(4) Given $\mathbb{C}^\vee \overset{\text{def}}{=} \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{R})$ (spanned by $\Re$ and $\Im$), give a nice interpretation of $\mathcal{O}_{\mathbb{C}}(U, \mathbb{C}^\vee)$:-) $\square$

Proposition 1.1.16: Let $U \subseteq \mathcal{A}$ be open and $n \in \mathbb{N} \cup \{\infty, \omega\}$.

(1) We have:

$$\mathscr{F}^n_{\mathcal{A}}(U, {}_{\mathcal{A}}M) = \mathscr{F}_{\mathcal{A}}(U, {}_{\mathcal{A}}M) \cap \mathscr{F}^n_{\mathbb{R}}(U, M). \tag{1.1.15}$$

Hence in particular:

$$\mathscr{C}^n_{\mathcal{A}}(U, {}_{\mathcal{A}}M) = \mathscr{F}_{\mathcal{A}}(U, {}_{\mathcal{A}}M) \cap \mathscr{C}^n(U, M). \tag{1.1.16}$$

[5] when ${}_{\mathcal{A}}M$ is specified instead of ρ;

[6] when $\mathcal{B} \in \mathsf{FdCAlg}(\mathcal{A})$ is specified instead of φ;

(2) Taylor Expansion: let $n < \infty$.

(i) Peano Remainder: if $f \in \mathscr{F}^n_\rho(U)$, then $\forall\, [Z, Z+H] \subseteq U$:

$$\begin{aligned} f(Z+H) &= \sum_{k=0}^{n} H^k \cdot \frac{f^{(k)}(Z)}{k!} + r_n(H), \\ r_n(H) &\coloneqq r_n(Z;H) = o(\|\rho(H)^n\|_{\mathrm{End}}) \text{ as } H \to 0 \end{aligned} \tag{1.1.17}$$

(ii) Integral Remainder: if $f \in \mathscr{C}^n_\rho(U)$, then $\forall\, [Z, Z+H] \subseteq U$:

$$\begin{aligned} f(Z+H) &= \sum_{k=0}^{n-1} H^k \cdot \frac{f^{(k)}(Z)}{k!} + R_n(H) = \sum_{k=0}^{n} H^k \cdot \frac{f^{(k)}(Z)}{k!} + r_n(H), \\ R_n(H) &\coloneqq R_n(Z;H) = \frac{H^n}{(n-1)!} \cdot \int_0^1 f^{(n)}(Z+tH)(1-t)^{n-1}\mathrm{d}t, \\ r_n(H) &\overset{\text{def}}{=} r_n(Z;H) = R_n(Z;H) - H^n \cdot \frac{f^{(n)}(Z)}{n!}. \end{aligned} \tag{1.1.18}$$

(3) If $\mathbb{K} = \mathbb{C}$, then $\mathcal{O}_\rho(U) = \mathscr{C}^\omega_\rho(U)$.

Proof: To (1): Since $\mathscr{F}_{\mathcal{A}}(U, {}_{\mathcal{A}}M) \subseteq \mathscr{F}_{\mathbb{R}}(U, M)$, we have $\mathscr{F}^1_{\mathcal{A}}(U, {}_{\mathcal{A}}M) \overset{\text{def}}{=} \mathscr{F}_{\mathcal{A}}(U, {}_{\mathcal{A}}M) = \mathscr{F}_{\mathcal{A}}(U, {}_{\mathcal{A}}M) \cap \mathscr{F}^1_{\mathbb{R}}(U, M)$, so we can assume without loss of generality that $n \geq 2$. Now let $f \in \mathscr{F}_{\mathcal{A}}(U, {}_{\mathcal{A}}M) \cap \mathscr{F}^n_{\mathbb{R}}(U, M)$. Writing $D_Z \coloneqq D$ for the total derivative in $Z \in U$ and $Z_1, \ldots, Z_k$ for *vector $\mathcal{A}$-variables*, we have by total $\mathbb{R}$-differentiability that

$$\forall\, 1 \leq k \leq n\ \forall\, Z \in U\colon \big((D^k_Z f)(Z)\big)(Z_1, \ldots, Z_k) \in \mathcal{L}^k_{\mathbb{R}}(\mathcal{A}, M),$$

(as a function in $Z_1, \ldots, Z_k$) and moreover

$$\big((D^{k+1}_Z f)(Z)\big)(Z_1, \ldots, Z_k, Z_{k+1}) = D_Z\big((D^k_Z f)(Z)(Z_1, \ldots, Z_k)\big)(Z_{k+1}).$$

We need to show that the condition $(Df)(Z) \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, {}_{\mathcal{A}}M)$, $Z \in U$, implies that in fact $\forall\, 1 \leq k \leq n\ \forall\, Z \in U\colon \big((D^k_Z f)(Z)\big)(Z_1, \ldots, Z_k) \in \mathcal{L}^k_{\mathcal{A}}(\mathcal{A}, M)$. For $1 \leq k \leq n-1$ it follows for example from Schwarz's theorem that $(D^k f)(Z) \in \mathrm{Sym}^k_{\mathbb{R}}(\mathcal{A}, M)$. For $k = n$ it follows from [HH15, Thm. 3.3.8, p. 732] that $\forall\, Z \in U\colon (D^n_Z f)(Z) \in \mathrm{Sym}^n_{\mathbb{R}}(\mathcal{A}, M)$: total differentiability of $(D^{n-1}_Z f)(Z)$ implies that the partial derivatives of $(D^{n-2}_Z f)(Z)$ are themselves totally differentiable, which suffices for the second partial derivatives of $(D^{n-2}_Z f)(Z)$ to commute and replaces the usual requirement of Schwarz's Theorem for the second partial derivatives to be continuous. The claim now follows by induction on k and the *symmetric* multilinearity. Indeed, for $k = 2$ we get

$$\begin{aligned} \forall\, A_1, A_2 \in \mathcal{A}\colon (D^2_Z f)(Z)(A_1 Z_1, A_2 Z_2) &= D_Z\big((D_Z f)(Z)(A_1 Z_1)\big)(A_2 Z_2) = \\ &= D_Z\big(A_1 (D_Z f)(Z)(Z_1)\big)(A_2 Z_2) = A_1 D_Z\big((D_Z f)(Z)(Z_1)\big)(A_2 Z_2) = \\ &= A_1 (D^2_Z f)(Z)(Z_1, A_2 Z_2) = A_1 (D^2_Z f)(Z)(A_2 Z_2, Z_1) = \\ &= A_1 D_Z\big((D_Z f)(Z)(A_2 Z_2)\big)(Z_1) = A_1 A_2 D_Z\big((D_Z f)(Z)(Z_2)\big)(Z_1) = \\ &= A_1 A_2 (D^2_Z f)(Z)(Z_2, Z_1) = A_1 A_2 (D^2_Z f)(Z)(Z_1, Z_2) \end{aligned}$$

since $(D_Z f)(Z)(Z_1)$ is $\mathcal{A}$-linear and $(D^2_Z f)(Z)(Z_1, Z_2)$ is symmetric in Z_1 and Z_2. The induction step follows completely analogously by swapping the Z_k and Z_{k+1} vectors.

To (2): If $\Phi_k \in \mathcal{L}^k_{\mathcal{A}}(\mathcal{A}, {}_{\mathcal{A}}M)$, then by definition $\forall\, Z \in \mathcal{A}\colon \Phi_k(Z, \ldots, Z) = Z^k \cdot \Phi(1_{\mathcal{A}}, \ldots, 1_{\mathcal{A}}) \eqqcolon Z^k \cdot v_k$, which proves the special shape of the Taylor expansion. The characterizations of the remainder terms R_n and r_n follow immediately from [Lan93, Ch. 13, §6] and the identity

$$\big((D^n f)(Z_0 + tH)\big)(\underbrace{H, \ldots, H}_{n \text{ times}}) = H^n \cdot f^{(n)}(Z_0 + tH)$$

by Eq. (1.1.8). (For $\mathscr{F}^n_\rho$ the symmetry of $D^n f$ follows as in (1).)

To (3): It suffices to prove that f is ρ-analytic at 0. If $f \in \mathscr{F}_\rho(U)$, then f is by definition totally differentiable with Df being $\mathcal{A}$-linear, hence in particular $\mathbb{C}$-linear, everywhere on U. It is standard that this already implies that f is analytic in several complex variables, hence in particular real-analytic. Thus we can write

$$f(Z) = f(0) + \sum_{k=1}^{\infty} \frac{1}{k!}(D^k f)(0)(\underbrace{Z, \ldots, Z}_{k\text{-times}}) =: f(0) + \sum_{k=1}^{\infty} \Phi_k(\underbrace{Z, \ldots, Z}_{k\text{-times}})$$

(convergent in some neighbourhood of 0), where $\Phi_k \in \mathrm{Sym}^k_{\mathcal{A}}(\mathcal{A}, {}_{\mathcal{A}}M)$ by (1). That the series has the shape as in Eq. (1.1.13) is immediate as in (2). □

Remarks:

(1) For $\mathbb{K} = \mathbb{C}$ we will give in Proposition 2.3.17 an *intrinsic* proof that $\mathcal{O}_\rho = \mathscr{C}^\omega_\rho$, that is, without relying on $\mathbb{C}$-linearity of Df, Cauchy's Integral Formula in one complex variable, or Hartog's theorem that *separately* holomorphic functions are (totally) holomorphic.

(2) Even if $\mathcal{A}$ has no $\mathbb{C}$-structure, it makes sense to introduce the notation $\mathcal{O}_{\mathcal{A}} := \mathscr{C}^\omega_{\mathcal{A}}$ in that case as well.

(3) In Proposition 1.1.16 the morphism case is clearly subsumed by the representation case since the statements are entirely "additive".

(4) A similar result is shown in [GM96] for the case $\varphi = \mathrm{id}_{\mathcal{A}}$ and n $\mathcal{A}$-variables instead of a single $\mathcal{A}$-variable, i.e. the *domain* of definition is the *free* $\mathcal{A}$-module $\mathcal{A}^n$ rather than $\mathcal{A}$ itself. The authors start from $\mathscr{L}^k_{\mathcal{A}} \cap \mathscr{C}^k$-type[7] of regularity, where, like Waterhouse, in the case $n = 1$ they call $\mathcal{A}$-differentiable what we call here *classically* $\mathcal{A}$-differentiable and for the case $n > 1$ they call $\mathcal{A}$-differentiable what we would call here *partially (or separately) classically* $\mathcal{A}$-differentiable, that is, $\mathscr{L}_{\mathcal{A}}$-differentiability in each $\mathcal{A}$-variable separately. In our notation, their Theorem I states that, given $U \subseteq \mathcal{A}^n$ open, $\mathscr{C}^\omega_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathcal{A}) = \mathscr{L}^\infty_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathcal{A}) \cap \mathscr{C}^\omega(U, \mathcal{A})$ (with the understanding that $\mathscr{L}^\infty_{\mathcal{A}}$ means $\mathscr{L}^\infty_{\mathcal{A}}$-differentiability in each $\mathcal{A}$-variable separately). Restricting attention to the case $n = 1$, a combination of the following two facts
- $\mathscr{L}_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathcal{A}) \cap \mathscr{C}^1(U, \mathcal{A}) = \mathscr{F}_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathcal{A}) \cap \mathscr{C}^1(U, \mathcal{A})$ (Proposition 1.1.14) and
- $\forall\, k \in \mathbb{N} \cup \{\infty, \omega\}$: $\mathscr{C}^k_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathcal{A}) = \mathscr{F}_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathcal{A}) \cap \mathscr{C}^k(U, \mathcal{A})$ (Proposition 1.1.16 (1))

immediately yields $\forall\, k \in \mathbb{N} \cup \{\infty, \omega\}$:

$$\begin{aligned}\mathscr{L}_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathcal{A}) \cap \mathscr{C}^k(U, \mathcal{A}) &= \mathscr{L}_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathcal{A}) \cap \mathscr{C}^1(U, \mathcal{A}) \cap \mathscr{C}^k(U, \mathcal{A}) = \\ &= \mathscr{F}_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathcal{A}) \cap \mathscr{C}^1(U, \mathcal{A}) \cap \mathscr{C}^k(U, \mathcal{A}) = \mathscr{F}_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathcal{A}) \cap \mathscr{C}^k(U, \mathcal{A}) = \mathscr{C}^k_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathcal{A})\end{aligned}$$

which is stronger than their result in the single-$\mathcal{A}$-variable case since it drops the higher $\mathscr{L}_{\mathcal{A}}$-differentiability condition altogether. We also find our line of argument neater. □

Problem 6: Does partial $\mathcal{A}$-differentiability in several $\mathcal{A}$-variables imply total $\mathcal{A}$-differentiability (like for SCVs) when $\mathcal{A}$ does not possess a complex structure? This problem is best suited for the treatment of the more general setting, in which the domain of definition is given by an (open subset of a) (not necessarily) free finite-dimensional $\mathcal{A}$-module. □

Corollary 1.1.17: Let $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$, $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$ a representation, and $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ a morphism.

(1) $\mathscr{L}^\infty_\rho = \mathscr{F}^{\mathrm{str},\infty}_\rho = \mathscr{F}^\infty_\rho = \mathscr{C}^\infty_\rho$.

(2) $\mathscr{L}^\infty_\varphi = \mathscr{F}^{\mathrm{str},\infty}_\varphi = \mathscr{F}^\infty_\varphi = \mathscr{C}^\infty_\varphi$.

Proof: We only prove (1) since (2) follows from it by taking $\rho := \rho_\varphi$. Let $U \subseteq \mathcal{A}$ be open. By

[7] strictly speaking, they start from "$\mathscr{L}^k_{\mathcal{A}}$ + *continuous* derivatives" type of regularity without actually showing that a $\mathscr{L}_{\mathcal{A}}$-differentiable functions are continuous, but since they cite [Wat92a], who, as already discussed, had shown that implication, it is safe to assume that they were aware of that and that their condition is just meant to say *continuity of the k-th $\mathscr{L}_{\mathcal{A}}$-derivative*;

Proposition 1.1.11 we have $\mathscr{L}_\rho(U) \subseteq \mathscr{F}_\rho(U) \subseteq \mathscr{F}_\mathbb{R}(U,M) \subseteq \mathscr{C}^0(U,M)$, hence $\mathscr{L}^2_\rho(U) \subseteq \mathscr{F}^2_\rho(U) \subseteq \mathscr{C}^1(U,M)$. Thus

$$\mathscr{L}^\infty_\rho(U) \overset{\text{def}}{=} \bigcap_{k=1}^\infty \mathscr{L}^k_\rho(U) = \mathscr{L}_\rho(U) \bigcap_{k=1}^\infty \mathscr{L}^{k+1}_\rho(U) = \mathscr{L}_\rho(U) \bigcap_{k=1}^\infty (\mathscr{L}^{k+1}_\rho(U) \cap \mathscr{C}^k(U,M)) =$$
$$= (\mathscr{L}_\rho(U) \cap \mathscr{C}^1(U,M)) \bigcap_{k=2}^\infty (\mathscr{L}^k_\rho(U) \cap \mathscr{C}^k(U,M)) = \mathscr{C}^1_\rho(U) \bigcap_{k=2}^\infty \mathscr{C}^k_\rho(U) \overset{\text{def}}{=} \mathscr{C}^\infty_\rho(U)$$

by Proposition 1.1.14. It follows analogously that $\mathscr{F}^\infty_\rho(U) = \mathscr{C}^\infty_\rho(U)$, which completes the proof. $\square$

Lemma 1.1.18 (Compatible Linear Maps)**:** Let $(V, \|\cdot\|_V), (W_1, \|\cdot\|_{W_1}), (W_2, \|\cdot\|_{W_2}) \in \mathsf{NrmFdVec}_\mathbb{K}$ and let $\Phi_i \in \mathrm{Hom}_\mathbb{K}(V, W_i)$, $i = 1, 2$. The following are equivalent:
  (i) $\exists\, C > 0\ \forall\, v \in V\colon\ \|\Phi_1(v)\|_{W_1} \le C\, \|\Phi_2(v)\|_{W_2}$
  (ii) $\mathrm{Ker}(\Phi_2) \subseteq \mathrm{Ker}(\Phi_1)$.

Proof: "(i) $\Rightarrow$ (ii)": Without loss of generality suppose that $\mathrm{Ker}(\Phi_2) \neq 0$. Let $v \in \mathrm{Ker}(\Phi_2)$, $v \neq 0$, and let $\varepsilon > 0$ be arbitrary. Then for any $t \in \mathbb{K}$ with $|t| < \delta(\varepsilon)/\, \|v\|_V > 0$ we have $tv \in \mathrm{Ker}(\Phi_2)$ and $\|tv\|_V < \delta(\varepsilon)$, hence $\|\Phi_1(tv)\|_{W_1} \le \varepsilon\, \|\Phi_2(tv)\|_{W_2} = 0 \Rightarrow tv \in \mathrm{Ker}(\Phi_1) \Rightarrow v \in \mathrm{Ker}(\Phi_1)$.

"(ii) $\Rightarrow$ (i)": Since $\mathrm{Ker}(\Phi_2) \subseteq \mathrm{Ker}(\Phi_1)$, we get a commutative diagram

$$\begin{tikzcd} V \arrow[r, "\Phi_1"] \arrow[d, two heads, "\Phi_2"'] & W_1 \\ \mathrm{Im}(\Phi_2) \arrow[ur, dashed, "\Psi"'] & \end{tikzcd}$$

where $\Psi(\Phi_2(v)) \coloneqq \Phi_1(v)$ is well-defined. Therefore, taking $C \coloneqq \|\Psi\|_{\mathrm{op}}$ (automatically bounded), we have $\forall\, v \in V\colon\ \|\Phi_1(v)\|_{W_1} = \|\Psi(\Phi_2(v))\|_{W_1} \le \|\Psi\|_{\mathrm{op}}\, \|\Phi_2(v)\|_{W_2} = C\, \|\Phi_2(v)\|_{W_2}$ as desired. $\square$

Consider again a representation $\rho\colon \mathcal{A} \to \mathrm{End}_\mathbb{K}(M)$ and a function $f\colon U \to M$. Then in particular tautologically also $M \in \mathsf{NrmLMod}(\mathrm{End}_\mathcal{A}(M))$. This clearly induces an action on functions given by $\forall\, \Phi \in \mathrm{End}_\mathcal{A}(M)\ \forall\, Z \in U\colon (\Phi f)(Z) \coloneqq \Phi(f(Z))$ and commuting with the $\mathcal{A}$-action. Furthermore, recall that $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ induces a restriction-of-scalars functor $\varphi^*\colon \mathsf{FdMod}(\mathcal{B}) \to \mathsf{FdMod}(\mathcal{A})$, ${}_\mathcal{B}M \mapsto {}_\mathcal{A}M$, via a pull-back of the representation λ of $\mathcal{B}$ to a representation of $\mathcal{A}$, that is, $\varphi^*\lambda \overset{\text{def}}{=} \lambda \circ \varphi\colon \mathcal{A} \to \mathcal{B} \to \mathrm{End}_\mathbb{K}(M)$.

Proposition 1.1.19: Let $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ be a morphism in $\mathsf{FdCAlg}_\mathbb{K}$ and $\rho\colon \mathcal{A} \to \mathrm{End}_\mathbb{K}(M)$ a finite-dimensional representation.
  (1) Constants: tautologically, $\varphi \in \mathscr{L}_\varphi(\mathcal{A}) \subseteq \mathscr{F}^{\mathrm{str}}_\varphi(\mathcal{A}) \subseteq \mathscr{F}_\varphi(\mathcal{A})$ with $\varphi'\colon \mathcal{A} \to \mathcal{B}$, $Z \mapsto 1_\mathcal{B}$.
  (2) $\mathscr{L}_\rho, \mathscr{F}^{\mathrm{str}}_\rho, \mathscr{F}_\rho$ are sheaves of left $\mathrm{End}_\mathcal{A}(M)$-modules on $|\mathcal{A}|$. More precisely:
    (i) Left $\mathrm{End}_\mathcal{A}(M)$-Action: if $\Phi \in \mathrm{End}_\mathcal{A}(M)$, then
      - $\forall\, f \in \mathscr{L}_\mathcal{A}(U, {}_\mathcal{A}M)\colon \Phi f \in \mathscr{L}_\mathcal{A}(U, {}_\mathcal{A}M)$
      - $\forall\, f \in \mathscr{F}^{\mathrm{str}}_\mathcal{A}(U, {}_\mathcal{A}M)\colon \Phi f \in \mathscr{F}^{\mathrm{str}}_\mathcal{A}(U, {}_\mathcal{A}M)$
      - $\forall\, f \in \mathscr{F}_\mathcal{A}(U, {}_\mathcal{A}M)\colon \Phi f \in \mathscr{F}_\mathcal{A}(U, {}_\mathcal{A}M)$

      with $(\Phi f)' = \Phi f'$ in all three cases.
    (ii) Additivity:
      - $\forall\, f_1, f_2 \in \mathscr{L}_\rho(U)\colon f_1 + f_2 \in \mathscr{L}_\rho(U)$
      - $\forall\, f_1, f_2 \in \mathscr{F}^{\mathrm{str}}_\rho(U)\colon f_1 + f_2 \in \mathscr{F}^{\mathrm{str}}_\rho(U)$
      - $\forall\, f_1, f_2 \in \mathscr{F}_\rho(U)\colon f_1 + f_2 \in \mathscr{F}_\rho(U)$

      with $(f_1 + f_2)' = f_1' + f_2'$ in all three cases.

(3) Mixed Leibniz Rule: if $\lambda\colon \mathcal{B} \to \mathrm{End}_{\mathbb{K}}(M)$ such that $\rho = \varphi^*\lambda \overset{\text{def}}{=} \lambda \circ \varphi$, that is

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\rho"] \arrow[d, "\varphi"'] & \mathrm{End}_{\mathbb{K}}(M) \\ \mathcal{B} \arrow[ur, dashed, "\lambda"'] & \end{tikzcd}$$

then

- $\mathscr{L}_\rho \in \mathsf{Mod}(\mathscr{L}_\varphi)$, i.e. $\forall\, f \in \mathscr{L}_\varphi(U)\ \forall\, g \in \mathscr{L}_\rho(U)\colon f \cdot g \in \mathscr{L}_\rho(U)$;
- $\mathscr{F}_\rho^{\mathrm{str}} \in \mathsf{Mod}(\mathscr{F}_\varphi^{\mathrm{str}})$, i.e. $\forall\, f \in \mathscr{F}_\varphi^{\mathrm{str}}(U)\ \forall\, g \in \mathscr{F}_\rho^{\mathrm{str}}(U)\colon f \cdot g \in \mathscr{F}_\rho^{\mathrm{str}}(U)$, **provided** $\mathrm{Ker}(\rho) = \mathrm{Ker}(\varphi)$;
- $\mathscr{F}_\rho \in \mathsf{Mod}(\mathscr{F}_\varphi)$, i.e. $\forall\, f \in \mathscr{F}_\varphi(U)\ \forall\, g \in \mathscr{F}_\rho(U)\colon f \cdot g \in \mathscr{F}_\rho(U)$;

with $(fg)' = f'g + fg'$ in all three cases.

(4) $\mathscr{L}_\varphi, \mathscr{F}_\varphi^{\mathrm{str}}, \mathscr{F}_\varphi$ are sheaves of $\mathcal{B}$-algebras on $|\mathcal{A}|$. More precisely:
  (i) $\mathcal{B}$-Action: if $b \in \mathcal{B}$, then
   - $\forall\, f \in \mathscr{L}_\varphi(U)\colon bf \in \mathscr{L}_\varphi(U)$
   - $\forall\, f \in \mathscr{F}_\varphi^{\mathrm{str}}(U)\colon bf \in \mathscr{F}_\varphi^{\mathrm{str}}(U)$
   - $\forall\, f \in \mathscr{F}_\varphi(U)\colon bf \in \mathscr{F}_\varphi(U)$

   with $(bf)' = bf'$ in all three cases.
  (ii) Leibniz Rule:
   - $\forall\, f_1, f_2 \in \mathscr{L}_\varphi(U)\colon f_1 f_2 \in \mathscr{L}_\varphi(U)$
   - $\forall\, f_1, f_2 \in \mathscr{F}_\varphi^{\mathrm{str}}(U)\colon f_1 f_2 \in \mathscr{F}_\varphi^{\mathrm{str}}(U)$
   - $\forall\, f_1, f_2 \in \mathscr{F}_\varphi(U)\colon f_1 f_2 \in \mathscr{F}_\varphi(U)$

   with $(f_1 f_2)' = f_1' f_2 + f_1 f_2'$ in all three cases.

(5) If $k \in \mathbb{N}$, then:
  (i) $\mathscr{L}_\rho^k$, $\mathscr{F}_\rho^{\mathrm{str},k}$, $\mathscr{F}_\rho^k$, and $\mathscr{C}_\rho^k$ are a sheaves of left $\mathrm{End}_{\mathcal{A}}(M)$-modules on $|\mathcal{A}|$;
  (ii) $\mathscr{L}_\varphi^k$, $\mathscr{F}_\varphi^{\mathrm{str},k}$, $\mathscr{F}_\varphi^k$, and $\mathscr{C}_\varphi^k$ are a sheaves of $\mathcal{B}$-algebras on $|\mathcal{A}|$.

(6) $\mathscr{C}_\rho^\infty$ and $\mathscr{C}_\rho^\omega$ are sheaves of left $\mathrm{End}_{\mathcal{A}}(M)$-modules on $|\mathcal{A}|$ with a distinguished "derivation" (here only a distinguished $\mathrm{End}_{\mathcal{A}}(M)$-linear map) $\partial\colon f \mapsto f'$.

(7) $\mathscr{C}_\varphi^\infty$ and $\mathscr{C}_\varphi^\omega$ are sheaves of differential $\mathcal{B}$-algebras on $|\mathcal{A}|$ with a distinguished derivation given by $\partial\colon f \mapsto f'$.

(8) If $\rho = \varphi^*\lambda$ as in (3), then $\forall\, k \in \mathbb{N} \cup \{\infty, \omega\}\colon \mathscr{C}_\rho^k \in \mathsf{Mod}(\mathscr{C}_\varphi^k)$. For $k \in \{\infty, \omega\}$ it also has a distinguished derivation $\partial\colon f \mapsto f'$ in the sense of (3). In other words, $(\mathscr{C}_\rho^\infty, \partial)$ and $(\mathscr{C}_\rho^\omega, \partial)$ are sheaves of differential modules over the sheaves of differential $\mathcal{B}$-algebras $(\mathscr{C}_\varphi^\infty, \partial)$ and $(\mathscr{C}_\varphi^\omega, \partial)$, respectively, or just differential $(\mathscr{C}_\varphi^\infty, \partial)$-, respectively $(\mathscr{C}_\varphi^\omega, \partial)$-modules for short on $|\mathcal{A}|$.

Proof: To (1): immediate from Definition 1.1.1 and Proposition 1.1.11.

To (2): To (i): These are special cases of Proposition 1.1.23 (2), so we reserve the proofs for there. The reader is free to verify that no circular reasoning arises that way.

To (ii): To ($\mathscr{L}_\rho$): this is immediate from Definition 1.1.1.

To ($\mathscr{F}_\rho^{\mathrm{str}}$): This is routine. Indeed write $f_i(Z_0 + H) = f_i(Z_0) + \rho(H) f_i'(Z_0) + r_i(H)$, where $\forall\, \varepsilon > 0\ \exists\, \delta_i = \delta_i(\varepsilon) > 0\ \forall\, \|H\|_{\mathcal{A}} < \delta_i(\varepsilon)\colon \|r_i(H)\|_M \le \varepsilon \|\rho(H)\|_{\mathrm{End}}$, $i = 1, 2$. It follows that $f_1(Z_0 + H) + f_2(Z_0 + H) = f_1(Z_0) + f_2(Z_0) + \rho(H)(f_1'(Z_0) + f_2'(Z_0)) + R(H)$, where $R(H) := r_1(H) + r_2(H)$. Let $\varepsilon > 0$ and define $\delta(\varepsilon) := \min\{\delta_1(\varepsilon/2), \delta_2(\varepsilon/2)\}$. Then $\forall\, \|H\|_{\mathcal{A}} < \delta(\varepsilon)\colon \|R(H)\|_M \le \|r_1(H)\|_M + \|r_2(H)\|_M \le \frac{\varepsilon}{2} \|\rho(H)\|_{\mathrm{End}} + \frac{\varepsilon}{2} \|\rho(H)\|_{\mathrm{End}} = \varepsilon \|\rho(H)\|_{\mathrm{End}}$ as desired.

To ($\mathscr{F}_\rho$): Completely analogous to the ($\mathscr{F}_\rho^{\mathrm{str}}$)-case by replacing the "$\|r_i(H)\|_M \le \varepsilon \|\rho(H)\|_{\mathrm{End}}$" condition by "$\|r_i(H)\|_M \le \varepsilon \|H\|_{\mathcal{A}}$".

To (3): To ($\mathscr{L}$): Put $D_\varphi := \mathcal{A}_\varphi^\times \cap (U - Z_0)$ and $D_\rho := \mathcal{A}_\rho^\times \cap (U - Z_0)$ for short and notice that $D_\rho \supseteq D_\varphi$. We have

$$\begin{aligned}
&\lim_{\substack{H\to 0\\ H\in D_\rho}} \rho(H)^{-1}(f(Z_0+H)\cdot g(Z_0+H) - f(Z_0)\cdot g(Z_0)) \overset{\text{def}}{=} \\
&\overset{\text{def}}{=} \lim_{\substack{H\to 0\\ H\in D_\rho}} \lambda(\varphi(H))^{-1}\big(\lambda(f(Z_0+H))g(Z_0+H) - \lambda(f(Z_0))g(Z_0)\big) = \\
&= \lim_{\substack{H\to 0\\ H\in D_\rho}} \lambda(\varphi(H))^{-1}\big(\lambda(f(Z_0+H)) - \lambda(f(Z_0))\big)g(Z_0+H) + \\
&+ \lim_{\substack{H\to 0\\ H\in D_\rho}} \lambda(\varphi(H))^{-1}\lambda(f(Z_0))\big(g(Z_0+H) - g(Z_0)\big) = \\
&= \lim_{\substack{H\to 0\\ H\in D_\varphi}} \lambda(\varphi(H)^{-1})\big(\lambda(f(Z_0+H)) - \lambda(f(Z_0))\big) \lim_{\substack{H\to 0\\ H\in D_\rho}} g(Z_0+H) + \\
&+ \lambda(f(Z_0)) \lim_{\substack{H\to 0\\ H\in D_\rho}} \rho(H)^{-1}\big(g(Z_0+H) - g(Z_0)\big) = \\
&= \lim_{\substack{H\to 0\\ H\in D_\varphi}} \lambda\Big(\varphi(H)^{-1}\big(f(Z_0+H) - f(Z_0)\big)\Big)g(Z_0) + \lambda(f(Z_0))g'(Z_0) = \\
&= \lambda(f'(Z_0))g(Z_0) + \lambda(f(Z_0))g'(Z_0) \overset{\text{def}}{=} f'(Z_0)\cdot g(Z_0) + f(Z_0)\cdot g'(Z_0),
\end{aligned}$$

where we have tacitly used Lemma 1.1.10 for replacing D_ρ by D_φ (for proving the existence of the limit rather than just calculating it) and the (automatic) continuity of λ.

To ($\mathscr{F}^{\text{str}}$): Write

$$\begin{aligned}
f(Z_0+H) &= f(Z_0) + \varphi(H)f'(Z_0) + r_f(H)\\
g(Z_0+H) &= g(Z_0) + \rho(H)g'(Z_0) + r_g(H),
\end{aligned}$$

where:

- $\forall\, \varepsilon > 0\ \exists\, \delta_f = \delta_f(\varepsilon) > 0\ \forall\, H \in \mathbb{B}_{\mathcal{A}}(0, \delta_f(\varepsilon))\colon\ \|r_f(H)\|_{\mathcal{B}} \le \varepsilon\, \|\varphi(H)\|_{\mathcal{B}}$;
- $\forall\, \varepsilon > 0\ \exists\, \delta_g = \delta_g(\varepsilon) > 0\ \forall\, H \in \mathbb{B}_{\mathcal{A}}(0, \delta_g(\varepsilon))\colon\ \|r_g(H)\|_{M} \le \varepsilon\, \|\rho(H)\|_{\text{End}}$.

Furthermore,

$$\begin{aligned}
&f(Z_0+H)\cdot g(Z_0+H) \overset{\text{def}}{=} \lambda(f(Z_0+H))g(Z_0+H) = \\
&= \big(\lambda(f(Z_0)) + \lambda(\varphi(H))\lambda(f'(Z_0)) + \lambda(r_f(H))\big)\big(g(Z_0) + \rho(H)g'(Z_0) + r_g(H)\big) = \\
&= \big(\lambda(f(Z_0)) + \rho(H)\lambda(f'(Z_0)) + \lambda(r_f(H))\big)\big(g(Z_0) + \rho(H)g'(Z_0) + r_g(H)\big) = \\
&= f(Z_0)g(Z_0) + (f(Z_0)g'(Z_0) + f'(Z_0)g(Z_0))\varphi(H) + R(H),
\end{aligned}$$

where $R(H) := \sum_{j=1}^{6} R_j(H)$ with $R_j(H)$, $1 \le j \le 6$, *listed below.* Now let $\varepsilon > 0$ be arbitrary. For each $R_j(H)$ we will construct a neighbourhood $\mathcal{N}_{j,\varepsilon} \ni 0$ such that $\forall\, H \in \mathcal{N}_{j,\varepsilon}\colon\ \|R_j(H)\|_M \le \frac{\varepsilon}{6}\, \|\rho(H)\|_{\text{End}}$, $1 \le j \le 6$. Namely:

(I) $R_1(H) := \rho(H)^2\lambda(f'(Z_0))g'(Z_0)$:

$$\|R_1(H)\|_{\mathcal{B}} \le \|\rho(H)\|_{\text{End}}^2 \left\|f'(Z_0)\right\|_{\mathcal{B}} \left\|g'(Z_0)\right\|_M.$$

If $f'(Z_0) = 0$ or $g'(Z_0) = 0$, then the desired inequality is trivially satisfied. Thus we can further assume that $f'(Z_0) \ne 0$ and $g'(Z_0) \ne 0$. Since ρ is (automatically) bounded, we can take $\mathcal{N}_{1,\varepsilon} := \rho^{-1}(\mathbb{B}_{\text{End}_{\mathbb{K}}(M)}(0, \varepsilon_1))$, where $\varepsilon_1 := \varepsilon/(6\,\|f'(Z_0)\|_{\mathcal{B}}\, \|g'(Z_0)\|_M)$. Indeed one verifies:

$$\begin{aligned}
\forall\, H \in \mathcal{N}_{1,\varepsilon}\colon\ \|R_1(H)\|_{\mathcal{B}} &\le \|\rho(H)\|_{\text{End}} \left\|f'(Z_0)\right\|_{\mathcal{B}} \left\|g'(Z_0)\right\|_M \|\rho(H)\|_{\text{End}} \le \\
&\le \varepsilon_1 \left\|f'(Z_0)\right\|_{\mathcal{B}} \left\|g'(Z_0)\right\|_M \|\rho(H)\|_{\text{End}} = \frac{\varepsilon}{6}\, \|\rho(H)\|_{\text{End}}.
\end{aligned}$$

(II) $R_2(H) \coloneqq \lambda(f(Z_0))r_g(H)$:

$$\|R_2(H)\|_M \leq \|f(Z_0)\|_{\mathcal{B}} \|r_g(H)\|_M .$$

If $f(Z_0) = 0$, then the desired inequality is trivially satisfied, so we can further assume that $f(Z_0) \neq 0$. Put $\varepsilon_2 \coloneqq \varepsilon/(6\|f(Z_0)\|_{\mathcal{B}})$ and take $\mathcal{N}_{2,\varepsilon} \coloneqq \mathbb{B}_{\mathcal{A}}(0, \delta_g(\varepsilon_2))$. Indeed one verifies:

$$\forall\, H \in \mathcal{N}_{2,\varepsilon}\colon\ \|R_2(H)\|_M \leq \|f(Z_0)\|_{\mathcal{B}} \|r_g(H)\|_M \leq \|f(Z_0)\|_{\mathcal{B}}\, \varepsilon_2 \|\rho(H)\|_{\mathrm{End}} = \frac{\varepsilon}{6} \|\rho(H)\|_{\mathrm{End}} .$$

(III) $R_3(H) \coloneqq \lambda(r_f(H))g(Z_0)$:

$$\|R_3(H)\|_M \leq \|r_f(H)\|_{\mathcal{B}} \|g(Z_0)\|_M$$

If $g(Z_0) = 0$, the desired inequality is trivially satisfied, so without loss of generality assume $g(Z_0) \neq 0$. Since $\mathrm{Ker}(\rho) \subseteq \mathrm{Ker}(\varphi)$, we have from Lemma 1.1.18 that $\exists\, C > 0\ \forall\, H \in \mathcal{A}$: $\|\varphi(H)\|_{\mathcal{B}} \leq C \|\rho(H)\|_{\mathrm{End}}$. Now put $\varepsilon_3 \coloneqq \varepsilon/(6C\|g(Z_0)\|_M)$ and take $\mathcal{N}_{3,\varepsilon} \coloneqq \mathbb{B}_{\mathcal{A}}(0, \delta_f(\varepsilon_3))$. Indeed one verifies:

$$\begin{aligned}\forall\, H \in \mathcal{N}_{3,\varepsilon}\colon\ \|R_3(H)\|_M &\leq \|r_f(H)\|_{\mathcal{B}} \|g(Z_0)\|_M \leq \varepsilon_3 \|\varphi(H)\|_{\mathcal{B}} \|g(Z_0)\|_M \leq \\ &\leq \varepsilon_3 C \|\rho(H)\|_{\mathrm{End}} \|g(Z_0)\|_M = \frac{\varepsilon}{6} \|\rho(H)\|_{\mathrm{End}} .\end{aligned}$$

We remark that this is the only step in the proof where the condition $\mathrm{Ker}(\rho) \subseteq \mathrm{Ker}(\varphi)$ is needed.

(IV) $R_4(H) \coloneqq \rho(H)\lambda(f'(Z_0))r_g(H)$:

$$\|R_4(H)\|_M \leq \|\rho(H)\|_{\mathrm{End}} \|f'(Z_0)\|_{\mathcal{B}} \|r_g(H)\|_M .$$

If $f'(Z_0) = 0$, then the desired inequality is trivially satisfied, so we can assume further that $f'(Z_0) \neq 0$. Since ρ is (automatically) continuous, now define $\mathcal{N}_{4,\varepsilon} \coloneqq \rho^{-1}(\mathbb{B}_{\mathrm{End}_{\mathbb{K}}(M)}(0, \varepsilon_4)) \cap \mathbb{B}_{\mathcal{A}}(0, \delta_g(\varepsilon_4))$, where $\varepsilon_4 \coloneqq (\varepsilon/(6\|f'(Z_0)\|_{\mathcal{B}}))^{1/2}$. Indeed one verifies:

$$\begin{aligned}\forall\, H \in \mathcal{N}_{4,\varepsilon}\colon\ \|R_4(H)\|_M &\leq \|\rho(H)\|_{\mathrm{End}} \|f'(Z_0)\|_{\mathcal{B}} \|r_g(H)\|_M \leq \\ &\leq \|\rho(H)\|_{\mathrm{End}} \|f'(Z_0)\|_{\mathcal{B}}\, \varepsilon_4 \|\rho(H)\|_{\mathrm{End}} \leq \\ &\leq \|\rho(H)\|_{\mathrm{End}} \|f'(Z_0)\|_{\mathcal{B}}\, \varepsilon_4^2 = \frac{\varepsilon}{6} \|\rho(H)\|_{\mathrm{End}} .\end{aligned}$$

(V) $R_5(H) \coloneqq \rho(H)\lambda(r_f(H))g'(Z_0)$:

$$\|R_5(H)\|_M \leq \|\rho(H)\|_{\mathrm{End}} \|r_f(H)\|_{\mathcal{B}} \|g'(Z_0)\|_M .$$

If $g'(Z_0) = 0$, then the desired inequality is trivially satisfied, so without loss of generality assume $g'(Z_0) \neq 0$. Now define $\mathcal{N}_{5,\varepsilon} \coloneqq \varphi^{-1}(\mathbb{B}_{\mathcal{B}}(0, \varepsilon_5)) \cap \mathbb{B}_{\mathcal{A}}(0, \delta_f(\varepsilon_5))$, where $\varepsilon_5 \coloneqq (\varepsilon/(6\|g'(Z_0)\|_M))^{1/2}$. Indeed one verifies:

$$\begin{aligned}\forall\, H \in \mathcal{N}_{5,\varepsilon}\colon\ \|R_5(H)\|_M &\leq \|\rho(H)\|_{\mathrm{End}} \|r_f(H)\|_{\mathcal{B}} \|g'(Z_0)\|_M \leq \\ &\leq \|\rho(H)\|_{\mathrm{End}}\, \varepsilon_5 \|\varphi(H)\|_{\mathcal{B}} \|g'(Z_0)\|_M \leq \\ &\leq \|\rho(H)\|_{\mathrm{End}}\, \varepsilon_5^2 \|g'(Z_0)\|_M = \frac{\varepsilon}{6} \|\rho(H)\|_{\mathrm{End}} .\end{aligned}$$

(VI) $R_6(H) \coloneqq \lambda(r_f(H))r_g(H)$:

$$\|R_6(H)\|_M \leq \|r_f(H)\|_{\mathcal{B}} \|r_g(H)\|_M .$$

Since φ is (automatically) continuous, take $\varepsilon_6 \coloneqq (\varepsilon/6)^{1/3}$ and $\mathcal{N}_{6,\varepsilon} \coloneqq \mathbb{B}_{\mathcal{A}}(0, \delta_f(\varepsilon_6)) \cap \mathbb{B}_{\mathcal{A}}(0, \delta_g(\varepsilon_6)) \cap \varphi^{-1}(\mathbb{B}_{\mathcal{B}}(0, \varepsilon_6))$. Indeed one verifies:

$$\begin{aligned}\forall\, H \in \mathcal{N}_{6,\varepsilon}\colon\ \|R_6(H)\|_M &\leq \|r_f(H)\|_{\mathcal{B}} \|r_g(H)\|_M \leq \varepsilon_6 \|\varphi(H)\|_{\mathcal{B}}\, \varepsilon_6 \|\rho(H)\|_{\mathrm{End}} \leq \\ &\leq \varepsilon_6^3 \|\rho(H)\|_{\mathrm{End}} = \frac{\varepsilon}{6} \|\rho(H)\|_{\mathrm{End}} .\end{aligned}$$

Finally, putting $\mathcal{N}_\varepsilon := \bigcap_{j=1}^6 \mathcal{N}_{j,\varepsilon} \ni 0$, we get

$$\forall\, H \in \mathcal{N}_\varepsilon \colon\ \|R(H)\|_M \leq \sum_{j=1}^{6} \|R_j(H)\|_M \leq \sum_{j=1}^{6} \frac{\varepsilon}{6} \|\rho(H)\|_{\mathrm{End}} = \varepsilon \|\rho(H)\|_{\mathrm{End}}$$

as desired.

To $(\mathscr{F})$: Write again

$$\begin{aligned} f(Z_0+H) &= f(Z_0) + \varphi(H) f'(Z_0) + r_f(H) \\ g(Z_0+H) &= g(Z_0) + \rho(H) g'(Z_0) + r_g(H), \end{aligned}$$

where this time:

- $\forall\, \varepsilon > 0\ \exists\, \delta_f = \delta_f(\varepsilon) > 0\ \forall\, H \in \mathbb{B}_\mathcal{A}(0, \delta_f(\varepsilon))\colon\ \|r_f(H)\|_\mathcal{B} \leq \varepsilon \|H\|_\mathcal{A}$;
- $\forall\, \varepsilon > 0\, \exists\, \delta_g = \delta_g(\varepsilon) > 0\ \forall\, H \in \mathbb{B}_\mathcal{A}(0, \delta_g(\varepsilon))\colon\ \|r_g(H)\|_M \leq \varepsilon \|H\|_\mathcal{A}$.

As in the previous case, we have

$$\begin{aligned} &f(Z_0+H) \cdot g(Z_0+H) \overset{\text{def}}{=} \lambda(f(Z_0+H)) g(Z_0+H) = \\ &= f(Z_0)g(Z_0) + (f(Z_0)g'(Z_0) + f'(Z_0)g(Z_0))\varphi(H) + R(H), \end{aligned}$$

where $R(H) := \sum_{j=1}^6 R_j(H)$ with $R_j(H)$, $1 \leq j \leq 6$, *listed below.* Now let $\varepsilon > 0$ be arbitrary. For each $R_j(H)$ we will construct again a neighbourhood $\mathcal{N}_{j,\varepsilon} \ni 0$ such that $\forall\, H \in \mathcal{N}_{j,\varepsilon}\colon\ \|R_j(H)\|_M \leq \frac{\varepsilon}{6} \|H\|_\mathcal{A}$, $1 \leq j \leq 6$. Namely:

(I) $R_1(H) := \rho(H)^2 \lambda(f'(Z_0)) g'(Z_0)$:

$$\|R_1(H)\|_\mathcal{B} \leq \|H\|_\mathcal{A}^2 \left\|f'(Z_0)\right\|_\mathcal{B} \left\|g'(Z_0)\right\|_M .$$

If $f'(Z_0) = 0$ or $g'(Z_0) = 0$, then the desired inequality is trivially satisfied. Thus we can further assume that $f'(Z_0) \neq 0$ and $g'(Z_0) \neq 0$. Now take $\mathcal{N}_{1,\varepsilon} := \mathbb{B}_\mathcal{A}(0, \varepsilon_1)$, where $\varepsilon_1 := \varepsilon/(6 \|f'(Z_0)\|_\mathcal{B} \|g'(Z_0)\|_M)$. Indeed one verifies:

$$\begin{aligned} \forall\, H \in \mathcal{N}_{1,\varepsilon}\colon\ \|R_1(H)\|_\mathcal{B} &\leq \|H\|_\mathcal{A} \left\|f'(Z_0)\right\|_\mathcal{B} \left\|g'(Z_0)\right\|_M \|H\|_\mathcal{A} \leq \\ &\leq \varepsilon_1 \left\|f'(Z_0)\right\|_\mathcal{B} \left\|g'(Z_0)\right\|_M \|H\|_\mathcal{A} = \frac{\varepsilon}{6} \|H\|_\mathcal{A} . \end{aligned}$$

(II) $R_2(H) := \lambda(f(Z_0)) r_g(H)$:

$$\|R_2(H)\|_M \leq \|f(Z_0)\|_\mathcal{B} \|r_g(H)\|_M .$$

If $f(Z_0) = 0$, then the desired inequality is trivially satisfied, so we can further assume that $f(Z_0) \neq 0$. Put $\varepsilon_2 := \varepsilon/(6 \|f(Z_0)\|_\mathcal{B})$ and take $\mathcal{N}_{2,\varepsilon} := \mathbb{B}_\mathcal{A}(0, \delta_g(\varepsilon_2))$. Indeed one verifies:

$$\forall\, H \in \mathcal{N}_{2,\varepsilon}\colon\ \|R_2(H)\|_M \leq \|f(Z_0)\|_\mathcal{B} \|r_g(H)\|_M \leq \|f(Z_0)\|_\mathcal{B}\, \varepsilon_2 \|H\|_\mathcal{A} = \frac{\varepsilon}{6} \|H\|_\mathcal{A} .$$

(III) $R_3(H) := \lambda(r_f(H)) g(Z_0)$:

$$\|R_3(H)\|_M \leq \|r_f(H)\|_\mathcal{B} \|g(Z_0)\|_M$$

If $g(Z_0) = 0$, the desired inequality is trivially satisfied, so without loss of generality assume $g(Z_0) \neq 0$. Now put $\varepsilon_3 := \varepsilon/(6 \|g(Z_0)\|_M)$ and take $\mathcal{N}_{3,\varepsilon} := \mathbb{B}_\mathcal{A}(0, \delta_f(\varepsilon_3))$. Indeed one verifies:

$$\forall\, H \in \mathcal{N}_{3,\varepsilon}\colon\ \|R_3(H)\|_M \leq \|r_f(H)\|_\mathcal{B} \|g(Z_0)\|_M \leq \varepsilon_3 \|H\|_\mathcal{A} \|g(Z_0)\|_M = \frac{\varepsilon}{6} \|H\|_\mathcal{A} .$$

(IV) $R_4(H) := \rho(H) \lambda(f'(Z_0)) r_g(H)$:

$$\|R_4(H)\|_M \leq \|H\|_\mathcal{A} \left\|f'(Z_0)\right\|_\mathcal{B} \|r_g(H)\|_M .$$

If $f'(Z_0) = 0$, then the desired inequality is trivially satisfied, so we can assume further that $f'(Z_0) \neq 0$. Since ρ is (automatically) continuous, now define $\mathcal{N}_{4,\varepsilon} := \mathbb{B}_\mathcal{A}(0, \varepsilon_4) \cap \mathbb{B}_\mathcal{A}(0, \delta_g(\varepsilon_4))$, where $\varepsilon_4 \coloneqq (\varepsilon/(6\,\|f'(Z_0)\|_{\mathcal{B}}))^{1/2}$. Indeed one verifies:

$$\begin{aligned}\forall\, H \in \mathcal{N}_{4,\varepsilon}\colon\ \|R_4(H)\|_M &\le \|H\|_{\mathcal{A}}\,\big\|f'(Z_0)\big\|_{\mathcal{B}}\,\|r_g(H)\|_M \le\\ &\le \|H\|_{\mathcal{A}}\,\big\|f'(Z_0)\big\|_{\mathcal{B}}\,\varepsilon_4\,\|H\|_{\mathcal{A}} \le\\ &\le \|H\|_{\mathcal{A}}\,\big\|f'(Z_0)\big\|_{\mathcal{B}}\,\varepsilon_4^2 = \frac{\varepsilon}{6}\,\|H\|_{\mathcal{A}}\,.\end{aligned}$$

(V) $R_5(H) \coloneqq \rho(H)\lambda(r_f(H))g'(Z_0)$:

$$\|R_5(H)\|_M \le \|H\|_{\mathcal{A}}\,\|r_f(H)\|_{\mathcal{B}}\,\big\|g'(Z_0)\big\|_M\,.$$

If $g'(Z_0) = 0$, then the desired inequality is trivially satisfied, so without loss of generality assume $g'(Z_0) \neq 0$. Now define $\mathcal{N}_{5,\varepsilon} \coloneqq \mathbb{B}_{\mathcal{A}}(0,\varepsilon_5)\cap\mathbb{B}_{\mathcal{A}}(0,\delta_f(\varepsilon_5))$, where $\varepsilon_5 \coloneqq (\varepsilon/(6\,\|g'(Z_0)\|_M))^{1/2}$. Indeed one verifies:

$$\begin{aligned}\forall\, H \in \mathcal{N}_{5,\varepsilon}\colon\ \|R_5(H)\|_M &\le \|H\|_{\mathcal{A}}\,\|r_f(H)\|_{\mathcal{B}}\,\big\|g'(Z_0)\big\|_M \le\\ &\le \|H\|_{\mathcal{A}}\,\varepsilon_5\,\|H\|_{\mathcal{A}}\,\big\|g'(Z_0)\big\|_M \le\\ &\le \|H\|_{\mathcal{A}}\,\varepsilon_5^2\,\big\|g'(Z_0)\big\|_M = \frac{\varepsilon}{6}\,\|H\|_{\mathcal{A}}\,.\end{aligned}$$

(VI) $R_6(H) \coloneqq \lambda(r_f(H))r_g(H)$:

$$\|R_6(H)\|_M \le \|r_f(H)\|_{\mathcal{B}}\,\|r_g(H)\|_M\,.$$

Take $\varepsilon_6 \coloneqq (\varepsilon/6)^{1/3}$ and $\mathcal{N}_{6,\varepsilon} \coloneqq \mathbb{B}_{\mathcal{A}}(0,\delta_f(\varepsilon_6))\cap\mathbb{B}_{\mathcal{A}}(0,\delta_g(\varepsilon_6))\cap\mathbb{B}_{\mathcal{A}}(0,\varepsilon_6)$. Indeed one verifies:

$$\begin{aligned}\forall\, H \in \mathcal{N}_{6,\varepsilon}\colon\ \|R_6(H)\|_M &\le \|r_f(H)\|_{\mathcal{B}}\,\|r_g(H)\|_M \le \varepsilon_6\,\|H\|_{\mathcal{A}}\,\varepsilon_6\,\|H\|_{\mathcal{A}} \le\\ &\le \varepsilon_6^3\,\|H\|_{\mathcal{A}} = \frac{\varepsilon}{6}\,\|H\|_{\mathcal{A}}\,.\end{aligned}$$

Finally, putting $\mathcal{N}_\varepsilon \coloneqq \bigcap_{j=1}^6 \mathcal{N}_{j,\varepsilon} \ni 0$, we get

$$\forall\, H \in \mathcal{N}_\varepsilon\colon\ \|R(H)\|_M \le \sum_{j=1}^6 \|R_j(H)\|_M \le \sum_{j=1}^6 \frac{\varepsilon}{6}\,\|\rho(H)\|_{\mathrm{End}} = \varepsilon\,\|\rho(H)\|_{\mathrm{End}}$$

as desired.

To (4): To (i): Completely analogous to (2)(i) for all three φ-differentiability classes since $\mathcal{B} \cong \mathrm{End}_{\mathcal{B}}(\mathcal{B})$ canonically.

To (ii): All three cases $\mathscr{L}_\varphi$, $\mathscr{F}_\varphi^{\mathrm{str}}$, and $\mathscr{F}_\varphi$ follow from (3) since $\mathscr{L}_\varphi = \mathscr{L}_{\lambda_\varphi}$, $\mathscr{F}_\varphi^{\mathrm{str}} = \mathscr{F}_{\lambda_\varphi}^{\mathrm{str}}$, and $\mathscr{F}_\varphi = \mathscr{F}_{\lambda_\varphi}$, respectively, for $\lambda_\varphi \overset{\mathrm{def}}{=} \lambda_{\mathcal{B}} \circ \varphi \overset{\mathrm{def}}{=} \varphi^*\lambda_{\mathcal{B}}$ and $\mathrm{Ker}(\lambda_\varphi) = \mathrm{Ker}(\varphi)$ by faithfulness of $\lambda_{\mathcal{B}}$.

To (5): follows from (2) and (4), respectively.

To (6): follows from (2).

To (7): follows from (4).

To (8): follows from (3). □

Remarks:

(1) Given $\rho = \lambda \circ \varphi$, we always have $\mathrm{Ker}(\varphi) \subseteq \mathrm{Ker}(\rho)$, so the non-trivial part of the condition $\mathrm{Ker}(\rho) = \mathrm{Ker}(\varphi)$ is the reverse inclusion $\mathrm{Ker}(\rho) \subseteq \mathrm{Ker}(\varphi)$. This is clearly equivalent to $\lambda|_{\mathrm{Im}(\varphi)}$ being injective. In other words, it suffices for λ to be bounded from below when restricted to $\mathrm{Im}(\varphi)$ rather than being bounded below everywhere.

(2) Observe that the "mixed" Leibniz rule does not apply to the "middle" regularity class $\mathscr{F}^{\mathrm{str}}$ without some further "commensurability" hypothesis needed in (III) such as the above one.

(3) Since every $M \in \mathsf{FdMod}(\mathcal{A})$ admits a complete (=indecomposable) decomposition ${}_{\mathcal{A}}M =$

$\bigoplus_{k=1}^m {}_{\mathcal{A}}M_k$, by Section 0.1.4 we have a canonical factorization $\rho = \lambda \circ \rho^{\mathrm{mor}}$, where $\rho^{\mathrm{mor}} = (\rho_k^{\mathrm{im}})_{1\leq k\leq m}\colon \mathcal{A} \to \mathcal{B} \overset{\mathrm{def}}{=} \prod_{k=1}^m \mathcal{B}_k$ for $\rho_k\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M_k)$ and $\mathcal{B}_k \overset{\mathrm{def}}{=} \rho_k(\mathcal{A})$, $1 \leq k \leq m$. Therefore, $(\mathscr{C}_\rho^\infty(U), \partial)$ is *canonically* a differential module over the differential $\mathcal{B}$-algebra $(\mathscr{C}_{\rho^{\mathrm{mor}}}^\infty(U), \partial)$. Likewise, $(\mathscr{C}_\rho^\omega(U), \partial)$ is *canonically* a differential module over the differential $\mathcal{B}$-algebra $(\mathscr{C}_{\rho^{\mathrm{mor}}}^\omega(U), \partial)$. Thus we can just speak of *the differential (left)* $\mathrm{End}_{\mathcal{A}}(M)$*-modules* $(\mathscr{C}_\rho^\infty(U), \partial)$ and $(\mathscr{C}_\rho^\omega(U), \partial)$ (since there is automatically an underlying differential algebra of our type anyway), or simply left $(\mathrm{End}_{\mathcal{A}}(M), \partial)$-modules (or similar). Note, however, that the $\mathrm{End}_{\mathcal{A}}(M)$-action need not commute with the $\mathcal{B}$-action in general (due to presence of off-diagonal elements if the decomposition of M is not $\mathrm{Hom}_{\mathcal{A}}$-orthogonal). □

Definition 1.1.20: Let $\mathcal{A} \in \mathsf{Alg}_{\mathbb{K}}$ and let $({}_{\mathcal{A}}M, \Phi_M)$ and $({}_{\mathcal{A}}N, \Phi_N)$ be two pairs of $M, N \in \mathsf{LMod}(\mathcal{A})$ equipped with distinguished endomorphisms $\Phi_M \in \mathrm{End}_{\mathcal{A}}(M)$ and $\Phi_N \in \mathrm{End}_{\mathcal{A}}(N)$. Then a morphism $\Psi\colon (M, \Phi_M) \to (N, \Phi_N)$ is just $\Psi \in \mathrm{Hom}_{\mathcal{A}}({}_{\mathcal{A}}M, {}_{\mathcal{A}}N)$ such that

$$\begin{array}{ccc} {}_{\mathcal{A}}M & \xrightarrow{\Psi} & {}_{\mathcal{A}}N \\ {\scriptstyle\Phi_M}\downarrow & & \downarrow{\scriptstyle\Phi_N} \\ {}_{\mathcal{A}}M & \xrightarrow{\Psi} & {}_{\mathcal{A}}N \end{array}$$

is a commutative diagram. □

For us the role of the distinguished module endomorphisms will be played of course by the "module derivation" $\partial\colon f \mapsto f'$.

Lemma 1.1.21: Let $a, b \geq 0$. Then the map $F\colon (0,\infty)^2 \to (0,\infty)$, $(x, y) \mapsto ax+by+xy$, is surjective.

Proof: Consider $ax + by + xy = z$ for some fixed $z > 0$. If $a = b = 0$, then clearly there exist $x, y > 0$ with $xy = z$. Without loss of generality suppose now that $b \neq 0$. We have $x = \frac{z-by}{a+y}$, well-defined and positive for all $y \in (0, \frac{z}{b})$. □

Recall that we now have $\mathscr{C}_\rho^1(U) \overset{\mathrm{def}}{=} \mathscr{F}_\rho(U) \cap \mathscr{C}^1(U, M) = \mathscr{F}_\rho^{\mathrm{str}}(U) \cap \mathscr{C}^1(U, M) = \mathscr{L}_\rho(U) \cap \mathscr{C}^1(U, M) \subseteq \mathscr{L}_\rho(U) \subseteq \mathscr{F}_\rho^{\mathrm{str}}(U) \subseteq \mathscr{F}_\rho(U)$. This can be used to establish "mixed" chain rules as follows:

Proposition 1.1.22 (Chain Rule with Functions)**:** Let $\mathcal{A} \overset{\varphi}{\to} \mathcal{B} \overset{\psi}{\to} \mathcal{C}$ and $M \in \mathsf{FdMod}(\mathcal{B})$. Let $U \subseteq \mathcal{A}$ and $V \subseteq \mathcal{B}$ be open and $f\colon U \to \mathcal{B}$ such that $f(U) \subseteq V$. We have:

(1) If $g \in \mathscr{F}_{\mathcal{B}}^{\mathrm{str}}(V, {}_{\mathcal{B}}M)$ and $f \in \mathscr{C}_{\mathcal{A}}^1(U, \mathcal{B})$, then $g \circ f \in \mathscr{F}_{\mathcal{A}}^{\mathrm{str}}(U, {}_{\mathcal{A}}M)$ and $\forall\, Z \in U\colon (g \circ f)'(Z) = \rho(f'(Z))g'(f(Z))$.
(2) In particular, if $g \in \mathscr{F}_\psi^{\mathrm{str}}(V)$ and $f \in \mathscr{C}_\varphi^1(U)$, then $g \circ f \in \mathscr{F}_{\psi\circ\varphi}^{\mathrm{str}}(U)$ and $\forall\, Z \in U\colon (g \circ f)'(Z) = \psi(f'(Z))g'(f(Z))$.
(3) If $g \in \mathscr{F}_{\mathcal{B}}(V, {}_{\mathcal{B}}M)$ and $f \in \mathscr{F}_{\mathcal{A}}(U, \mathcal{B})$, then $g \circ f \in \mathscr{F}_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$ and $\forall\, Z \in U\colon (g \circ f)'(Z) = \rho(f'(Z))g'(f(Z))$.
(4) In particular, if $g \in \mathscr{F}_\psi(V)$ and $f \in \mathscr{F}_\varphi(U)$, then $g \circ f \in \mathscr{F}_{\psi\circ\varphi}(U)$ and $\forall\, Z \in U\colon (g \circ f)'(Z) = \psi(f'(Z))g'(f(Z))$.

Proof: We begin by setting up some general notations used in the proofs of both (1) and (3). Let $\rho\colon \mathcal{B} \to \mathrm{End}_{\mathbb{K}}(M)$ denote the corresponding representation for $M \in \mathsf{FdMod}(\mathcal{B})$. Put $W_0 := f(Z_0) \in \mathcal{B}$, $B := f'(Z_0) \in \mathcal{B}$, $v := g'(W_0) \in M$ for short, and r_f and r_g for the remainder terms of f and g at Z_0 and W_0, respectively. Set $K_H := \varphi(H)B + r_f(H) \to 0$ as $H \to 0$ and $R(H) := \rho(r_f(H))v + r_g(K_H)$. Then

$$g(f(Z_0 + H)) = g(W_0 + K_H) = g(W_0) + \rho(K_H)v + r_g(K_H) = g(W_0) + \rho(\varphi(H))\rho(B)v + R(H).$$

Moreover:

- φ (automatically) bounded, hence $\forall\, H \in \mathcal{A}$: $\|\varphi(H)\|_{\mathcal{B}} \le \|H\|_{\mathcal{A}}$;
- ρ (automatically) bounded, hence $\forall\, K \in \mathcal{B}$: $\|\rho(K)\|_{\mathrm{op}} \le \|K\|_{\mathcal{B}}$;

(cf. Lemma 0.3.24). We now proceed with the proofs.

To (1): For a "model" of $\mathscr{C}^1_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathcal{B})$ it is probably most convenient to use $\mathscr{F}^1_{\varphi}(U) \cap \mathscr{C}^1(U,\mathcal{B})$. We need to show that $\|R(H)\|_M = o(\|\rho(\varphi(H))\|_{\mathrm{End}})$ as $H \to 0$. But first we need to fix some further notations:

- $f \in \mathscr{F}_{\varphi}(U)$, hence $\forall\, \varepsilon_{\mathcal{B}} > 0\ \exists\, \delta_{\mathcal{A}} = \delta_{\mathcal{A}}(\varepsilon_{\mathcal{B}}) > 0\ \forall\, \|H\|_{\mathcal{A}} < \delta_{\mathcal{A}}$: $\|r_f(H)\|_{\mathcal{B}} \le \varepsilon_{\mathcal{B}} \|H\|_{\mathcal{A}}$;
- $f' \in \mathscr{C}(U,\mathcal{B})$ and $\mathcal{A}$ is locally convex, hence
$$\forall\, \varepsilon'_{\mathcal{B}} > 0\ \exists\, \delta'_{\mathcal{A}} = \delta'_{\mathcal{A}}(\varepsilon'_{\mathcal{B}})\ \forall\, \|H\|_{\mathcal{A}} < \delta'_{\mathcal{A}}\colon\ \left\|f'(Z_0+H) - f'(Z_0)\right\|_{\mathcal{B}} < \varepsilon'_{\mathcal{B}}$$
with $[Z_0, Z_0+H] \subseteq U$;
- $g \in \mathscr{F}^{\mathrm{str}}_{\rho}(V)$, hence $\forall\, \varepsilon_M > 0\ \exists\, \delta_{\mathcal{B}} = \delta_{\mathcal{B}}(\varepsilon_M) > 0\ \forall\, \|K\|_{\mathcal{B}} < \delta_{\mathcal{B}}$: $\|r_g(K)\|_M \le \varepsilon_M \|\rho(K)\|_{\mathrm{End}}$.

Now define
$$\delta := \delta(\varepsilon_{\mathcal{B}}, \varepsilon'_{\mathcal{B}}, \varepsilon_M) := \min\left\{\frac{\delta_{\mathcal{B}}(\varepsilon_M)}{\|B\|_{\mathcal{B}} + \varepsilon_{\mathcal{B}}}, \delta_{\mathcal{A}}(\varepsilon_{\mathcal{B}}), \delta'_{\mathcal{A}}(\varepsilon'_{\mathcal{B}})\right\} > 0$$
and
$$\varepsilon := \varepsilon(\varepsilon_{\mathcal{B}}, \varepsilon'_{\mathcal{B}}, \varepsilon_M) := \varepsilon(\varepsilon'_{\mathcal{B}}, \varepsilon_M) := \|v\|_E\, \varepsilon'_{\mathcal{B}} + \|B\|_{\mathcal{B}}\, \varepsilon_M + \varepsilon'_{\mathcal{B}}\varepsilon_M > 0.$$
We are going to show that $\forall\, \|H\|_{\mathcal{A}} < \delta$: $\|R(H)\|_M \le \varepsilon \|\rho(\varphi(H))\|_{\mathrm{End}}$. Since $\varepsilon\colon (0,\infty)^3 \to (0,\infty)$ is *surjective* by Lemma 1.1.21, this would complete the proof. Firstly, we have
$$\begin{aligned}\forall\, \|H\|_{\mathcal{A}} < \delta\colon\ \|K_H\|_{\mathcal{B}} &\overset{\mathrm{def}}{=} \|\varphi(H)B + r_f(H)\|_{\mathcal{B}} \le \|B\|_{\mathcal{B}} \|\varphi(H)\|_{\mathcal{B}} + \|r_f(H)\|_{\mathcal{B}} \le \\ &\le \|B\|_{\mathcal{B}} \|\varphi(H)\|_{\mathcal{B}} + \varepsilon_{\mathcal{B}} \|H\|_{\mathcal{A}} \le (\|B\|_{\mathcal{B}} + \varepsilon_{\mathcal{B}}) \|H\|_{\mathcal{A}} < \delta_{\mathcal{B}}(\varepsilon_M).\end{aligned}$$
Secondly, if $\|H\|_{\mathcal{A}} < \delta'_{\mathcal{A}}(\varepsilon'_{\mathcal{B}})$, then also $\forall\, t \in [0,1]$: $\|tH\|_{\mathcal{A}} = t\|H\|_{\mathcal{A}} \le \|H\|_{\mathcal{A}} < \delta'_{\mathcal{A}}(\varepsilon'_{\mathcal{B}})$. Hence by Lemma 1.1.12
$$\begin{aligned}\forall\, \|H\|_{\mathcal{A}} < \delta\colon\ \|\rho(r_f(H))\|_{\mathrm{End}} &= \left\|\rho(\varphi(H))\rho\Big(\int_0^1 (f'(Z_0+tH) - f'(Z_0))\mathrm{d}t\Big)\right\|_{\mathrm{End}} \le \\ &\le \|\rho(\varphi(H))\|_{\mathrm{End}} \int_0^1 \left\|f'(Z_0+tH) - f'(Z_0)\right\|_{\mathcal{B}} \mathrm{d}t \le \\ &\le \|\rho(\varphi(H))\|_{\mathrm{End}} \int_0^1 \varepsilon'_{\mathcal{B}} \mathrm{d}t = \varepsilon'_{\mathcal{B}} \|\rho(\varphi(H))\|_{\mathrm{End}}\end{aligned}$$
Thus, we obtain
$$\begin{aligned}\forall\, \|H\|_{\mathcal{A}} < \delta\colon\ \|R(H)\|_M &\overset{\mathrm{def}}{=} \|\rho(r_f(H))v + r_g(K_H)\|_M \le \|\rho(r_f(H))\|_{\mathrm{End}} \|v\|_M + \|r_g(K_H)\|_M \le \\ &\le \|v\|_M \|\rho(r_f(H))\|_{\mathrm{End}} + \varepsilon_M \|\rho(K_H)\|_{\mathrm{End}} \overset{\mathrm{def}}{=} \\ &\overset{\mathrm{def}}{=} \|v\|_M \|\rho(r_f(H))\|_{\mathrm{End}} + \varepsilon_M \|\rho(\varphi(H))\rho(B) + \rho(r_f(H))\|_{\mathrm{End}} \le \\ &\le \|v\|_M \|\rho(r_f(H))\|_{\mathrm{End}} + \varepsilon_M \|\rho(\varphi(H))\|_{\mathrm{End}} \|B\|_{\mathcal{B}} + \varepsilon_M \|\rho(r_f(H))\|_{\mathrm{End}} = \\ &= \varepsilon_M \|B\|_{\mathcal{B}} \|\rho(\varphi(H))\|_{\mathrm{End}} + (\|v\|_M + \varepsilon_M) \|\rho(r_f(H))\|_{\mathrm{End}} \le \\ &\le \varepsilon_M \|B\|_{\mathcal{B}} \|\rho(\varphi(H))\|_{\mathrm{End}} + (\|v\|_M + \varepsilon_M)\varepsilon'_{\mathcal{B}} \|\rho(\varphi(H))\|_{\mathrm{End}} = \\ &= (\varepsilon_M \|B\|_{\mathcal{B}} + \|v\|_M\, \varepsilon'_{\mathcal{B}} + \varepsilon_M\varepsilon'_{\mathcal{B}}) \|\rho(\varphi(H))\|_{\mathrm{End}} \overset{\mathrm{def}}{=} \varepsilon \|\rho(\varphi(H))\|_{\mathrm{End}}\end{aligned}$$
as desired.

To (2): special case of (1).

To (3): We need to show that $\|R(H)\|_M = o(\|H\|_{\mathcal{A}})$ as $H \to 0$. The proof is similar to that of (1). This time we have:

- $f \in \mathscr{F}_\varphi(U)$, hence $\forall\, \varepsilon_\mathcal{B} > 0\ \exists\, \delta_\mathcal{A} = \delta_\mathcal{A}(\varepsilon_\mathcal{B}) > 0\ \forall \|H\|_\mathcal{A} < \delta_\mathcal{A}\colon\ \|r_f(H)\|_\mathcal{B} \leq \varepsilon_\mathcal{B} \|H\|_\mathcal{A}$;
- $g \in \mathscr{F}_\rho(V)$, hence $\forall\, \varepsilon_M > 0\ \exists\, \delta_\mathcal{B} = \delta_\mathcal{B}(\varepsilon_M) > 0\ \forall \|K\|_\mathcal{B} < \delta_\mathcal{B}\colon\ \|r_g(K)\|_M \leq \varepsilon_M \|K\|_\mathcal{B}$.

Now define

$$\delta := \delta(\varepsilon_\mathcal{B}, \varepsilon_M) := \min\left\{\frac{\delta_\mathcal{B}(\varepsilon_M)}{\|B\|_\mathcal{B} + \varepsilon_\mathcal{B}}, \delta_\mathcal{A}(\varepsilon_\mathcal{B})\right\} > 0$$

and

$$\varepsilon := \varepsilon(\varepsilon_\mathcal{B}, \varepsilon_M) := \varepsilon_\mathcal{B} \|v\|_M + \varepsilon_M \|B\|_\mathcal{B} + \varepsilon_\mathcal{B}\varepsilon_M.$$

We are going to show that $\forall \|H\|_\mathcal{A} < \delta\colon\ \|R(H)\|_M \leq \varepsilon \|H\|_\mathcal{A}$. As in (1) this would imply the claim since $\varepsilon\colon (0,\infty)^2 \to (0,\infty)$ is *surjective* by Lemma 1.1.21. Again as in (1), we have:

$$\forall \|H\|_\mathcal{A} < \delta\colon\ \|K_H\|_\mathcal{B} \overset{\text{def}}{=} \|\varphi(H)B + r_f(H)\|_\mathcal{B} \leq (\|B\|_\mathcal{B} + \varepsilon_\mathcal{B}) \|H\|_\mathcal{A} < \delta_\mathcal{B}(\varepsilon_M).$$

We thus obtain:

$$\begin{aligned}\forall \|H\|_\mathcal{A} < \delta\colon\ \|R(H)\|_M &\overset{\text{def}}{=} \|\rho(r_f(H))v + r_g(K_H)\|_M \leq \|\rho(r_f(H))\|_{\text{End}} \|v\|_M + \|r_g(K_H)\|_M \leq \\ &\leq \|r_f(H)\|_\mathcal{B} \|v\|_M + \varepsilon_M \|K_H\|_\mathcal{B} \overset{\text{def}}{=} \\ &\overset{\text{def}}{=} \|r_f(H)\|_\mathcal{B} \|v\|_M + \varepsilon_M \|\varphi(H)B + r_f(H)\|_\mathcal{B} \leq \\ &\leq \varepsilon_\mathcal{B} \|H\|_\mathcal{A} \|v\|_M + \varepsilon_M \|H\|_\mathcal{A} \|B\|_\mathcal{B} + \varepsilon_M\varepsilon_\mathcal{B} \|H\|_\mathcal{A} = \\ &= (\varepsilon_\mathcal{B} \|v\|_M + \varepsilon_M \|B\|_\mathcal{B} + \varepsilon_\mathcal{B}\varepsilon_M) \|H\|_\mathcal{A} = \varepsilon \|H\|_\mathcal{A}\end{aligned}$$

To (4): special case of (3). □

Remarks:

(1) In other words, the composition of representation- and morphism-differentiable functions behaves reasonably *functorially*. However, unconditional functoriality in that sense requires a certain minimum amount of regularity.

(2) Despite Proposition 1.1.14, we have picked $\mathscr{F}_\mathcal{A} \cap \mathscr{C}^1$ and $\mathscr{F}_\varphi \cap \mathscr{C}^1$ as "models" for $\mathscr{C}^1_\mathcal{A}$ and $\mathscr{C}^1_\varphi$ in order to emphasize the most convenient proof approach.

(3) By restricting to pre- and post-compositions with homomorphisms only, we get composition closedness for tighter regularity classes (details follow). □

Proposition 1.1.23 (Chain Rule with Homomorphisms)**:** Let $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ be a morphism and let $U \subseteq \mathcal{A}$, $V \subseteq \mathcal{B}$ be open.

(1) Pre-composition with a morphism $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$: given $\varphi(U) \subseteq V$ and $M \in \mathsf{FdMod}(\mathcal{B})$, we have:
  (i) If $g \in \mathscr{L}_\mathcal{B}(V, {}_\mathcal{B}M)$, then $g \circ \varphi \in \mathscr{L}_\mathcal{A}(U, {}_\mathcal{A}M)$ and $\forall\, Z \in U\colon (g \circ \varphi)(Z) = g'(\varphi(Z))$.
  (ii) If $g \in \mathscr{F}^{\text{str}}_\mathcal{B}(V, {}_\mathcal{B}M)$, then $g \circ \varphi \in \mathscr{F}^{\text{str}}_\mathcal{A}(U, {}_\mathcal{A}M)$ and $\forall\, Z \in U\colon (g \circ \varphi)(Z) = g'(\varphi(Z))$.
  (iii) If $g \in \mathscr{F}_\mathcal{B}(V, {}_\mathcal{B}M)$, then $g \circ \varphi \in \mathscr{F}_\mathcal{A}(U, {}_\mathcal{A}M)$ and $\forall\, Z \in U\colon (g \circ \varphi)(Z) = g'(\varphi(Z))$.

(2) Post-composition with a module homomorphism $\Phi \in \text{Hom}_\mathcal{A}({}_\mathcal{A}M, {}_\mathcal{A}N)$:
  (i) If $f \in \mathscr{L}_\mathcal{A}(U, {}_\mathcal{A}M)$, then $\Phi \circ f \in \mathscr{L}_\mathcal{A}(U, {}_\mathcal{A}N)$ and $\forall\, Z \in U\colon (\Phi \circ f)'(Z) = \Phi(f'(Z))$.
  (ii) If $f \in \mathscr{F}^{\text{str}}_\mathcal{A}(U, {}_\mathcal{A}M)$ **and** $\text{Ann}_\mathcal{A}(N) \subseteq \text{Ann}_\mathcal{A}(M)$, then $\Phi \circ f \in \mathscr{F}^{\text{str}}_\mathcal{A}(U, {}_\mathcal{A}N)$ and $\forall\, Z \in U\colon (\Phi \circ f)'(Z) = \Phi(f'(Z))$.
  (iii) If $f \in \mathscr{F}_\mathcal{A}(U, {}_\mathcal{A}M)$, then $\Phi \circ f \in \mathscr{F}_\mathcal{A}(U, {}_\mathcal{A}N)$ and $\forall\, Z \in U\colon (\Phi \circ f)'(Z) = \Phi(f'(Z))$.

(3) Post-composition with an algebra morphism $\mathcal{B} \xrightarrow{\psi} \mathcal{C}$:
  (i) If $f \in \mathscr{L}_\varphi(U)$, then $\psi \circ f \in \mathscr{L}_{\psi\circ\varphi}(U)$ and $\forall\, Z \in U\colon (\psi \circ f)'(Z) = \psi(f'(Z))$.
  (ii) If $f \in \mathscr{F}^{\text{str}}_\varphi(U)$ **and** $\psi|_{\text{Im}(\varphi)}$ **is injective**, then $\psi \circ f \in \mathscr{F}^{\text{str}}_{\psi\circ\varphi}(U)$ and $\forall\, Z \in U\colon (\psi \circ f)'(Z) = \psi(f'(Z))$.
  (iii) If $f \in \mathscr{F}_\varphi(U)$, then $\psi \circ f \in \mathscr{F}_{\psi\circ\varphi}(U)$ and $\forall\, Z \in U\colon (\psi \circ f)'(Z) = \psi(f'(Z))$.

Proof: To (1): To (i): Denote the corresponding representation by $\rho\colon \mathcal{B} \to \text{End}_\mathbb{K}(M)$ and put

$D \coloneqq \mathcal{A}^{\times}_{\rho\circ\varphi} \cap (U - Z_0)$ and $E \coloneqq \mathcal{B}^{\times}_{\rho} \cap (V - \varphi(Z_0))$ for short. Since $\varphi(D) \subseteq E$, and $g \in \mathscr{C}_\rho(V)$, we have

$$\begin{aligned}&\lim_{\substack{H\to 0\\ H\in D}} \rho(\varphi(H))^{-1}\big((g\circ\varphi)(Z_0+H) - (g\circ\varphi)(Z_0)\big) =\\ &=\lim_{\substack{H\to 0\\ H\in D}} \rho(\varphi(H))^{-1}\big(g(\varphi(Z_0)+\varphi(H)) - g(\varphi(Z_0))\big) =\\ &= \lim_{\substack{K\to 0\\ K\in E}} \rho(K)^{-1}\big(g(\varphi(Z_0)+K) - g(\varphi(Z_0))\big) = g'(\varphi(Z_0)),\end{aligned}$$

where $K \coloneqq \varphi(H) \in E$ and $K \to 0$ as $H \to 0$ by automatic continuity of φ.

To (ii): special case of Proposition 1.1.22 (2) since tautologically $\varphi \in \mathscr{F}^1_\varphi(U)$.

To (iii): special case of Proposition 1.1.22 (4) since tautologically $\varphi \in \mathscr{F}_\varphi(U)$.

To (2): Let $\rho_M\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$ and $\rho_N\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(N)$ denote the corresponding representations.

To (i): Put $D \coloneqq \mathcal{A}^{\times}_{\rho_N} \cap (U - Z_0)$ and $E \coloneqq \mathcal{A}^{\times} \cap (U - Z_0)$ for short and observe that $E \subseteq D$. Using Lemma 1.1.10 we have

$$\begin{aligned}&\lim_{\substack{H\to 0\\ H\in D}} \rho_N(H)^{-1}\big((\Phi\circ f)(Z_0+H) - (\Phi\circ f)(Z_0)\big) = \lim_{\substack{H\to 0\\ H\in E}} \rho_N(H)^{-1}\Phi(f(Z_0+H) - f(Z_0)) =\\ &= \lim_{\substack{H\to 0\\ H\in E}} H^{-1}\cdot\Phi(f(Z_0+H) - f(Z_0)) = \lim_{\substack{H\to 0\\ H\in E}} \Phi\Big(H^{-1}\cdot\big(f(Z_0+H) - f(Z_0)\big)\Big) = \Phi(f'(Z_0)).\end{aligned}$$

since $\Phi \in \mathrm{Hom}_{\mathcal{A}}({}_{\mathcal{A}}M, {}_{\mathcal{A}}N)$ is (automatically) continuous and $f \in \mathscr{L}_{\rho_M}(U)$.

To (ii): Let $\varepsilon > 0$ be arbitrary. Since $\Phi \in \mathrm{Hom}_{\mathcal{A}}(M, N)$, we have

$$\Phi(f(Z_0+H)) = \Phi(f(Z_0)) + H\cdot\Phi(f'(Z_0)) + \Phi(r(H)),$$

where $\exists\,\delta = \delta(\varepsilon) > 0\ \forall\,\|H\|_{\mathcal{A}} < \delta\colon\ \|r(H)\|_M \le \varepsilon\,\|\rho_M(H)\|_{\mathrm{End}_{\mathbb{K}}(M)}$. Since $\mathrm{Ann}_{\mathcal{A}}(N) \subseteq \mathrm{Ann}_{\mathcal{A}}(M)$, i.e. $\mathrm{Ker}(\rho_N) \subseteq \mathrm{Ker}(\rho_M)$, we have that $\exists\, C > 0\ \forall\, H \in \mathcal{A}\colon\ \|\rho_M(H)\|_{\mathrm{End}_{\mathbb{K}}(M)} \le C\,\|\rho_N(H)\|_{\mathrm{End}_{\mathbb{K}}(N)}$. Without loss of generality $\Phi \ne 0$ and define $\delta_1(\varepsilon) \coloneqq \delta(\frac{\varepsilon}{C\|\Phi\|_{\mathrm{op}}})$. Then

$$\forall\,\|H\|_{\mathcal{A}} \le \delta_1(\varepsilon)\colon\ \|\Phi(r(H))\|_N \le \|\Phi\|_{\mathrm{op}}\,\|r(H)\|_M \le \|\Phi\|_{\mathrm{op}}\,\frac{\varepsilon}{C\,\|\Phi\|_{\mathrm{op}}}\,\|\rho_M(H)\|_{\mathrm{End}} \le \varepsilon\,\|\rho_N(H)\|_{\mathrm{End}}$$

since Φ is (automatically) bounded.

To (iii): Similarly to (ii), let again $\varepsilon > 0$ be arbitrary and consider $\Phi(r(H))$, where this time $\exists\,\delta = \delta(\varepsilon) > 0\ \forall\,\|H\|_{\mathcal{A}} < \delta\colon\ \|r(H)\|_M \le \varepsilon\,\|H\|_{\mathcal{A}}$. We have $\forall\,\|H\|_{\mathcal{A}} < \delta(\varepsilon/\|\Phi\|_{\mathrm{op}})\colon\ \|\Phi(r(H))\|_N \le \|\Phi\|_{\mathrm{op}}\,\|r(H)\|_M \le \|\Phi\|_{\mathrm{op}}\,\frac{\varepsilon}{\|\Phi\|_{\mathrm{op}}}\,\|H\|_{\mathcal{A}} = \varepsilon\,\|H\|_{\mathcal{A}}$ as desired.

To (3): To (i): Put $D \coloneqq \mathcal{A}^{\times}_{\psi\circ\varphi} \cap (U - Z_0)$ and $E \coloneqq \mathcal{A}^{\times}_{\varphi} \cap (U - Z_0)$ for short and observe that $E \subseteq D$. Using Lemma 1.1.10, we obtain

$$\begin{aligned}\lim_{\substack{H\to 0\\ H\in D}} \frac{(\psi\circ f)(Z_0+H) - (\psi\circ f)(Z_0)}{\psi(\varphi(H))} &= \lim_{\substack{H\to 0\\ H\in E}} \frac{\psi(f(Z_0+H) - f(Z_0))}{\psi(\varphi(H))} =\\ &= \lim_{\substack{H\to 0\\ H\in E}} \psi\bigg(\frac{f(Z_0+H) - f(Z_0)}{\varphi(H)}\bigg) = \psi(f'(Z_0))\end{aligned}$$

since $\varphi(H) \in \mathcal{B}^{\times}$ now, ψ (automatically) continuous, and $f \in \mathscr{L}_\varphi(U)$.

To (ii): Write $f(Z_0+H) = f(Z_0) + \varphi(H)f'(Z_0) + r(H)$, where $\forall\,\varepsilon > 0\ \exists\,\delta = \delta(\varepsilon) > 0\ \forall\,\|H\|_{\mathcal{A}} < \delta\colon\ \|r(H)\|_{\mathcal{B}} \le \varepsilon\,\|\varphi(H)\|_{\mathcal{B}}$. We need to show that $\|\psi(r(H))\|_{\mathcal{C}} = o(\|(\psi\circ\varphi)(H)\|_{\mathcal{C}})$ as $H \to 0$. Since $\psi_{\mathrm{Im}(\varphi)}$ is injective, we have $\mathrm{Ker}(\psi\circ\varphi) \subseteq \mathrm{Ker}(\varphi)$, hence by Lemma 1.1.18 $\exists\, C > 0\ \forall\, H \in$

$\mathcal{A}$: $\|\varphi(H)\|_{\mathcal{B}} \leq C\,\|(\psi\circ\varphi)(H)\|_{\mathcal{C}}$. Let $\varepsilon > 0$ arbitrary. We get

$$\forall\,\|H\|_{\mathcal{A}} \leq \delta(\varepsilon/C)\colon\ \|\psi(r(H))\|_{\mathcal{C}} \leq \|r(H)\|_{\mathcal{B}} \leq \varepsilon\,\|\varphi(H)\|_{\mathcal{B}} \leq \varepsilon\,\|(\psi\circ\varphi)(H)\|_{\mathcal{C}}$$

as required.

To (iii): special case of Proposition 1.1.22 (4) since tautologically $\psi \in \mathscr{F}_{\psi}(\mathcal{B})$. □

Remark: Once again the "middle" regularity class $\mathscr{F}^{\mathrm{str}}$ requires an additional compatibility condition in order to satisfy the desirable properties the other two classes do. □

We now establish the connection between the coordinate-free ρ-derivative $f'(Z)$ and the Jacobian matrix $(J_{\mathbb{K}}f)(Z)$. Fixing $\mathbb{K}$-bases $\mathcal{A} \cong \mathbb{K}^n$ and $M \cong \mathbb{K}^m$, the regular representation $\rho_{\mathcal{A}}\colon \mathcal{A} \hookrightarrow \mathrm{End}_{\mathbb{K}}(\mathcal{A})$, $A \mapsto (\hat{A}\colon a \mapsto A\cdot a)$, becomes a map $\rho_{\mathcal{A}}\colon \mathcal{A} \hookrightarrow \mathrm{M}_n(\mathbb{K})$ and the $\mathcal{A}$-module "regular representation" $\Upsilon_M\colon M \hookrightarrow \mathrm{Hom}_{\mathbb{K}}(\mathcal{A}, M)$, $v \mapsto (\hat{v}\colon a \mapsto a\cdot v)$, becomes a map $\Upsilon_M\colon M \hookrightarrow \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^m) = \mathbb{K}^{m\times n}$. In the special case ${}_{\mathcal{A}}M \coloneqq {}_{\mathcal{A}}\mathcal{B} \cong \mathbb{K}^m$ given by a morphism $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ in $\mathsf{FdCAlg}_{\mathbb{K}}$, the " regular module representation" $\Upsilon_{\mathcal{B}}\colon \mathcal{B} \hookrightarrow \mathrm{Hom}_{\mathbb{K}}(\mathcal{A}, \mathcal{B}) \cong \mathbb{K}^{m\times n}$ is given by $b \mapsto \rho_{\mathcal{B}}(b)\circ\varphi$ (not to be confused with $\rho_{\varphi} \overset{\mathrm{def}}{=} \rho_{\mathcal{B}}\circ\varphi$) with $\varphi\colon \mathbb{K}^n \to \mathbb{K}^m$ understood.

Lemma 1.1.24 (ρ-Derivative vs. Jacobian of f)**:** With the above notations let $Z_0 \in \mathcal{A}$.

(1) Representation Setting: if $f\colon \mathcal{A} \to {}_{\mathcal{A}}M$ is ρ-differentiable at Z_0, then

$$\Upsilon_M(f'(Z_0)) = (J_{\mathbb{K}}f)(Z_0) \in \mathbb{K}^{m\times n}. \tag{1.1.19}$$

(2) Relative Setting: if $f\colon \mathcal{A} \to \mathcal{B}$ is φ-differentiable at Z_0, then

$$\Upsilon_{\mathcal{B}}(f'(Z_0)) = \rho_{\mathcal{B}}(f'(Z_0))\circ\varphi = (J_{\mathbb{K}}f)(Z_0) \in \mathbb{K}^{m\times n}. \tag{1.1.20}$$

(3) Absolute Setting: if $f\colon \mathcal{A} \to \mathcal{A}$ is $\mathcal{A}$-differentiable at Z_0, then

$$\rho_{\mathcal{A}}(f'(Z_0)) = (J_{\mathbb{K}}f)(Z_0) \in \mathrm{M}_n(\mathbb{K}). \tag{1.1.21}$$

Proof: We will only prove (1) since (2) follows from (1) and (3) is a special case of (2). Let $\{a_1, \ldots, a_n\}$ be a $\mathbb{K}$-basis for $\mathcal{A}$ and $\{v_1, \ldots, v_m\}$ be a $\mathbb{K}$-basis for ${}_{\mathcal{A}}M$. Accordingly, write $Z = z^1a_1 + \cdots + z^na_n$ and $f = f^1v_1 + \cdots + f^mv_m$ for the respective coordinates. We have

$$\forall\, 1 \leq j \leq n\colon a_j\cdot f'(Z_0) \overset{\mathrm{def}}{=} \frac{\partial f}{\partial z^j}(Z_0) = \frac{\partial}{\partial z^j}\Big|_{Z_0}\sum_{k=1}^{m} f^kv_k = (v_1, \ldots, v_m)\cdot\frac{\partial}{\partial z^j}\Big|_{Z_0}\begin{pmatrix} f^1\\ \vdots\\ f^m\end{pmatrix}$$

Thus, putting them all together, we obtain $\Upsilon_M(f'(Z_0))(a_1, \ldots, a_n) = (v_1, \ldots, v_m)\cdot(J_{\mathbb{K}}f)(Z_0)$, i.e. $(J_{\mathbb{K}}f)(Z_0)$ is the representing matrix of $\Upsilon_M(f'(Z_0))$. □

Examples (Commutative Associative Structures on $\mathbb{R}^2$)**:**

(1) Complex Numbers: The regular representation of $\mathbb{C} \overset{\mathrm{def}}{=} \mathbb{R}[X]/(X^2+1)$ with respect to the standard basis is given by

$$\rho_{\mathbb{C}}\colon \mathbb{C} \hookrightarrow \mathrm{M}_2(\mathbb{R}),\ a+ib \mapsto \begin{pmatrix} a & -b\\ b & a\end{pmatrix}.$$

Hence, if $f = f(x,y) = u(x,y) + iv(x,y)$ is complex-differentiable at $z_0 \in \mathbb{C}$, then

$$(J_{\mathbb{R}}f)(z_0) = \begin{pmatrix} \frac{\partial u}{\partial x}(z_0) & \frac{\partial u}{\partial y}(z_0)\\ \frac{\partial v}{\partial x}(z_0) & \frac{\partial v}{\partial y}(z_0)\end{pmatrix} = \rho_{\mathbb{C}}(f'(z_0)) = \begin{pmatrix} a & -b\\ b & a\end{pmatrix},$$

which in turn immediately implies the Cauchy–Riemann equations at $z_0 \in \mathbb{C}$.

(2) Hyperbolic Numbers: The regular representation of $\mathbb{B} \coloneqq \mathbb{B}_{\mathbb{R}} \overset{\mathrm{def}}{=} \mathbb{R}[X]/(X^2-1)$ with respect

to the standard basis is given by

$$\rho_{\mathbb{B}}\colon \mathbb{B} \hookrightarrow \mathrm{M}_2(\mathbb{R}),\ a + jb \mapsto \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

Hence, if $f = f(x, y) = u(x, y) + jv(x, y)$ is $\mathbb{B}$-differentiable at $z_0 \in \mathbb{B}$, then

$$(J_{\mathbb{R}}f)(z_0) = \begin{pmatrix} \frac{\partial u}{\partial x}(z_0) & \frac{\partial u}{\partial y}(z_0) \\ \frac{\partial v}{\partial x}(z_0) & \frac{\partial v}{\partial y}(z_0) \end{pmatrix} = \rho_{\mathbb{B}}(f'(z_0)) = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

which in turn immediately implies the following Cauchy–Riemann(–Scheffers) PDE system at $z_0 \in \mathbb{B}$ for f:

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0),\ \frac{\partial v}{\partial x}(z_0) = \frac{\partial u}{\partial y}(z_0).$$

(3) (Real) Dual Numbers: The regular representation of $\mathbb{D} := \mathbb{D}_{\mathbb{R}} := \mathbb{D}\mathbb{R} := \mathbb{R}[X]/(X^2) = \mathbb{R}[\epsilon]$, $\epsilon \neq 0$, $\epsilon^2 = 0$, with respect to the standard basis is given by

$$\rho_{\mathbb{D}}\colon \mathbb{D} \hookrightarrow \mathrm{M}_2(\mathbb{R}),\ a + ib \mapsto \begin{pmatrix} a & 0 \\ b & a \end{pmatrix}.$$

Hence, if $f = f(x, y) = u(x, y) + iv(x, y)$ is $\mathbb{D}$-differentiable at $z_0 \in \mathbb{D}$, then

$$(J_{\mathbb{R}}f)(z_0) = \begin{pmatrix} \frac{\partial u}{\partial x}(z_0) & \frac{\partial u}{\partial y}(z_0) \\ \frac{\partial v}{\partial x}(z_0) & \frac{\partial v}{\partial y}(z_0) \end{pmatrix} = \rho_{\mathbb{D}}(f'(z_0)) = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix},$$

which in turn immediately implies the following Cauchy–Riemann(–Scheffers) PDE system at $z_0 \in \mathbb{B}$ for f:

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0),\ \frac{\partial u}{\partial y}(z_0) = 0$$

In summary, the regular representation of $\mathcal{A}$ provides an easy way to obtain the corresponding generalized Cauchy–Riemann(–Scheffers) equaltions. This will be continued in some more detail in some more detail in Section 1.3. □

We finish the section with the useful observation that $\mathcal{A}$-holomorphic maps behave nicely in the algebraic category as well:

Corollary 1.1.25 (Jacobian Problem for $\mathcal{A}$-holomorphic regular maps)**:** Let $\mathcal{A} = \prod_{j=1}^{N}(\mathcal{A}_j, \mathfrak{m}_j, \mathbb{K})$ and identify $\mathcal{A} \cong \mathbb{K}^n$ as vector spaces. Then the Jacobian Conjecture holds for $\mathcal{A}$-holomorphic regular maps: if $P\colon \mathbb{K}^n \to \mathbb{K}^n$, $P(z_1, \dots, z_n) = (P_1(z_1, \dots, z_n), \dots, P_n(z_1, \dots, z_n))$, is regular and $\mathcal{A}$-holomorphic with $\det J_{\mathbb{K}}P = \mathrm{const} \neq 0$, then P is biregular.

Proof: Since $\mathcal{A}_j/\mathfrak{m}_j \cong \mathbb{K}$ for all $1 \leq j \leq N$, we can identify $\mathcal{A}$ with its *regular lower-triangular representation*: a change of basis for $\mathcal{A}$ translates into a composition of P with invertible linear maps; in fact, even the value of the constant $\det J_{\mathbb{K}}P$ itself does not change. Now, $P'(Z) = (J_{\mathbb{K}}P)(Z)$ by Lemma 1.1.24, hence $\det P'(Z) = \det(J_{\mathbb{K}}P)(Z)$. Write $P(Z) = (Q_1(Z_1), \dots, Q_N(Z_N))$ with respect to the decomposition of $\mathcal{A}$. Then also $P'(Z) = (Q_1'(Z_1), \dots, Q_N'(Z_N))$, hence

$$\det P'(Z) = \prod_{j=1}^{N} \det Q_j'(Z_j) \in \mathbb{K}^\times = \mathbb{K}[z_1, \dots, z_n]^\times.$$

Therefore it suffices to assume that $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{K})$. Thus,

$$\forall\, 1 \leq i < j \leq n\colon \frac{\partial P_i}{\partial z_j} = 0 \text{ and } \forall\, 1 \leq i \leq n\colon \frac{\partial P_i}{\partial z_i} = \frac{\partial P_1}{\partial z_1},$$

i.e. $\forall\, 1 \leq i \leq n\colon P_i = P_i(z_1, \dots, z_i)$ is a polynomial depending only on the first i variables. Furthermore,

$$0 \neq \text{const} = \det(J_{\mathbb{K}}P)(Z) = \det P'(Z) = \det \frac{\partial P}{\partial z_1}(Z) = \left(\frac{\partial P_1}{\partial z_1}\right)^n \Rightarrow P_1 = \alpha z_1 + \beta$$

for some $\alpha \in \mathbb{K}^\times$ and $\beta \in \mathbb{K}$, and more generally, $P_i = \alpha z_i + p_i(z_1, \ldots, z_{i-1})$ for some $p_i \in \mathbb{K}[z_1, \ldots, z_{i-1}]$. Setting $w_i \coloneqq P_i$, one can now write the inverse map recursively as

$$z_1 = \frac{1}{\alpha}(w_1 - \beta),\ z_i = \frac{1}{\alpha}(w_i - p_i(z_1, \ldots, z_{i-1})),\ 2 \leq i \leq n$$

This completes the proof. □

Remarks:

(1) The proof can be easily modified to accomodate the "mixed residue field" case $\mathbb{K}_j \coloneqq \mathcal{A}_j/\mathfrak{m}_j \in \{\mathbb{R}, \mathbb{C}\}$, $1 \leq j \leq N$.

(2) This is a special instance of the well-known case of triangular $J_{\mathbb{K}}P$.

1.2. Canonical Factorizations & Extensions

Recall Definition 0.1.33.

Proposition 1.2.1 (Factorization through epimorphisms)**:** Let $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$ be a representation, $\varphi\colon \mathcal{A} \to \mathcal{B}$ a morphism in $\mathsf{FdCAlg}_{\mathbb{K}}$, $U \subseteq \mathcal{A}$ open, and $k \in \mathbb{N}$.

(1) If $\rho = \bar{\rho} \circ \pi$ for some epimorphism $\pi\colon \mathcal{A} \twoheadrightarrow \bar{\mathcal{A}}$ in $\mathsf{FdCAlg}_{\mathbb{K}}$, then $\forall\, f \in \mathscr{F}^{\mathrm{str},k}_{\rho}(U)\ \exists_1\, \bar{f} \in \mathscr{F}^{\mathrm{str},k}_{\bar{\rho}}(\pi(U))\colon f = \bar{f} \circ \pi$, and in that case $f' = \bar{f}' \circ \pi$. In other words, we have an implication of commutative diagrams:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\rho"] \arrow[d, "\pi"'] & \mathrm{End}_{\mathbb{K}}(M) \\ \bar{\mathcal{A}} \arrow[ur, "\bar{\rho}"'] & \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\pi"'] & M \\ \pi(U) \arrow[ur, dashed, "\exists_1 \bar{f}"'] & \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} U \arrow[r, "f'"] \arrow[d, "\pi"'] & M \\ \pi(U) \arrow[ur, "\bar{f}'"'] & \end{tikzcd}$$

It follows that every $f \in \mathscr{F}^{\mathrm{str},k}_{\rho}(U)$ admits a canonical $\mathscr{F}^{\mathrm{str},k}_{\rho}$-differentiable extension f^{π} to $U^{\pi} \overset{\mathrm{def}}{=} U + \mathrm{Ker}(\pi)$, and we have canonical isomorphisms of left $\mathrm{End}_{\mathcal{A}}(M)$-modules:

$$\begin{gathered} \mathscr{F}^{\mathrm{str},k}_{\rho}(U) \xrightarrow{\cong} \mathscr{F}^{\mathrm{str},k}_{\bar{\rho}}(\pi(U)) \xrightarrow{\cong} \mathscr{F}^{\mathrm{str},k}_{\rho}(U^{\pi}) \\ f \mapsto \bar{f} \mapsto f^{\pi} := \pi^*(\bar{f}) \overset{\mathrm{def}}{=} \bar{f} \circ \pi. \end{gathered} \tag{1.2.1}$$

(2) Under the hypothesis of (1), if additionally $f \in \mathscr{L}^{k}_{\rho}(U)$, then in fact $\bar{f} \in \mathscr{L}^{k}_{\bar{\rho}}(\pi(U))$. It follows that every $f \in \mathscr{L}^{k}_{\rho}(U)$ admits a canonical $\mathscr{L}^{k}_{\rho}$-differentiable extension f^{π} to $U^{\pi} \overset{\mathrm{def}}{=} U + \mathrm{Ker}(\pi)$, and we have canonical isomorphisms of left $\mathrm{End}_{\mathcal{A}}(M)$-modules:

$$\begin{gathered} \mathscr{L}^{k}_{\rho}(U) \xrightarrow{\cong} \mathscr{L}^{k}_{\bar{\rho}}(\pi(U)) \xrightarrow{\cong} \mathscr{L}^{k}_{\rho}(U^{\pi}) \\ f \mapsto \bar{f} \mapsto f^{\pi} := \pi^*(\bar{f}). \end{gathered} \tag{1.2.2}$$

(3) If $\varphi = \bar{\varphi} \circ \pi$ for some epimorphism $\pi\colon \mathcal{A} \twoheadrightarrow \bar{\mathcal{A}}$ in $\mathsf{FdCAlg}_{\mathbb{K}}$, then $\forall\, f \in \mathscr{F}^{\mathrm{str},k}_{\varphi}(U)\ \exists_1\, \bar{f} \in \mathscr{F}^{\mathrm{str},k}_{\bar{\varphi}}(\pi(U))\colon f = \bar{f} \circ \pi$ and $\forall\, f \in \mathscr{L}^{k}_{\varphi}(U)\ \exists_1\, \bar{f} \in \mathscr{L}^{k}_{\bar{\varphi}}(\pi(U))\colon f = \bar{f} \circ \pi$. In that case $f' = \bar{f}' \circ \pi$. In other words, we have an implication of commutative diagrams:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \mathcal{B} \\ \bar{\mathcal{A}} \arrow[ur, "\bar{\varphi}"'] & \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\pi"'] & \mathcal{B} \\ \pi(U) \arrow[ur, dashed, "\exists_1 \bar{f}"'] & \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} U \arrow[r, "f'"] \arrow[d, "\pi"'] & \mathcal{B} \\ \pi(U) \arrow[ur, "\bar{f}'"'] & \end{tikzcd}$$

It follows that every $f \in \mathscr{F}^{\mathrm{str},k}_{\varphi}(U)$ or $\mathscr{L}^{k}_{\varphi}(U)$ admits a canonical $\mathscr{F}^{\mathrm{str},k}_{\varphi}$- or respectively $\mathscr{L}^{k}_{\varphi}$-differentiable extension f^{π} to $U^{\pi} \overset{\mathrm{def}}{=} U + \mathrm{Ker}(\pi)$, and we have canonical isomorphisms of $\mathcal{B}$-algebras:

$$\begin{gathered} \mathscr{F}^{\mathrm{str},k}_{\varphi}(U) \xrightarrow{\cong} \mathscr{F}^{\mathrm{str},k}_{\bar{\varphi}}(\pi(U)) \xrightarrow{\cong} \mathscr{F}^{\mathrm{str},k}_{\varphi}(U^{\pi}) \\ \mathscr{L}^{k}_{\varphi}(U) \xrightarrow{\cong} \mathscr{L}^{k}_{\bar{\varphi}}(\pi(U)) \xrightarrow{\cong} \mathscr{L}^{k}_{\varphi}(U^{\pi}) \\ f \mapsto \bar{f} \mapsto f^{\pi} := \pi^*(\bar{f}). \end{gathered} \tag{1.2.3}$$

(4) If $\varphi\colon \mathcal{A} \twoheadrightarrow \mathcal{B}$ itself epic, then $\forall\, f \in \mathscr{F}^{\mathrm{str},k}_{\varphi}(U)$ or $\mathscr{L}^{k}_{\varphi}(U)\ \exists_1\, \bar{f} \in \mathscr{F}^{\mathrm{str},k}_{\mathcal{B}}(\varphi(U))$ or $\mathscr{L}^{k}_{\mathcal{B}}(\varphi(U))\colon f = \bar{f} \circ \varphi$, and in that cae $f' = \bar{f}' \circ \varphi$. Moreover, f admits a canonical $\mathscr{F}^{\mathrm{str},k}_{\varphi}$-, respectively $\mathscr{L}^{k}_{\varphi}$-differentiable extension f^{φ} to $U^{\varphi} \overset{\mathrm{def}}{=} U + \mathrm{Ker}(\varphi)$, and we have canonical isomorphisms of $\mathcal{B}$-algebras:

$$\begin{gathered} \mathscr{F}^{\mathrm{str},k}_{\varphi}(U) \xrightarrow{\cong} \mathscr{F}^{\mathrm{str},k}_{\mathcal{B}}(\varphi(U)) \xrightarrow{\cong} \mathscr{F}^{\mathrm{str},k}_{\varphi}(U^{\varphi}) \\ \mathscr{L}^{k}_{\varphi}(U) \xrightarrow{\cong} \mathscr{L}^{k}_{\mathcal{B}}(\varphi(U)) \xrightarrow{\cong} \mathscr{L}^{k}_{\varphi}(U) \\ f \mapsto \bar{f} \mapsto f^{\varphi} := \varphi^*(\bar{f}). \end{gathered} \tag{1.2.4}$$

Proof: It suffices to prove the claims for $k = 1$.

To (1): If $\rho\colon \mathcal{A} \overset{\pi}{\twoheadrightarrow} \bar{\mathcal{A}} \overset{\bar{\rho}}{\to} \operatorname{End}_{\mathbb{K}}(M)$ is such a decomposition, then $\bar{\mathcal{A}} \cong \mathcal{A}/\mathfrak{a}$, where $\mathfrak{a} := \operatorname{Ker}\pi \subseteq \operatorname{Ker}\rho$. It is illuminating to first consider the stronger regularity class $\mathscr{C}^1_\rho = \mathscr{F}^{\mathrm{str}}_\rho \cap \mathscr{C}^1$ for f.

Fix $Z_0 \in U$ to be an arbitrary point. Since $\mathcal{A}$ is automatically locally convex, there exists $\delta_0 > 0$ such that $\forall \|H\|_{\mathcal{A}} < \delta_0\colon [Z_0, Z_0 + H] \subseteq U$. Thus, if $f \in \mathscr{C}^1_\rho(U)$, then by Corollary 1.1.13 we have that $\forall H \in \mathbb{B}_{\mathcal{A}}(0, \delta_0) \cap \mathfrak{a}\colon f(Z_0 + H) = f(Z_0)$ as $\mathfrak{a} \subseteq \operatorname{Ker}\rho$. Since f is $\mathscr{C}^1_\rho$ everywhere on U, it follows by successive translations by sufficiently small $H \in \mathfrak{a}$ that $\forall H \in (U - Z_0) \cap \mathfrak{a}\colon f(Z_0 + H) = f(Z_0)$. In other words, if $Z \equiv W \pmod{\mathfrak{a}}$ for some $Z, W \in U$, then also $f(Z) = f(W)$. Therefore f induces a well-defined function $\bar{f}\colon \pi(U) \to M$, $\bar{f}(\bar{Z}) := f(Z)$, where observe that $\pi(U)$ itself is open since π is an open map by virtue of being a projection.

In fact, to show the existence of $\bar{f}$ we don't need to argue by way of integration and don't need $f \in \mathscr{C}^1_\rho(U)$, but only $f \in \mathscr{F}^{\mathrm{str}}_\rho(U)$. Indeed, since f is ρ-differentiable at Z_0, we have $f(Z_0 + H) = f(Z_0) + \rho(H)f'(Z_0) + r(H)$, where $r \in \mathscr{C}(U - Z_0, M)$ (since f and ρ are continuous) such that $\forall \varepsilon > 0\, \exists\, \delta = \delta(\varepsilon) > 0\, \forall \|H\|_{\mathcal{A}} < \delta(\varepsilon)\colon \|r(H)\|_M \le \varepsilon \|\rho(H)\|_{\mathrm{End}}$. Fixing some $\varepsilon_0 > 0$ and taking $\delta_0 := \delta(\varepsilon_0)$, we therefore have $\forall H \in \mathbb{B}_{\mathcal{A}}(0, \delta_0) \cap \mathfrak{a}\colon \|r(H)\|_M \le \varepsilon_0 \|\rho(H)\|_{\mathrm{End}} = 0$, hence again $\forall H \in \mathbb{B}_{\mathcal{A}}(0, \delta_0) \cap \mathfrak{a}\colon f(Z_0 + H) = f(Z_0) + \rho(H)f'(Z_0) + r(H) = f(Z_0)$.

Either way, we still need to prove $\bar{f} \in \mathscr{F}^{\mathrm{str}}_{\bar{\rho}}(\pi(U))$, though. To this end we show first that $\bar{r}\colon \pi(U - Z_0) = \pi(U) - \bar{Z}_0 \to M$, $\bar{H} \mapsto r(H)$, is a well-defined function. Indeed, if we fix $H_0 \in U - Z_0$, then as before $\exists\, \delta_0 > 0\, \forall K \in \mathbb{B}_{\mathcal{A}}(0, \delta_0) \cap \mathfrak{a}\colon H_0 + K \in U - Z_0$ and $r(H_0 + K) \overset{\mathrm{def}}{=} f(Z_0 + H_0 + K) - f(Z_0) - \rho(H_0 + K)f'(Z_0) = f(Z_0 + H_0) - f(Z_0) - \rho(H_0)f'(Z_0) \overset{\mathrm{def}}{=} r(H_0)$, hence $\bar{r}$ is well-defined. Thus:

$$\begin{aligned} \forall H \in U - Z_0\colon \bar{f}(\bar{Z}_0 + \bar{H}) &\overset{\mathrm{def}}{=} f(Z_0 + H) = f(Z_0) + \rho(H)f'(Z_0) + r(H) = \\ &= \bar{f}(\bar{Z}_0) + \bar{\rho}(\bar{H})f'(Z_0) + \bar{r}(\bar{H}) \end{aligned} \tag{1.2.5}$$

Now let $\varepsilon > 0$ be arbitrary. Since $\pi(\mathbb{B}_{\mathcal{A}}(0, \delta(\varepsilon))) \ni 0$ is open, there exists $\bar{\delta} = \bar{\delta}(\varepsilon) > 0$ such that $\pi(\mathbb{B}_{\mathcal{A}}(0, \delta(\varepsilon))) \supseteq \mathbb{B}_{\bar{\mathcal{A}}}(0, \bar{\delta}(\varepsilon))$. In particular, every class $\bar{H} \in \bar{\mathcal{A}}$ with $\|\bar{H}\|_{\bar{\mathcal{A}}} < \bar{\delta}(\varepsilon)$ is represented by $H \in \mathcal{A}$ with $\|H\|_{\mathcal{A}} < \delta(\varepsilon)$. The claim now follows by the $\mathfrak{a}$-periodicity of $r(H)$ and $\rho(H)$. More precisely:

$$\forall \|\bar{H}\|_{\bar{\mathcal{A}}} < \bar{\delta}(\varepsilon)\colon \|\bar{r}(\bar{H})\|_M \overset{\mathrm{def}}{=} \|r(H)\|_M \le \varepsilon \|\rho(H)\|_{\mathrm{End}} = \varepsilon \|\bar{\rho}(\bar{H})\|_{\mathrm{End}},$$

i.e. $\|\bar{r}(\bar{H})\|_M = o(\|\bar{\rho}(\bar{H})\|_{\mathrm{End}})$ as $\bar{H} \to 0$, which proves that $\bar{f} \in \mathscr{F}^{\mathrm{str}}_{\bar{\rho}}(\pi(U))$, and Eq. (1.2.5) shows that $\bar{f}'(\bar{Z}_0) = f'(Z_0)$. We can summarize this schematically via the following implication of commutative diagrams:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\rho"] \arrow[d, "\pi"'] & \operatorname{End}_{\mathbb{K}}(M) \\ \bar{\mathcal{A}} \arrow[ur, dashed, "\bar{\rho}"'] & \end{tikzcd} \quad\Longrightarrow\quad \begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\pi"'] & M \\ \pi(U) \arrow[ur, dashed, "\exists_1 \bar{f}"'] & \end{tikzcd} \quad\Longrightarrow\quad \begin{tikzcd} U \arrow[r, "f'"] \arrow[d, "\pi"'] & M \\ \pi(U) \arrow[ur, "\bar{f}'"'] & \end{tikzcd}$$

By surjectivity of π, $\bar{f}$ is indeed the unique function with this property, and we obtain a homomorphism of left $\operatorname{End}_{\mathcal{A}}(M)$-modules

$$\begin{aligned} \Phi\colon \mathscr{F}^{\mathrm{str}}_\rho(U) &\to \mathscr{F}^{\mathrm{str}}_{\bar{\rho}}(\pi(U)) \\ f &\mapsto \bar{f}. \end{aligned}$$

Conversely, by Proposition 1.1.23 we have a well-defined homomorphism of left $\operatorname{End}_{\mathcal{A}}(M)$-modules

$$\begin{aligned} \Psi\colon \mathscr{F}^{\mathrm{str}}_{\bar{\rho}}(\pi(U)) &\to \mathscr{F}^{\mathrm{str}}_\rho(U) \\ g &\mapsto g \circ \pi|_U. \end{aligned}$$

Clearly, $\Psi(\Phi(f)) = f$, and $\Phi(\Psi(g)) = g$ follows from the uniqueness property / surjectivity of π, hence $\mathscr{F}_\rho^{\mathrm{str}}(U) \cong \mathscr{F}_{\bar\rho}^{\mathrm{str}}(\pi(U))$. Redefining Ψ as

$$\begin{aligned}\Psi\colon \mathscr{F}_{\bar\rho}^{\mathrm{str}}(\pi(U)) &\to \mathscr{F}_\rho^{\mathrm{str}}(U^\pi)\\ g &\mapsto g\circ\pi,\end{aligned}$$

we obtain similarly $\mathscr{F}_\rho^{\mathrm{str}}(U^\pi) \cong \mathscr{F}_{\bar\rho}^{\mathrm{str}}(\pi(U))$. In particular, every $f \in \mathscr{F}_\rho^{\mathrm{str}}(U)$ has a unique $\mathscr{F}_\rho^{\mathrm{str}}$-differentiable extension to U^π.

To (2): Since $\mathscr{L}_\rho(U) \subseteq \mathscr{F}_\rho^{\mathrm{str}}(U)$, it follows from (1) that there indeed exists a unique $f \in \mathscr{F}_{\bar\rho}^{\mathrm{str}}(\pi(U))$ such that $f = \bar f\circ\pi$. We want to show that $\bar f \in \mathscr{L}_{\bar\rho}(\pi(U))$. To this end, fix $\bar Z_0 \in \pi(U)$ and let $(\bar H_n)_{n\in\mathbb{N}} \subset \bar{\mathcal{A}}_{\bar\rho}^\times$ be arbitrary with $\bar H_n \to 0_{\bar{\mathcal{A}}}$ as $n\to\infty$, i.e. $\left\|\bar H_n\right\|_{\bar{\mathcal{A}}} \to 0$. We claim that $(\bar H_n)_{n\in\mathbb{N}}$ lifts to a sequence $(H_n)_{n\in\mathbb{N}} \subset \mathcal{A}_\rho^\times$ with $H_n \to 0_{\mathcal{A}}$ as $n\to\infty$. Indeed, $\left\|\bar H_n\right\|_{\bar{\mathcal{A}}} \overset{\mathrm{def}}{=} \inf\limits_{H\in\bar H_n} \|H\|_{\mathcal{A}} = \mathrm{dist}_{\mathcal{A}}(\bar H_n, 0_{\mathcal{A}})$ with $\bar H_n \overset{\mathrm{def}}{=} H_n + \mathfrak{a} \subseteq \mathcal{A}$ being a closed subset of the finite-dimensional $\mathbb{K}$-vector space $\mathcal{A}$, hence the infimum is attained by some $H_n \in \bar H_n$ and $\|H_n\|_{\mathcal{A}} = \left\|\bar H_n\right\|_{\bar{\mathcal{A}}} \to 0$, i.e. $H_n \to 0$ as $n\to\infty$. It remains to show that $H_n \in \mathcal{A}_\rho^\times$. But this is immediate from $\mathcal{A}_\rho^\times = \pi^{-1}(\bar{\mathcal{A}}_{\bar\rho}^\times)$. Keeping the notations from the proof of (1), we now have

$$\lim_{n\to\infty} \bar\rho(\bar H_n)^{-1}\bar r(\bar H_n) = \lim_{n\to\infty} \rho(H_n)^{-1} r(H_n) = 0$$

since f is $\mathscr{L}_\rho$-differentiable at $Z_0 \in U$. As $(\bar H_n)_{n\in\mathbb{N}} \subset \bar{\mathcal{A}}_{\bar\rho}^\times$ was arbitrary, this shows that $\bar f$ is $\mathscr{L}_{\bar\rho}$-differentiable at $\bar Z_0$.

To (3): follows from (1) and (2) since $\mathscr{F}_\varphi^{\mathrm{str}} = \mathscr{F}_{\rho_\varphi}^{\mathrm{str}}$ and $\mathscr{L}_\varphi = \mathscr{L}_{\rho_\varphi}$.

To (4): special case of (3) with $\pi \coloneqq \varphi$ and $\bar\varphi \coloneqq \mathrm{id}_{\mathcal{B}}$. □

Corollary 1.2.2: Let $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$ be a representation, $\varphi\colon \mathcal{A}\to\mathcal{B}$ a morphism in $\mathsf{FdCAlg}_{\mathbb{K}}$, $U \subseteq \mathcal{A}$ open, and $k \in \mathbb{N}\cup\{\infty,\omega\}$.

(1) If $\rho = \bar\rho\circ\pi$ for some epimorphism $\pi\colon \mathcal{A} \twoheadrightarrow \bar{\mathcal{A}}$ in $\mathsf{FdCAlg}_{\mathbb{K}}$, then $\forall f \in \mathscr{C}_\rho^k(U)\ \exists_1 \bar f \in \mathscr{C}_{\bar\rho}^k(\pi(U))\colon f = \bar f\circ\pi$, in which case $f' = \bar f'\circ\pi$, and we have canonical isomorphisms of left $\mathrm{End}_{\mathcal{A}}(M)$-modules:

$$\begin{aligned}\mathscr{C}_\rho^k(U) \xrightarrow{\cong} \mathscr{C}_{\bar\rho}^k(\pi(U)) &\xrightarrow{\cong} \mathscr{C}_\rho^k(U^\pi)\\ f \mapsto \bar f &\mapsto f^\pi \coloneqq \pi^*(\bar f) \overset{\mathrm{def}}{=} \bar f\circ\pi.\end{aligned} \tag{1.2.6}$$

For $k \in \{\infty,\omega\}$ these are isomorphisms of *differential* left $\mathrm{End}_{\mathcal{A}}(M)$-modules.

(2) If $\varphi = \bar\varphi\circ\pi$ for some epimorphism $\pi\colon \mathcal{A} \twoheadrightarrow \bar{\mathcal{A}}$ in $\mathsf{FdCAlg}_{\mathbb{K}}$, then $\forall f \in \mathscr{C}_\varphi^k(U)\ \exists_1 \bar f \in \mathscr{C}_{\bar\varphi}^k(\pi(U))\colon f = \bar f\circ\pi$, in which case $f' = \bar f'\circ\pi$, and we have canonical isomorphisms of $\mathcal{B}$-algebras:

$$\begin{aligned}\mathscr{C}_\varphi^k(U) \xrightarrow{\cong} \mathscr{C}_{\bar\varphi}^k(\pi(U)) &\xrightarrow{\cong} \mathscr{C}_\varphi^k(U^\pi)\\ f \mapsto \bar f &\mapsto f^\pi \coloneqq \pi^*(\bar f).\end{aligned} \tag{1.2.7}$$

For $k \in \{\infty,\omega\}$ these are isomorphisms of *differential* $\mathcal{B}$-algebras.

(3) If $\varphi\colon \mathcal{A} \twoheadrightarrow \mathcal{B}$ itself epic, then $\forall f \in \mathscr{C}_\varphi^k(U)\ \exists_1 \bar f \in \mathscr{C}_{\mathcal{B}}^k(\varphi(U))\colon f = \bar f\circ\varphi$, in which case $f' = \bar f'\circ\varphi$, and we have canonical isomorphisms of $\mathcal{B}$-algebras:

$$\begin{aligned}\mathscr{C}_\varphi^k(U) \xrightarrow{\cong} \mathscr{C}_{\mathcal{B}}^k(\varphi(U)) &\xrightarrow{\cong} \mathscr{C}_\varphi^k(U^\varphi)\\ f \mapsto \bar f &\mapsto f^\varphi \coloneqq \varphi^*(\bar f).\end{aligned} \tag{1.2.8}$$

For $k \in \{\infty,\omega\}$ these are isomorphisms of *differential* $\mathcal{B}$-algebras. □

Remarks:

(1) Factorization property does not appear to hold for the top regularity class $\mathscr{F}_\rho$, though. Even though $\mathscr{F}_\rho^{\mathrm{str}}$ previously behaved worse than its counter-parts, it is clear from the proof that

it is the conceptually "correct" choice for Proposition 1.2.1.

(2) In the proof of (2) it was convenient that we had allowed for the more general dense subset $\mathcal{A}_\rho^\times \subset \mathcal{A}$ in Definition 1.1.1 as $\pi^{-1}(\bar{\mathcal{A}}^\times)$ can be bigger than $\mathcal{A}^\times$.

(3) In the case of epic $\mathcal{A} \overset{\varphi}{\twoheadrightarrow} \mathcal{B}$ (hence $\dim_{\mathbb{K}} \mathcal{B} \leq \dim_{\mathbb{K}} \mathcal{A}$) Proposition 1.2.1 thus gives an affirmative answer to the lifting question stated previously.

(4) Canonical reduction to faithful representations: we have

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\rho"] \arrow[d, "\rho^{\mathrm{im}}"'] & \mathrm{End}_{\mathbb{K}}(M) \\ \rho(\mathcal{A}) \arrow[ur, hook, dashed, "\lambda"'] & \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\rho"'] & M \\ \rho(U) \arrow[ur, dashed, "\exists_1 \bar{f}"'] & \end{tikzcd}$$

where λ is the canonical faithful representation (inclusion) of the image $\rho(\mathcal{A})$ and $\bar{f} \in \mathscr{F}_\lambda^{\mathrm{str}}(\rho(U))$. Up to isomorphism, this is of course the same as:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\rho"] \arrow[d, "\pi"'] & \mathrm{End}_{\mathbb{K}}(M) \\ \mathcal{A}/\operatorname{Ann}_{\mathcal{A}}(M) \arrow[ur, hook, dashed, "\lambda"'] & \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\pi"'] & M \\ \pi(U) \arrow[ur, dashed, "\exists_1 \bar{f}"'] & \end{tikzcd}$$

It follows from Proposition 1.2.1 that f extends (classically or strictly) ρ-differentiably to

$$f^\rho \colon U^\rho \to M, \tag{1.2.9}$$

where $U^\rho \overset{\mathrm{def}}{=} \rho^{-1}(\rho(U)) \overset{\mathrm{def}}{=} U + \mathrm{Ker}(\rho)$. Finally, observe that since λ is faithful, we have $\mathscr{F}_\lambda^{\mathrm{str}} = \mathscr{F}_\lambda$ by Proposition 1.1.11.

(5) A non-trivial $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$ won't be epic in general due to the non-commutativity of $\mathrm{End}_{\mathbb{K}}(M)$, whereas replacing ρ by the naturally surjective corestriction $\rho^{\mathrm{im}}\colon \mathcal{A} \twoheadrightarrow \rho(\mathcal{A})$ (or any other commutative corestriction) does not recover the representation definition, as already observed. Instead, the better approach would be to treat $\mathscr{F}_\rho$ as a module over a sheaf $\mathscr{F}_\varphi$ of algebras for an appropriate (and a practical) choice of φ, e.g. $\varphi := \rho^{\mathrm{mor}}$. $\square$

Consider now the local case $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{K})$. Then in particular its canonical character $\chi_{\mathcal{A}}\colon \mathcal{A} \twoheadrightarrow \mathbb{K}$, which is of course a morphism in $\mathsf{FdCAlg}_{\mathbb{K}}$, turns the residue field $\mathbb{K}$ into an $\mathcal{A}$-module ${}_{\mathcal{A}}\mathbb{K}$ in the usual way.

Definition 1.2.3 (Spectral Component)**:** Let $(\mathcal{A}, \mathfrak{m}, \mathbb{K})$, $(\mathcal{B}, \mathfrak{n}, \mathbb{K}) \in \mathsf{LocArt}_{\mathbb{K}}$, $U \subseteq \mathcal{A}$ open, and $f\colon U \to \mathcal{B}$ some function. Then

$$f^{\mathrm{sp}} := \chi_{\mathcal{B}} \circ f\colon U \to \mathbb{K}$$

will be called the *spectral component* of f.

Remark: If we take a $\mathbb{K}$-basis $\{b_1 = 1_{\mathcal{B}}, b_2, \dots, b_m\}$ of $\mathcal{B} = (\mathcal{B}, \mathfrak{n}, \mathbb{K})$ and write $f = \sum_{k=1}^m f_k b_k$ accordingly, then clearly $f_1 \overset{\mathrm{def}}{=} f^{\mathrm{sp}}$. $\square$

Corollary 1.2.4: Let $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{K}) \in \mathsf{LocArt}_{\mathbb{K}}$ and $U \subseteq \mathcal{A}$ open. If $f \in \mathscr{L}_{\mathcal{A}}(U)$, then there exists a unique $\bar{f} \in \mathscr{L}_{\mathbb{K}}(U^{\mathrm{sp}}, \mathbb{K})$ such that $f^{\mathrm{sp}} = \bar{f} \circ \chi_{\mathcal{A}}$, $(f^{\mathrm{sp}})' = \bar{f}' \circ \chi_{\mathcal{A}}$, and $f^{\mathrm{sp}} \in \mathscr{L}_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathbb{K})$ extends $\mathscr{L}_{\mathcal{A}}$-differentiably uniquely to $f^{\mathrm{sp}} \in \mathscr{L}_{\mathcal{A}}(\widehat{U}, {}_{\mathcal{A}}\mathbb{K})$. Moreover, we have a canonical isomorphisms

$$\begin{gathered} \mathscr{L}_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathbb{K}) \xrightarrow{\cong} \mathscr{L}_{\mathbb{K}}(U^{\mathrm{sp}}, \mathbb{K}) \xrightarrow{\cong} \mathscr{L}_{\mathcal{A}}(\widehat{U}, {}_{\mathcal{A}}\mathbb{K}) \\ f \mapsto \bar{f} \mapsto \bar{f} \circ \chi_{\mathcal{A}} \end{gathered} \tag{1.2.10}$$

of differential $\mathbb{K}$-algebras.

Proof: Since $f \in \mathscr{L}_{\mathcal{A}}(U)$, it follows from Proposition 1.1.23 that $f^{\mathrm{sp}} \in \mathscr{L}_{\chi_{\mathcal{A}}}(U) \overset{\mathrm{def}}{=} \mathscr{L}_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathbb{K})$. Since

$\chi_{\mathcal{A}}$ is epic, it further follows from Proposition 1.2.1 that $f^{\mathrm{sp}} = \bar{f} \circ \chi_{\mathcal{A}}$ for a unique $\bar{f} \in \mathscr{L}_{\mathbb{K}}(U^{\mathrm{sp}}, \mathbb{K})$ with f^{sp} extending $\mathscr{L}_{\chi_{\mathcal{A}}}$-differentiably uniquely from U to $U^{\chi_{\mathcal{A}}} \overset{\mathrm{def}}{=} U + \operatorname{Ker}(\chi_{\mathcal{A}}) = U + \mathfrak{m} \overset{\mathrm{def}}{=} \widehat{U}$ and that we have a canonical isomorphism $\mathscr{L}_{\mathcal{A}}(\widehat{U}, {}_{\mathcal{A}}\mathbb{K}) \cong \mathscr{L}_{\mathbb{K}}(U^{\mathrm{sp}}, \mathbb{K})$ of differential $\mathbb{K}$-algebras. □

Lemma-Definition 1.2.5 (π-Topology of $\mathcal{A}$)**:** Let $\mathcal{A} \overset{\pi}{\twoheadrightarrow} \mathcal{B}$ be an epimorphism in $\mathsf{FdCAlg}_{\mathbb{K}}$ and denote by $\mathcal{T}^{\mathrm{an}}_{\mathcal{A}}$ the analytic topology on $\mathcal{A}$. Then

$$\mathcal{T}^{\mathrm{ker}}_{\mathcal{A}} := \mathcal{T}^{\mathrm{sat}}_{\mathcal{A}} := \mathcal{T}^{\pi}_{\mathcal{A}} := \{U^{\pi} \overset{\mathrm{def}}{=} \pi^{-1}(\pi(U)) \overset{\mathrm{def}}{=} U + \operatorname{Ker}(\pi) : U \in \mathcal{T}^{\mathrm{an}}_{\mathcal{A}}\} \subseteq \mathcal{T}^{\mathrm{an}}_{\mathcal{A}}$$

will be called the *π-saturation topology*, or *π-analytic topology*, or the *$\operatorname{Ker}(\pi)$-topology*, or simply the *π-topology* on $\mathcal{A}$.

Proof: We need to show that $\mathcal{T}^{\pi}_{\mathcal{A}}$ is indeed a topology. Clearly, $\varnothing, \mathcal{A} \in \mathcal{T}^{\pi}_{\mathcal{A}}$. Moreover, if $\{U^{\pi}_i\}_{i\in I} \subseteq \mathcal{T}^{\pi}_{\mathcal{A}}$ is an arbitrary family, then

$$\bigcup_{i\in I} U^{\pi}_i \overset{\mathrm{def}}{=} \bigcup_{i\in I} \pi^{-1}(\pi(U_i)) = \pi^{-1}\Big(\bigcup_{i\in I} \pi(U_i)\Big) = \pi^{-1}\Big(\pi\Big(\bigcup_{i\in I} U_i\Big)\Big) \overset{\mathrm{def}}{=} \Big(\bigcup_{i\in I} U_i\Big)^{\pi} \in \mathcal{T}^{\pi}_{\mathcal{A}}.$$

Finally, let $U^{\pi}_1, U^{\pi}_2 \in \mathcal{T}^{\pi}_{\mathcal{A}}$ and consider $W := \pi(U_1) \cap \pi(U_2) \subseteq \mathcal{B}$. Since U_1, U_2, and π are open, so is W, hence so is $V := \pi^{-1}(W) \subseteq \mathcal{A}$ (by (automatic) continuity of π), and by surjectivity of π we have $\pi(V) \overset{\mathrm{def}}{=} \pi(\pi^{-1}(W)) = W$. Thus $U^{\pi}_1 \cap U^{\pi}_2 \overset{\mathrm{def}}{=} \pi^{-1}(\pi(U_1)) \cap \pi^{-1}(\pi(U_2)) = \pi^{-1}(\pi(U_1) \cap \pi(U_2)) \overset{\mathrm{def}}{=} \pi^{-1}(W) = \pi^{-1}(\pi(V)) \in \mathcal{T}^{\pi}_{\mathcal{A}}$. □

Definition 1.2.6: Let $\rho\colon \mathcal{A} \to \operatorname{End}_{\mathbb{K}}(M)$ be a representation. Then

$$\mathcal{T}^{\mathrm{ker}}_{\mathcal{A}} := \mathcal{T}^{\mathrm{sat}}_{\mathcal{A}} := \mathcal{T}^{\rho}_{\mathcal{A}} := \mathcal{T}^{\rho^{\mathrm{im}}}_{\mathcal{A}} \overset{\mathrm{def}}{=} \{U^{\rho} \overset{\mathrm{def}}{=} \rho^{-1}(\rho(U)) \overset{\mathrm{def}}{=} U + \operatorname{Ker}(\rho) : U \in \mathcal{T}^{\mathrm{an}}_{\mathcal{A}}\}$$

will be called the *ρ-saturation topology*, or the *ρ-analytic topology*, or the *$\operatorname{Ker}(\rho)$-topology*, or the *annihilator topology* (module point of view referencing $\operatorname{Ann}_{\mathcal{A}}(M)$), or simply the *$\rho$-topology* on $\mathcal{A}$.

Lemma 1.2.7: Let $\rho\colon \mathcal{A} \to \operatorname{End}_{\mathbb{K}}(M)$ be a representation. For every epimorphism $\pi\colon \mathcal{A} \twoheadrightarrow \bar{\mathcal{A}}$ compatible with ρ in the sense of Proposition 1.2.1 we have $U^{\pi} \subseteq U^{\rho}$. That is, given a (classically or strictly) ρ-differentiable function $f\colon U \to M$, the extension f^{ρ} is maximal among all the extensions f^{π} as π varies through compatible epimorphisms.

Proof: As before, we can write $\bar{\mathcal{A}} = \mathcal{A}/\mathfrak{a}$ for some $\mathfrak{a} \lhd \mathcal{A}$ with $\mathfrak{a} \subseteq \operatorname{Ker}(\rho) \overset{\mathrm{def}}{=} \operatorname{Ann}_{\mathcal{A}}(M)$. Thus we have a commutative diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\rho"] \arrow[d, "\pi"] \arrow[dd, bend right, "\rho^{\mathrm{im}}"'] & \operatorname{End}_{\mathbb{K}}(M) \\
\mathcal{A}/\mathfrak{a} \arrow[ur, "\bar{\rho}"] \arrow[d, "\sigma"] & \\
\mathcal{A}/\operatorname{Ann}_{\mathcal{A}}(M) \arrow[uur, hook, "\lambda"'] &
\end{tikzcd}$$

Hence

$$\begin{aligned}
U^{\rho} \overset{\mathrm{def}}{=} \rho^{-1}(\rho(U)) &= (\rho^{\mathrm{im}})^{-1}(\rho^{\mathrm{im}}(U)) = (\sigma \circ \pi)^{-1}((\sigma \circ \pi)(U)) = \pi^{-1}(\sigma^{-1}(\sigma(\pi(U)))) = \\
&= \pi^{-1}(\pi(U)^{\sigma}) \supseteq \pi^{-1}(\pi(U)) \overset{\mathrm{def}}{=} U^{\pi}
\end{aligned}$$

since $\pi(U)^{\sigma} \supseteq \pi(U)$. □

Lemma 1.2.8 (Little-o respects products)**:** Let $(X, p), (X_1, p_1), \ldots, (X_n, p_n) \in \mathsf{Top}_*$, let $(V_1, \|\cdot\|_{V_1}), \ldots, (V_n, \|\cdot\|_{V_n}), (W_1, \|\cdot\|_{W_1}), \ldots, (W_n, \|\cdot\|_{W_n}) \in \mathsf{NrmVec}_{\mathbb{K}}$, and put $V := \prod_{i=1}^n V_i$ and $W := \prod_{i=1}^n W_i$.

(1) Internal Products: Let $f_i\colon X \to V_i$, $g_i\colon X \to W_i$, be functions, $1 \le i \le n$, and put $f := (f_1, \ldots, f_n)\colon X \to V$ and $g := (g_1, \ldots, g_n)\colon X \to W$. If $\forall 1 \le i \le n\colon \|f_i(x)\|_{V_i} = o(\|g_i(x)\|_{W_i})$

as $x \to p$, then also $\|f(x)\|_V = o(\|g(x)\|_W)$ as $x \to p$.

(2) External Products: Let $f_i\colon X_i \to V_i$, $g_i\colon X_i \to W_i$, be functions such that $\|f_i(x_i)\|_{V_i} = o(\|g_i(x_i)\|_{W_i})$ as $x_i \to p_i$, $1 \leq i \leq n$. If $(X,p) = \prod_{i=1}^n (X_i, p_i)$ and $f := \times_{i=1}^n f_i\colon X \to V$ and $g := \times_{i=1}^n g_i\colon X \to W$, then also $\|f(x)\|_V = o(\|g(x)\|_W)$ as $x \to p$.

Proof: We will denote by $\mathcal{N} \ni p$ various neighbourhoods. Since the o-conditions are independent of the choice of norms, we are free to equip V and W with the respective *product* norms (= *max*-norms) (see also Lemma-Definition 0.3.25).

To (1): Let $\varepsilon > 0$ be arbitrary. By assumption we have that $\forall\, 1 \leq i \leq n\ \exists \mathcal{N}_{i,\varepsilon} \ni p\ \forall\, x \in \mathcal{N}_{i,\varepsilon}\colon \|f_i(x)\|_{V_i} \leq \varepsilon \|g_i(x)\|_{W_i}$. Put $\mathcal{N}_\varepsilon := \bigcap_{i=1}^n \mathcal{N}_{i,\varepsilon} \ni p$. Then $\forall\, x \in \mathcal{N}_\varepsilon\ \forall\, 1 \leq i \leq n\colon \|f_i(x)\|_{V_i} \leq \varepsilon \|g_i(x)\|_{W_i}$, hence $\forall\, x \in \mathcal{N}_\varepsilon\colon \max\limits_{1 \leq i \leq n} \|f_i(x)\|_{V_i} \leq \varepsilon \max\limits_{1 \leq j \leq n} \|g_j(x)\|_{W_j}$ as required.

To (2): This immediately follows from (1) since once again *external* products can be written as *internal* ones. More precisely, if $\mathrm{pr}_i\colon (X,p) \twoheadrightarrow (X_i, p_i)$ denotes the corresponding projection, define $f^i := f_i \circ \mathrm{pr}_i\colon X \to V_i$ and $g^i := g_i \circ \mathrm{pr}_i\colon X \to W_i$, $1 \leq i \leq n$, to be the corresponding coordinate functions of f and g, respectively. Since $x \to p$ implies $x_i \to p_i$, we have $\left\|f^i(x)\right\|_{V_i} \overset{\text{def}}{=} \|f_i(x_i)\|_{V_i} = o(\|g_i(x_i)\|_{W_i}) \overset{\text{def}}{=} o(\left\|g^i(x)\right\|_{W_i})$ as $x \to p$, $1 \leq i \leq n$. □

Proposition 1.2.9 (Naturality with respect to products)**:** Let $n \in \mathbb{N}$.

(1) Internal Direct Products (Differentiability of the Components): Let $U \subseteq \mathcal{A}$ be open.

(i) Representation Setting: Let ${}_{\mathcal{A}}M = \bigoplus_{k=1}^m {}_{\mathcal{A}}M_k$ in $\mathsf{Mod}(\mathcal{A})$ and $f = (f_1, \ldots, f_m)\colon U \to M$ for the respective components. Then:

- $(\forall\, 1 \leq k \leq m\colon f_k \in \mathscr{L}^n_{\mathcal{A}}(U, {}_{\mathcal{A}}M_k)) \Leftrightarrow f \in \mathscr{L}^n_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$;
- $(\forall\, 1 \leq k \leq m\colon f_k \in \mathscr{F}^{\mathrm{str},n}_{\mathcal{A}}(U, {}_{\mathcal{A}}M_k)) \Rightarrow f \in \mathscr{F}^{\mathrm{str},n}_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$;
- $(\forall\, 1 \leq k \leq m\colon f_k \in \mathscr{F}^n_{\mathcal{A}}(U, {}_{\mathcal{A}}M_k)) \Leftrightarrow f \in \mathscr{F}^n_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$.

In all three cases we have:

$$(\oplus_{k=1}^m f_k(Z))' = \oplus_{k=1}^m f_k'(Z). \tag{1.2.11}$$

It follows that we have equalies

$$\begin{aligned} \bigoplus_{k=1}^m \mathscr{L}^n_{\mathcal{A}}(-, {}_{\mathcal{A}}M_k) &= \mathscr{L}^n_{\mathcal{A}}\Big(-, \bigoplus_{k=1}^m {}_{\mathcal{A}}M_k\Big) \\ \bigoplus_{k=1}^m \mathscr{F}^n_{\mathcal{A}}(-, {}_{\mathcal{A}}M_k) &= \mathscr{F}^n_{\mathcal{A}}\Big(-, \bigoplus_{k=1}^m {}_{\mathcal{A}}M_k\Big) \end{aligned} \tag{1.2.12}$$

of sheaves of left $\mathrm{End}_{\mathcal{A}}(M)$-modules, but only an inclusion

$$\bigoplus_{k=1}^m \mathscr{F}^{\mathrm{str},n}_{\mathcal{A}}(-, {}_{\mathcal{A}}M_k) \hookrightarrow \mathscr{F}^{\mathrm{str},n}_{\mathcal{A}}\Big(-, \bigoplus_{k=1}^m {}_{\mathcal{A}}M_k\Big) \tag{1.2.13}$$

of sheaves of $\prod_{k=1}^m \mathrm{End}_{\mathcal{A}}(M_k)$-modules.

(ii) Morphism Setting: Let $\mathcal{B} = \prod_{k=1}^m \mathcal{B}_k$ in $\mathsf{FdCAlg}_{\mathbb{K}}$ and write $\varphi = (\varphi_1, \ldots, \varphi_m)\colon \mathcal{A} \to \mathcal{B}$ and $f = (f_1, \ldots, f_m)\colon U \to \mathcal{B}$ for the respective components. Then:

- $(\forall\, 1 \leq k \leq m\colon f_k \in \mathscr{L}^n_{\varphi_k}(U)) \Leftrightarrow f \in \mathscr{L}^n_{\varphi}(U)$;
- $(\forall\, 1 \leq k \leq m\colon f_k \in \mathscr{F}^{\mathrm{str},n}_{\varphi_k}(U)) \Rightarrow f \in \mathscr{F}^{\mathrm{str},n}_{\varphi}(U)$;
- $(\forall\, 1 \leq k \leq m\colon f_k \in \mathscr{F}^n_{\varphi_k}(U)) \Leftrightarrow f \in \mathscr{F}^n_{\varphi}(U)$.

In all three cases we have:

$$(f_1(Z), \ldots, f_m(Z))' = (f_1'(Z), \ldots, f_m'(Z)). \tag{1.2.14}$$

It follows that we have equalities

$$\prod_{k=1}^m \mathscr{L}^n_{\varphi_k} = \mathscr{L}^n_{\varphi} \text{ and } \prod_{k=1}^m \mathscr{F}^n_{\varphi_k} = \mathscr{F}^n_{\varphi}, \tag{1.2.15}$$

but only an inclusion

$$\prod_{k=1}^m \mathscr{F}^{\mathrm{str},n}_{\varphi_k} \hookrightarrow \mathscr{F}^{\mathrm{str},n}_{\varphi} \tag{1.2.16}$$

of sheaves $\mathcal{B}$-algebras.

(iii) Diagonal embeddings: In particular, if $\Delta\colon \mathcal{A} \hookrightarrow \mathcal{A}^m$ is the obvious diagonal morphism and $f = (f_1, \ldots, f_m)\colon U \to \mathcal{A}^m$, we have:
- $(\forall 1 \le k \le m\colon f_k \in \mathscr{L}^n_{\mathcal{A}}(U)) \Leftrightarrow f \in \mathscr{L}^n_{\Delta}(U)$;
- $(\forall 1 \le k \le m\colon f_k \in \mathscr{F}^{\mathrm{str},n}_{\mathcal{A}}(U)) \Leftrightarrow f \in \mathscr{F}^{\mathrm{str},n}_{\Delta}(U)$;
- $(\forall 1 \le k \le m\colon f_k \in \mathscr{F}^n_{\mathcal{A}}(U)) \Leftrightarrow f \in \mathscr{F}^n_{\Delta}(U)$.

(2) External Direct Products: Let $\{\mathcal{A}_k\}_{1\le k\le m} \subset \mathsf{FdCAlg}_{\mathbb{K}}$ be a family, $U_k \subseteq \mathcal{A}_k$, $1 \le k \le m$, be open, and put $\mathcal{A} \coloneqq \prod_{k=1}^m \mathcal{A}_k$ and $U \coloneqq \prod_{k=1}^m U_k \subseteq \mathcal{A}$.

(i) Representation Setting: let $\rho_k\colon \mathcal{A}_k \to \mathrm{End}_{\mathbb{K}}(M_k)$, $1 \le k \le m$, be a corresponding family of representations and put $M \coloneqq \bigtimes_{k=1}^m {}_{\mathcal{A}_k}M_k$. Then:
- $(\forall 1 \le k \le m\colon f_k \in \mathscr{L}^n_{\mathcal{A}_k}(U_k, {}_{\mathcal{A}_k}M_k)) \Leftrightarrow f \in \mathscr{L}^n_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$;
- $(\forall 1 \le k \le m\colon f_k \in \mathscr{F}^{\mathrm{str},n}_{\mathcal{A}_k}(U_k, {}_{\mathcal{A}_k}M_k)) \Rightarrow f \in \mathscr{F}^{\mathrm{str},n}_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$;
- $(\forall 1 \le k \le m\colon f_k \in \mathscr{F}^n_{\mathcal{A}_k}(U_k, {}_{\mathcal{A}_k}M_k)) \Rightarrow f \in \mathscr{F}^n_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$.

In all three cases we have:

$$(\times_{k=1}^m f_k(Z_k))' = \times_{k=1}^m f_k'(Z_k). \tag{1.2.17}$$

In other words, we have an equality

$$\prod_{k=1}^m \mathscr{L}^n_{\mathcal{A}_k}(U_k, {}_{\mathcal{A}_k}M_k) = \mathscr{L}^n_{\mathcal{A}}\Big(\prod_{k=1}^m U_k, \bigtimes_{k=1}^m {}_{\mathcal{A}_k}M_k\Big), \tag{1.2.18}$$

but only inclusions

$$\begin{aligned} \prod_{k=1}^m \mathscr{F}^{\mathrm{str},n}_{\mathcal{A}_k}(U_k, {}_{\mathcal{A}_k}M_k) &\hookrightarrow \mathscr{F}^{\mathrm{str},n}_{\mathcal{A}}\Big(\prod_{k=1}^m U_k, \bigtimes_{k=1}^m {}_{\mathcal{A}_k}M_k\Big) \\ \prod_{k=1}^m \mathscr{F}^n_{\mathcal{A}_k}(U_k, {}_{\mathcal{A}_k}M_k) &\hookrightarrow \mathscr{F}^n_{\mathcal{A}}\Big(\prod_{k=1}^m U_k, \bigtimes_{k=1}^m {}_{\mathcal{A}_k}M_k\Big) \end{aligned} \tag{1.2.19}$$

of left $\prod_{k=1}^m \mathrm{End}_{\mathcal{A}_k}(M_k)$-modules.

(ii) Morphism Setting: Let $\varphi_k\colon \mathcal{A}_k \to \mathcal{B}_k$, $1 \le k \le m$, be a corresponding family of morphisms in $\mathsf{FdCAlg}_{\mathbb{K}}$ and put $\mathcal{B} \coloneqq \prod_{k=1}^m \mathcal{B}_k$. Then:
- $(\forall 1 \le k \le m\colon f_k \in \mathscr{L}^n_{\varphi_k}(U_k)) \Leftrightarrow f \in \mathscr{L}_{\varphi}(U)$;
- $(\forall 1 \le k \le m\colon f_k \in \mathscr{F}^{\mathrm{str},n}_{\varphi_k}(U_k)) \Rightarrow f \in \mathscr{F}^{\mathrm{str},n}_{\varphi}(U)$;
- $(\forall 1 \le k \le m\colon f_k \in \mathscr{F}^n_{\varphi_k}(U_k)) \Rightarrow f \in \mathscr{F}^n_{\varphi}(U)$.

In all three cases we have:

$$(\times_{k=1}^m f_k(Z_k))' = \times_{k=1}^m f_k'(Z_k). \tag{1.2.20}$$

In other words, we have equalities

$$\prod_{k=1}^m \mathscr{L}^n_{\varphi_k}(U_k) = \mathscr{L}^n_{\times_{k=1}^m \varphi_k}\Big(\prod_{k=1}^m U_k\Big), \tag{1.2.21}$$

but only inclusions

$$\prod_{k=1}^{m} \mathscr{F}^{\mathrm{str},n}_{\varphi_k}(U_k) \hookrightarrow \mathscr{F}^{\mathrm{str},n}_{\times_{k=1}^{m}\varphi_k}\Big(\prod_{k=1}^{m} U_k\Big) \text{ and } \prod_{k=1}^{m} \mathscr{F}^{n}_{\varphi_k}(U_k) \hookrightarrow \mathscr{F}^{n}_{\times_{k=1}^{m}\varphi_k}\Big(\prod_{k=1}^{m} U_k\Big) \quad (1.2.22)$$

of $\mathcal{B}$-algebras.

(3) Canonical Extension for External Products: Let $\{\mathcal{A}_k\}_{1\le k\le m} \subset \mathsf{FdCAlg}_{\mathbb{K}}$ be a family, put $\mathcal{A} := \prod_{k=1}^m \mathcal{A}_k$, and let $U \subseteq \mathcal{A}$ be open and $\widetilde{U} \stackrel{\text{def}}{=} \widetilde{U}_{([m])} \stackrel{\text{def}}{=} U_{[m]}$ its projection closure (with respect to the given product).

(i) Representation Setting: If $M_k \in \mathsf{FdMod}(\mathcal{A}_k)$, $1 \le k \le m$, is a corresponding family of modules and $M := \bigtimes_{k=1}^m {}_{\mathcal{A}_k}M_k$, then every $f \in \mathscr{L}_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$ admits a unique $\mathscr{L}_{\mathcal{A}}$-differentiable extension $\tilde{f}\colon \widetilde{U} \to {}_{\mathcal{A}}M$. Moreover, we have a canonical isomorphism

$$\mathscr{L}^n_{\mathcal{A}}(U, {}_{\mathcal{A}}M) \cong \mathscr{L}^n_{\mathcal{A}}(\widetilde{U}, {}_{\mathcal{A}}M). \quad (1.2.23)$$

of left $\prod_{k=1}^m \mathrm{End}_{\mathcal{A}_k}(M_k)$-modules.

(ii) Morphism Setting: If $\mathcal{A}_k \xrightarrow{\varphi_k} \mathcal{B}_k$, $1 \le k \le m$, is a corresponding family of morphisms in $\mathsf{FdCAlg}_{\mathbb{K}}$, $\mathcal{B} := \prod_{k=1}^m \mathcal{B}_k$, and $\varphi := \times_{k=1}^m \varphi_k$, then every $f \in \mathscr{L}_{\varphi}(U)$ admits a unique $\mathscr{L}_{\varphi}$-differentiable extension $\tilde{f}\colon \widetilde{U} \to \mathcal{B}$. Moreover, we have a canonical isomorphism

$$\mathscr{L}^n_{\varphi}(U) \cong \mathscr{L}^n_{\varphi}(\widetilde{U}). \quad (1.2.24)$$

of $\mathcal{B}$-algebras.

Proof: It suffices to prove the claims for $n = 1$.

To (1): To (i): Observe that the action of $\mathrm{End}_{\mathcal{A}}(M)$ is indeed well-defined by Eq. (0.1.16) and Proposition 1.1.23. In the same vein, in the $\mathscr{F}^{\mathrm{str}}$-case we only have an action of the "diagonal" product $\prod_{k=1}^m \mathrm{End}_{\mathcal{A}}(M_k)$.

"$\Leftarrow$": Apply Proposition 1.1.23 (2) to $\mathrm{pr}_k^M\colon M \twoheadrightarrow M_k$, $1 \le k \le m$.

"$\Rightarrow$": As we are dealing with different images of H rather than a single H, the proof perhaps involves slightly more details than one would normally expect in this situation. Write $\rho = (\rho_1, \dots, \rho_m)\colon \mathcal{A} \to \prod_{k=1}^m \mathrm{End}_{\mathbb{K}}(M_k) \hookrightarrow \mathrm{End}_{\mathbb{K}}(M)$ for the respective "components" of ρ. Put $r_k\colon U - Z_0 \to M_k$, $r_k(H) := f_k(Z_0 + H) - f_k(Z_0) - \rho_k(H) f_k'(Z_0)$, $1 \le k \le m$, set $f'(Z_0) := (f_1'(Z_0), \dots, f_m'(Z_0))$, and define $r\colon U - Z_0 \to M$ by $r(H) := (r_1(H), \dots, r_m(H)) \stackrel{\text{def}}{=} f(Z_0 + H) - f(Z_0) - \rho(H) f'(Z_0)$. We are going to show that $r(H)$ satisfies the corresponding vanishing condition for each ρ-differentiability class $\mathscr{L}$, $\mathscr{F}^{\mathrm{str}}$, and $\mathscr{F}$, respectively. For the class $\mathscr{L}_\rho$ observe that $D := \mathcal{A}_\rho^\times \cap (U - Z_0) = (\bigcap_{k=1}^m \mathcal{A}_{\rho_k}^\times) \cap (U - Z_0) = \bigcap_{k=1}^m \big(\mathcal{A}_{\rho_k}^\times \cap (U - Z_0)\big)$ (see Lemma 0.1.40 (3)) is still a densely defined neighbourhood at 0 for each r_k, $1 \le k \le m$, separately. Hence, invoking Lemma 1.1.10, we have

$$\lim_{\substack{H\to 0\\ H\in D}} \rho(H)^{-1} r(H) = \lim_{\substack{H\to 0\\ H\in D}} (\rho_1(H)^{-1} r_1(H), \dots, \rho_m(H)^{-1} r_m(H)) = 0,$$

as required. For the ρ-differentiability class $\mathscr{F}^{\mathrm{str}}$ we have by definition that $\|r_k(H)\|_{M_k} = o(\|\rho_k(H)\|_{\mathrm{End}})$ as $H \to 0$, $1 \le k \le m$. It now follows from Lemma 1.2.8 (1) that $\|r(H)\|_M = o(\|\rho(H)\|_{\mathrm{End}})$ as $H \to 0$. Similarly for the class $\mathscr{F}$, it follows from Lemma 1.2.8 (1) that $\|r(H)\|_M = o(\|(H, \dots, H)\|_{\mathcal{A}^m}) = o(\|H\|_{\mathcal{A}})$ as $H \to 0$.

To (ii): "$\Leftarrow$": Apply Proposition 1.1.23 (3) to $\mathrm{pr}_k^{\mathcal{B}}\colon \mathcal{B} \twoheadrightarrow \mathcal{B}_k$, $1 \le k \le m$.

"$\Rightarrow$": We get

$$\prod_{k=1}^m \mathscr{F}^{\mathrm{str}}_{\varphi_k} = \prod_{k=1}^m \mathscr{F}^{\mathrm{str}}_{\varphi_k^{\mathrm{rep}}} \overset{(i)}{\hookrightarrow} \mathscr{F}^{\mathrm{str}}_{\iota_{\mathrm{cor}}\circ(\varphi_1^{\mathrm{rep}},\dots,\varphi_m^{\mathrm{rep}})} = \mathscr{F}^{\mathrm{str}}_{(\varphi_1,\dots,\varphi_m)^{\mathrm{rep}}} = \mathscr{F}^{\mathrm{str}}_{(\varphi_1,\dots,\varphi_m)} \stackrel{\text{def}}{=} \mathscr{F}^{\mathrm{str}}_{\varphi}$$

by Eq. (1.1.9) and Lemma 0.1.20, where we have explicitly included the *identification* ι_{cor} to be completely rigorous (but otherwise usually omit) in the representation notation to match the module notation (cf. Section 0.1.4). Analogously for the differentiability classes $\mathscr{L}$ and $\mathscr{F}$ with the inclusion replaced by equality by (i).

To (iii): The $\mathscr{L}$ and $\mathscr{F}$ cases follow immediately from (ii) since $\Delta \overset{\mathrm{def}}{=} (\mathrm{id}_{\mathcal{A}}, \dots, \mathrm{id}_{\mathcal{A}})$. The "$\Rightarrow$"-direction of the $\mathscr{F}^{\mathrm{str}}$ also follows from (ii). Conversely, since Δ is injective, we also have $\mathscr{F}^{\mathrm{str}}_{\Delta} = \mathscr{F}_{\Delta}$ by Proposition 1.1.11 (and $\mathscr{F}^{\mathrm{str}}_{\mathcal{A}} \overset{\mathrm{def}}{=} \mathscr{F}_{\mathcal{A}}$).

To (2): We put for short $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$ for the corresponding representation (see also Section 0.1.3)

To (i): "$\Rightarrow$ / $\subseteq$": For f_k in the respective ρ_k-differentiability class over U_k write $r_k\colon U_k - Z_k \to M_k$, $r_k(H_k) := f_k(Z_k + H_k) - f_k(Z_k) - \rho_k(H_k) f_k'(Z_k)$, $1 \le k \le m$. Now put $Z := (Z_1, \dots, Z_m)$ and $H := (H_1, \dots, H_m)$, and for $Z \in U$ define $f(Z) := \times_{k=1}^m f_k(Z_k)$, $f'(Z) := \times_{k=1}^m f_k'(Z_k)$, and $r(H) := \times_{k=1}^m r_k(H_k)$. That is, $r\colon \prod_{k=1}^m (U_k - Z_k) = U - Z \to {}_{\mathcal{A}}M$, $r(H) = f(Z+H) - f(Z) - \rho(H) f'(Z)$. Moreover, for the $\mathscr{L}$-class observe that $\mathcal{A}^{\times}_{\rho} = \prod_{k=1}^m \mathcal{A}^{\times}_{k,\rho_k}$, hence putting $D := \mathcal{A}^{\times}_{\rho} \cap (U - Z)$ and $D_k := \mathcal{A}^{\times}_{k,\rho_k} \cap (U_k - Z_k)$ for short, we have $D = \prod_{k=1}^m D_k$. Thus

$$\lim_{\substack{H \to 0 \\ H \in D}} \rho(H)^{-1} r(H) = \lim_{\substack{H \to 0 \\ H \in \prod_{k=1}^m D_k}} \times_{k=1}^m \big(\rho_k(H_k)^{-1} r_k(H_k)\big) = \times_{k=1}^m \lim_{\substack{H_k \to 0 \\ H_k \in D_k}} \rho_k(H_k)^{-1} r_k(H_k) = 0$$

as required. For the classes $\mathscr{F}^{\mathrm{str}}$ and $\mathscr{F}$ we have $\|r_k(H_k)\|_{M_k} = o(\|\rho_k(H_k)\|_{\mathrm{End}})$ and $\|r_k(H_k)\|_{M_k} = o(\|H_k\|_{\mathcal{A}_k})$, respectively, $1 \le k \le m$. It now follows from Lemma 1.2.8 (2) that $\|r(H)\|_M = o(\|\rho(H)\|_{\mathrm{End}})$ for $\mathscr{F}^{\mathrm{str}}$ and $\|r(H)\|_M = o(\|H\|_{\mathcal{A}})$ for $\mathscr{F}$.

"$\Leftarrow$ / $\supseteq$": Let $f \in \mathscr{L}_{\rho}(U) = \mathscr{L}_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$ and let $f^k := \mathrm{pr}^M_k \circ f \in \mathscr{L}_{\mathcal{A}}(U, {}_{\mathcal{A}}M_k) \overset{\mathrm{def}}{=} \mathscr{L}_{\rho^k}(U)$, $1 \le k \le m$, be the corresponding components of f (cf. Proposition 1.1.23), where

$$\rho^k\colon \mathcal{A} \xrightarrow{\rho^{\mathrm{cor}}} \prod_{\ell=1}^m \mathrm{End}_{\mathbb{K}}(M_\ell) \xrightarrow{\mathrm{pr}^{\mathrm{End}}_k} \mathrm{End}_{\mathbb{K}}(M_k),\ 1 \le k \le m,$$

denote the respective components of ρ. Since, by definition, $\rho^k = \rho_k \circ \mathrm{pr}^{\mathcal{A}}_k$ for the epimorphism $\mathrm{pr}^{\mathcal{A}}_k\colon \mathcal{A} \twoheadrightarrow \mathcal{A}_k$, it follows from Proposition 1.2.1 that $\exists_1 f_k \in \mathscr{L}_{\rho_k}(U_k) \overset{\mathrm{def}}{=} \mathscr{L}_{\mathcal{A}_k}(U_k, {}_{\mathcal{A}_k}M_k)\colon f^k = f_k \circ \mathrm{pr}^{\mathcal{A}}_k$, $1 \le k \le m$. Schematically:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\rho^k"] \arrow[d, "\mathrm{pr}^{\mathcal{A}}_k"'] & \mathrm{End}_{\mathbb{K}}(M_k) \\ \mathcal{A}_k \arrow[ur, "\rho_k"'] & \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} U \arrow[r, "f^k"] \arrow[d, "\mathrm{pr}^{\mathcal{A}}_k"'] & M_k \\ U_k \arrow[ur, dashed, "\exists_1 f_k"'] & \end{tikzcd}$$

By construction it is immediate that $\times_{k=1}^m f_k = f$ on $U = U_1 \times \cdots \times U_m$.

To (ii): Similarly to (1)(ii), we get

$$\prod_{k=1}^m \mathscr{F}_{\varphi_k}(U_k) = \prod_{k=1}^m \mathscr{F}_{\varphi_k^{\mathrm{rep}}}(U_k) \overset{\text{(i)}}{\hookrightarrow} \mathscr{F}_{\times_{k=1}^m \varphi_k^{\mathrm{rep}}}\Big(\prod_{k=1}^m U_k\Big) = \mathscr{F}_{(\times_{k=1}^m \varphi_k)^{\mathrm{rep}}}\Big(\prod_{k=1}^m U_k\Big) = \mathscr{F}_{\times_{k=1}^m \varphi_k}\Big(\prod_{k=1}^m U_k\Big)$$

by Eq. (1.1.9) and Lemma 0.1.20 (where this time we have omitted ι_{cor}) and likewise for $\mathscr{F}^{\mathrm{str}}$. Analogously for the differentiability class $\mathscr{L}$ with the inclusion replaced by equality by (i).

To (3): To (i): One completes the same construction as in (2)(i), but with an arbitrary U in place of a basic open product set. Schematically, we have

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\rho^k"] \arrow[d, "\mathrm{pr}_k^{\mathcal{A}}"'] & \mathrm{End}_{\mathbb{K}}(M_k) \\ \mathcal{A}_k \arrow[ur, "\rho_k"'] & \end{tikzcd} \implies \begin{tikzcd} U \arrow[r, "f^k"] \arrow[d, "\mathrm{pr}_k^{\mathcal{A}}"'] & M_k \\ U_k \arrow[ur, dashed, "\exists_1 f_k"'] & \end{tikzcd} \implies \begin{tikzcd} U \arrow[r, "f"] \arrow[d, hook] & M \\ \widetilde{U} \arrow[ur, dashed, "\exists_1 \tilde{f} := \times_{k=1}^m f_k"'] & \end{tikzcd} \tag{1.2.25}$$

The uniqueness of $\tilde{f}$ follows from the construction procedure since any $\mathscr{L}_{\mathcal{A}}$-differentiable extension admits the same pre-projection and post-projection properties. Finally, $\mathscr{L}_{\mathcal{A}}(U, {}_{\mathcal{A}}M) \cong \mathscr{L}_{\mathcal{A}}(\widetilde{U}, {}_{\mathcal{A}}M)$ is given by $f \mapsto \tilde{f}$ with inverse $\mathscr{L}_{\mathcal{A}}(\widetilde{U}, {}_{\mathcal{A}}M) \to \mathscr{L}_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$, $g \mapsto g|_U$.

To (ii): follows from (i) the usual way. $\square$

Remarks:

(1) Observe again the symmetric asymmetry between $\mathscr{F}^{\mathrm{str}}$ and $\mathscr{F}$, which traces back to Proposition 1.1.23 and Proposition 1.2.1: *post-composition* with projections doesn't apply (unconditionally) to $\mathscr{F}^{\mathrm{str}}$, whereas *pre-compositional factorization* with respect to projections doesn't apply to $\mathscr{F}$ (naively at least, as far as we know). However, $\mathscr{L}$ behaves well in both cases. Of course, all of that disappears once one passes to continuous ρ-differentiability.

(2) Given $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$ and ${}_{\mathcal{A}}M = \bigoplus_{k=1}^m {}_{\mathcal{A}}M_k$ in $\mathsf{FdMod}(\mathcal{A})$ with corresponding representations $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$ and $\rho_k\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M_k)$, $1 \le k \le m$, we have by Lemma-Definition 0.1.15 a commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\rho"] \arrow[d, "\rho^{\mathrm{mor}}"'] & \mathrm{End}_{\mathbb{K}}(M) \\ \mathcal{B} \arrow[ur, hook, "\iota_{\mathrm{mor}}"'] & \end{tikzcd}$$

where $\mathcal{B} \overset{\text{def}}{=} \prod_{k=1}^m \mathcal{B}_k$ with $\mathcal{B}_k \overset{\text{def}}{=} \rho_k(\mathcal{A})$, $1 \le k \le m$. Since $\mathrm{Ker}(\rho) = \mathrm{Ker}(\rho^{\mathrm{mor}})$ (that is, the representation $\iota^{\mathrm{mor}}\colon \mathcal{B} \hookrightarrow \mathrm{End}_{\mathbb{K}}(\mathcal{B})$ is faithful), it follows from Proposition 1.1.19 that $\mathscr{L}_\rho \in \mathsf{Mod}(\mathscr{L}_{\rho^{\mathrm{mor}}})$, $\mathscr{F}_\rho^{\mathrm{str}} \in \mathsf{Mod}(\mathscr{F}_{\rho^{\mathrm{mor}}}^{\mathrm{str}})$, and $\mathscr{F}_\rho \in \mathsf{Mod}(\mathscr{F}_{\rho^{\mathrm{mor}}})$ (and moreover the mixed Leibniz rule applies). On the other hand, $\rho^{\mathrm{mor}} \overset{\text{def}}{=} (\rho_k^{\mathrm{im}})_{1 \le k \le m}$ (as morphisms), so Proposition 1.2.9 gives us in turn $\mathsf{Mod}(\mathscr{L}_{\rho^{\mathrm{mor}}}) = \mathsf{Mod}(\prod_{k=1}^m \mathscr{L}_{\rho_k^{\mathrm{im}}}) \cong \prod_{k=1}^m \mathsf{Mod}(\mathscr{L}_{\rho_k^{\mathrm{im}}})$ and $\mathsf{Mod}(\mathscr{F}_{\rho^{\mathrm{mor}}}) = \mathsf{Mod}(\prod_{k=1}^m \mathscr{F}_{\rho_k^{\mathrm{im}}}) \cong \prod_{k=1}^m \mathsf{Mod}(\mathscr{F}_{\rho_k^{\mathrm{im}}})$.

(3) Continuing (2), by Eq. (0.1.15) we also have a commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\rho"] \arrow[d, hook, "\Delta"'] & \mathrm{End}_{\mathbb{K}}(M) \\ \mathcal{A}^m \arrow[ur, "\lambda := \iota_{\mathrm{cor}} \circ \times_{k=1}^m \rho_k"'] & \end{tikzcd}$$

hence $\mathscr{L}_\rho \in \mathsf{Mod}(\mathscr{L}_\Delta) = \mathsf{Mod}(\mathscr{L}_{\mathcal{A}}^{\times m}) \cong \mathsf{Mod}(\mathscr{L}_{\mathcal{A}})^m$ and $\mathscr{F}_\rho \in \mathsf{Mod}(\mathscr{F}_\Delta) = \mathsf{Mod}(\mathscr{F}_{\mathcal{A}}^{\times m}) \cong \mathsf{Mod}(\mathscr{F}_{\mathcal{A}})^m$ by Proposition 1.2.9 (1)(iii). $\square$

Corollary 1.2.10: Let $n \in \mathbb{N} \cup \{\infty, \omega\}$.

(1) Internal Products: Let $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$.

(i) Representation Setting: if $\{{}_{\mathcal{A}}M_k\}_{1 \le k \le m} \subset \mathsf{Mod}(\mathcal{A})$, then

$$\mathscr{C}_{\mathcal{A}}^n\Big(-, \bigoplus_{k=1}^m {}_{\mathcal{A}}M_k\Big) = \bigoplus_{k=1}^m \mathscr{C}_{\mathcal{A}}^n(-, {}_{\mathcal{A}}M_k) \tag{1.2.26}$$

as sheaves of left $\prod_{k=1}^m \mathrm{End}_{\mathcal{A}}(M_k)$-modules. For $n \in \{\infty, \omega\}$ this is an equality of sheaves of left $(\prod_{k=1}^m \mathrm{End}_{\mathcal{A}}(M_k), \partial)$-modules.

(ii) Morphism Setting: if $\mathcal{A} \xrightarrow{\varphi_k} \mathcal{B}_k$, $1 \le k \le m$, is a family of morphisms in $\mathsf{FdCAlg}_{\mathbb{K}}$ and

$\mathcal{B} := \prod_{k=1}^m \mathcal{B}_k$, then

$$\mathscr{C}^n_{\mathcal{A}}\Big(-, \prod_{k=1}^m {}_{\mathcal{A}}\mathcal{B}_k\Big) = \prod_{k=1}^m \mathscr{C}^n_{\mathcal{A}}(-, {}_{\mathcal{A}}\mathcal{B}_k). \tag{1.2.27}$$

as sheaves of $\mathcal{B}$-algebras. For $n \in \{\infty, \omega\}$ this is an equality of sheaves of *differential* $\mathcal{B}$-algebras.

(2) External Products: Let $\mathcal{A} := \prod_{k=1}^m \mathcal{A}_k$ in $\mathsf{FdCAlg}_{\mathbb{K}}$ and $U_k \subseteq \mathcal{A}_k$ open, $1 \le k \le m$.

(i) Representation Setting: if $\{{}_{\mathcal{A}_k}M_k\}_{1\le k\le m} \subset \mathsf{FdMod}$, then

$$\mathscr{C}^n_{\prod_{k=1}^m \mathcal{A}_k}\Big(\prod_{k=1}^m U_k, \bigtimes_{k=1}^m {}_{\mathcal{A}_k}M_k\Big) = \prod_{k=1}^m \mathscr{C}^n_{\mathcal{A}_k}(U_k, {}_{\mathcal{A}_k}M_k) \tag{1.2.28}$$

as left $\prod_{k=1}^m \operatorname{End}_{\mathcal{A}_k}(M_k)$-modules. For $n \in \{\infty, \omega\}$ this is an equality of left $(\prod_{k=1}^m \operatorname{End}_{\mathcal{A}_k}(M_k), \partial)$-modules. Moreover, if $M := \bigtimes_{k=1}^m {}_{\mathcal{A}_k}M_k$ and $U \subseteq \mathcal{A}$ is an arbitrary open subset, then we have a canonical isomorphism

$$\mathscr{C}^n_{\mathcal{A}}(U, {}_{\mathcal{A}}M) \cong \mathscr{C}^n_{\mathcal{A}}(\widetilde{U}, {}_{\mathcal{A}}M) \tag{1.2.29}$$

of left $\prod_{k=1}^m \operatorname{End}_{\mathcal{A}_k}(M_k)$-modules. For $n \in \{\infty, \omega\}$ this is also an isomorphism of left $(\prod_{k=1}^m \operatorname{End}_{\mathcal{A}_k}(M_k), \partial)$-modules.

(ii) Morphism Setting: If $\mathcal{A}_k \xrightarrow{\varphi_k} \mathcal{B}_k$ and $\mathcal{B} := \prod_{k=1}^m \mathcal{B}_k$, then

$$\mathscr{C}^n_{\times_{k=1}^m \varphi_k}\Big(\prod_{k=1}^m U_k\Big) = \prod_{k=1}^m \mathscr{C}^n_{\varphi_k}(U_k) \tag{1.2.30}$$

as $\mathcal{B}$-algebras. For $n \in \{\infty, \omega\}$ this is an equality of differential $\mathcal{B}$-algebras. Moreover, if $U \subseteq \mathcal{A}$ is an arbitrary open subset, then we have a canonical isomorphism

$$\mathscr{C}^n_{\mathcal{A}}(U, {}_{\mathcal{A}}\mathcal{B}) \cong \mathscr{C}^n_{\mathcal{A}}(\widetilde{U}, {}_{\mathcal{A}}\mathcal{B}) \tag{1.2.31}$$

of $\mathcal{B}$-algebras. For $n \in \{\infty, \omega\}$ this is also an isomorphism of differential $\mathcal{B}$-algebras. □

Corollary 1.2.11: Let $\mathcal{A} = \prod_{j=1}^n \mathcal{A}_j$ in $\mathsf{FdCAlg}_{\mathbb{K}}$, $U_j \subseteq \mathcal{A}_j$ open, $1 \le j \le n$, and $k \in \mathbb{N} \cup \{\infty, \omega\}$. Then we have equalities

$$\mathscr{L}^k_{\mathcal{A}}\Big(\prod_{j=1}^n U_j\Big) = \prod_{j=1}^n \mathscr{L}^k_{\mathcal{A}_j}(U_j) \text{ and } \mathscr{C}^k_{\mathcal{A}}\Big(\prod_{j=1}^n U_j\Big) = \prod_{j=1}^n \mathscr{C}^k_{\mathcal{A}_j}(U_j) \tag{1.2.32}$$

of $\mathcal{A}$-algebras. For $k \in \{\infty, \omega\}$ these are equalities of *differential* $\mathcal{A}$-algebras. Moreover, if $U \subseteq \mathcal{A}$ is open and $f \in \mathscr{L}^k_{\mathcal{A}}(U)$ or $f \in \mathscr{C}^k_{\mathcal{A}}(U)$, then f admits a unique $\mathscr{L}^k_{\mathcal{A}}$-, respectively $\mathscr{C}^k_{\mathcal{A}}$-regular extension $\tilde{f} \colon \widetilde{U} \to \mathcal{A}$, and we have canonical isomorphisms

$$\mathscr{L}^k_{\mathcal{A}}(U) \cong \mathscr{L}^k_{\mathcal{A}}(\widetilde{U}) \text{ and } \mathscr{C}^k_{\mathcal{A}}(U) \cong \mathscr{C}^k_{\mathcal{A}}(\widetilde{U}) \tag{1.2.33}$$

of $\mathcal{A}$-algebras. For $k \in \{\infty, \omega\}$ these are isomorphisms of *differential* $\mathcal{A}$-algebras. □

Example: Consider the so called bi-real[8]/bi-complex numbers[9] $\mathbb{B} := \mathbb{B}_{\mathbb{K}} := \mathbb{B}\mathbb{K} := \mathbb{K}[X]/(X^2-1) = \mathbb{K}[j]$, $j^2 = 1$, for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We have a well-known isomorphism

$$\begin{aligned} \varphi \colon \mathbb{B} &\to \mathbb{K} \times \mathbb{K} \\ z_1 + z_2 j &\mapsto (z_1 + z_2, z_1 - z_2) \end{aligned}$$

[8]in the case $\mathbb{K} = \mathbb{R}$ the algebra $\mathbb{B}\mathbb{R}$ is also called *split-complex numbers*, or *hyperbolic numbers*, or *double numbers*, or *paracomplex numbers* in the literature, which is why we've decided to unify all these names by adding yet another name to the list, namely *"bi-real"* – it somehow *sounds* more appropriate;

[9]a bi-complex variable is sometimes also called a *motor* variable in the literature;

of $\mathbb{K}$-algebras with the inverse given by

$$\begin{aligned}\varphi^{-1}\colon \mathbb{K}\times\mathbb{K} &\to \mathbb{B}\\ (w_1,w_2) &\mapsto \frac{w_1+w_2}{2}+\frac{w_1-w_2}{2}j.\end{aligned}$$

For $U\subseteq\mathbb{B}$ open and $f\in\mathscr{L}_{\mathbb{B}}(U)$ consider $g:=\varphi\circ f\circ\varphi^{-1}\colon \varphi(U)\to\mathbb{K}\times\mathbb{K}$. By Proposition 1.1.23 we have that $g\in\mathscr{L}_{\mathbb{K}\times\mathbb{K}}(\varphi(U))$, hence

$$\begin{aligned}g\colon \widetilde{\varphi(U)} &\to \mathbb{K}\times\mathbb{K}\\ g(w_1,w_2) &= (g_1(w_1),g_2(w_2))\end{aligned}$$

with $g_i\colon \varphi(U)_i\to\mathbb{K}$ being $\mathbb{K}$-differentiable, $i=1,2$. Thus f extends to $f\colon \varphi^{-1}(\widetilde{\varphi(U)})\to\mathbb{B}$ and

$$\begin{aligned}f(z_1,z_2) &= (\varphi^{-1}\circ g\circ\varphi)(z_1,z_2) = (\varphi^{-1}\circ g)(z_1+z_2,z_1-z_2) =\\ &= \varphi^{-1}(g_1(z_1+z_2),g_2(z_1-z_2)) =\\ &= \frac{g_1(z_1+z_2)+g_2(z_1-z_2)}{2}+j\frac{g_1(z_1+z_2)-g_2(z_1-z_2)}{2}\end{aligned}\tag{1.2.34}$$

In other words, up to *isomorphic* change of variables, the function theory of $\mathscr{L}_{\mathbb{B}}$-differentiable functions is just an *external product* of two copies of the theory of ordinary $\mathbb{K}$-differentiable functions. Despite that, as is well-known, both $\mathbb{BR}$ and $\mathbb{BC}$ manage to offer rich theories (both geometrically[10] and operator-theoretically), so one should not underestimate the power of linear changes of variables. (We remark though that $\mathrm{Aut}_{\mathbb{K}}(\mathbb{B})\cong\mathfrak{S}_2$, generated by $\mathrm{id}_{\mathbb{B}}$ and the transposition $\tau\colon \mathbb{K}\times\mathbb{K}\to\mathbb{K}\times\mathbb{K}$, $(w_1,w_2)\mapsto(w_2,w_1)$, or equivalently the "conjugation" $\tau\colon z_1+jz_2\mapsto z_1-jz_2$, as both factors are isomorphic to each other and trivial over the ground field.) Indeed, we give now a very short proof of the *hyperbolic* Euler's formula $e^{jt}=\cosh(t)+j\sinh(t)$ without appealing to analyticity[11] and manipulations of power series. Consider the $\mathbb{K}\times\mathbb{K}$-differentiable component-wise exponential

$$\begin{aligned}\exp_{\mathbb{K}\times\mathbb{K}}\colon \mathbb{K}\times\mathbb{K} &\to \mathbb{K}\times\mathbb{K}\\ (w_1,w_2) &\mapsto (e^{w_1},e^{w_2}).\end{aligned}$$

Then simply

$$e^{z_1+jz_2}\overset{\text{def}}{=}\exp_{\mathbb{B}}(z_1+jz_2)=(\varphi^{-1}\circ\exp_{\mathbb{K}\times\mathbb{K}}\circ\varphi)(z_1,z_2)=\frac{e^{z_1+z_2}+e^{z_1-z_2}}{2}+j\frac{e^{z_1+z_2}-e^{z_1-z_2}}{2},$$

hence in particular taking $z_1:=0$ and $z_2:=t$ yields

$$e^{jt}=\frac{e^t+e^{-t}}{2}+j\frac{e^t-e^{-t}}{2}\overset{\text{def}}{=}\cosh(t)+j\sinh(t)$$

as desired. $\mathbb{B}$ is obviously the simplest non-trivial example for Corollary 1.2.11 and, more generally, the subsequent Proposition 1.2.12 below. For the benefit of the interested reader we have included some further useful references on $\mathbb{B}$ – that hopefully demonstrate its richness – in the bibliography at the end (see for example [MR98] and [GGM02] for $\mathbb{BR}$ and [CRS12], [Alp+13] and [Lun+15] for $\mathbb{BC}$). □

Given $f\in\mathscr{L}_\rho(U)$, following the above train of thought and the *standard factorization* of ρ, we are going to establish a *general corresponding standard factorization* for f, which in turn leads to a natural ρ-differentiable extension of f. To this end, suppose that $\mathcal{A}=\prod_{j=1}^n\mathcal{A}_j=\prod_{j=1}^n(\mathcal{A}_j,\mathfrak{m}_j,\mathbb{K}_j)$ in $\mathsf{FdCAlg}_{\mathbb{K}}$ (block decomposition by locality of the factors), $\mathbb{K}_j\in\{\mathbb{R},\mathbb{C}\}$, $1\le j\le n$, and that

[10] upon endowing $\mathbb{B}$ with an additional, metric structure; in view of Problem 3 we propose that one should thus speak of *a category of metric algebras*;

[11] strictly speaking, this would still need some kind of finite-dimensional *operator calculus over* $\mathbb{B}$ *for* $\mathbb{B}$*-valued functions* in order to rigorously well-define the extension of $\exp_{\mathbb{K}}$ to $\mathbb{B}$ – for our purposes here it suffices to take more of an *ad hoc* view, that is, we *assume* that one has already defined what $\exp_{\mathbb{B}}$ should mean, which, regardless of one's approach, unsurprisingly coincides with the obvious choice;

${}_{\mathcal{A}}M = \bigoplus_{k=1}^{m} {}_{\mathcal{A}}M_k$ in $\mathsf{FdMod}(\mathcal{A})$, with corresponding representations $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$ and $\rho_k\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M_k)$, $1 \le k \le m$, is also a *complete* decomposition. Then by Corollary 0.1.83 there exist a *fiber-grouping* index mapping $j \coloneqq j_\rho \coloneqq \tau_\rho\colon [m] \to [n]$ and an *external morphism product* $\bar\rho = \times_{k=1}^{m} \bar\rho_k$ of representations $\bar\rho_k\colon \mathcal{A}_{j(k)} \to \mathrm{End}_{\mathbb{K}}(M_k)$ induced by $\rho_k\colon \mathcal{A} \to \mathrm{End}(M_k)$, $1 \le k \le m$, via

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\rho"] \arrow[d, "\mathrm{pr}^{\mathcal{A}}_{j(k)}"'] & \mathrm{End}_{\mathbb{K}}(M_k) \\ \mathcal{A}_{j(k)} \arrow[ur, dashed, "\exists_1\, \bar\rho_k"'] & \end{tikzcd}$$

such that $\rho = \iota_{\mathrm{cor}} \circ \bar\rho \circ \Pi_\rho$, where $\Pi_\rho \overset{\mathrm{def}}{=} (\mathrm{pr}^{\mathcal{A}}_{j(1)}, \dots, \mathrm{pr}^{\mathcal{A}}_{j(m)})$. Analogously for a morphism $\varphi\colon \mathcal{A} = \prod_{j=1}^{n} (\mathcal{A}_j, \mathfrak{m}_j, \mathbb{K}_j) \to \mathcal{B} = \prod_{k=1}^{m} (\mathcal{B}_k, \mathfrak{n}_k, \mathbb{L}_k)$ with $\mathbb{L}_k \in \{\mathbb{R}, \mathbb{C}\}$, $1 \le k \le m$. Now, if $U_{j(k)} \subseteq \mathcal{A}_{j(k)}$ is any open subset and $\bar f_k \in \mathscr{L}_{\bar\rho_k}(U_{j(k)})$ is any such function, $1 \le k \le m$, then it follows from Proposition 1.1.23 and Proposition 1.2.9 that $f \coloneqq (\times_{k=1}^{m} \bar f_k) \circ \Pi_\rho$ defines a $\mathscr{L}_\rho$-differentiable[12] function

$$\begin{tikzcd} \bigcap_{k=1}^{m} \mathrm{pr}^{-1}_{j(k)}(U_{j(k)}) \arrow[r, "f"] \arrow[d, "\Pi_\rho"'] & {}_{\mathcal{A}}M = \prod_{k=1}^{m} {}_{\mathcal{A}}M_k \\ \bar U \arrow[ur, "\times_{k=1}^{m} \bar f_k"'] & \end{tikzcd}$$

where we have put $\bar U \coloneqq \prod_{k=1}^{m} U_{j(k)} \subseteq \bar{\mathcal{A}} \coloneqq \prod_{k=1}^{m} \mathcal{A}_{j(k)}$ (open) for short. Conversely, we will show below that all $\mathscr{L}_\rho$-differentiable functions arise this way. Moreover, this naturally gives rise to our closure operations from Lemma-Definition 0.1.84:

$$\Pi_\rho^{-1}(\bar U) \overset{\mathrm{def}}{=} (\mathrm{pr}^{\mathcal{A}}_{j(k)})^{-1}_{1\le k\le m}\Big(\prod_{k=1}^{m} U_{j(k)}\Big) = \bigcap_{k=1}^{m} \mathrm{pr}^{-1}_{j(k)}(U_{j(k)}) \overset{\mathrm{def}}{=} \widetilde U_\rho \overset{\mathrm{def}}{=} \widetilde U \tag{1.2.35}$$

and similarly,

$$\widehat U \overset{\mathrm{def}}{=} \widehat U_\rho \overset{\mathrm{def}}{=} \bigcap_{k=1}^{m} \chi^{-1}_{j(k)}(U_{j(k)}) = \Sigma_\rho^{-1}\Big(\prod_{k=1}^{m} U^{\mathrm{sp}}_{j(k)}\Big). \tag{1.2.36}$$

Likewise for the morphism setting $\mathscr{L}_\varphi$.

Proposition 1.2.12 (Standard Factorization)**:** With the above notations let $U \subseteq \mathcal{A}$ be open.

(1) Representation Setting: If $f \in \mathscr{L}_\rho(U)$, then there exists a unique $\bar f = \times_{k=1}^{m} \bar f_k \in \mathscr{L}_{\bar\rho}(\bar U)$ so that we have the following implication of commutative diagrams:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\rho"] \arrow[d, "\Pi_\rho"'] & \mathrm{End}_{\mathbb{K}}(M) \\ \bar{\mathcal{A}} \arrow[ur, dashed, "\exists_1\, \bar\rho"'] & \end{tikzcd} \Longrightarrow \begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\Pi_\rho"'] & M \\ \bar U \arrow[ur, dashed, "\exists_1\, \bar f"'] & \end{tikzcd} \Longrightarrow \begin{tikzcd} \widetilde U \arrow[r, dashed, "\exists_1\, \tilde f"] \arrow[d, "\Pi_\rho"'] & M \\ \bar U \arrow[ur, "\bar f"'] & \end{tikzcd}$$

That is, $f \in \mathscr{L}_\rho(U)$ extends $\mathscr{L}_\rho$-differentiably uniquely to a $\tilde f \in \mathscr{L}_\rho(\widetilde U)$. It follows that

$$\mathscr{L}_\rho(U) \cong \mathscr{L}_\rho(\widetilde U) \tag{1.2.37}$$

as left $(\mathrm{End}_{\mathcal{A}}(M), \partial)$-modules.

(2) Morphism Setting: If $f \in \mathscr{L}_\varphi(U)$, then there exists a unique $\bar f = \times_{k=1}^{m} \bar f_k \in \mathscr{L}_{\bar\varphi}(\bar U)$ so that we have the following implication of commutative diagrams:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\varphi"] \arrow[d, "\Pi_\varphi"'] & \mathcal{B} \\ \bar{\mathcal{A}} \arrow[ur, dashed, "\exists_1\, \bar\varphi"'] & \end{tikzcd} \Longrightarrow \begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\Pi_\varphi"'] & \mathcal{B} \\ \bar U \arrow[ur, dashed, "\exists_1\, \bar f"'] & \end{tikzcd} \Longrightarrow \begin{tikzcd} \widetilde U \arrow[r, dashed, "\exists_1\, \tilde f"] \arrow[d, "\Pi_\varphi"'] & \mathcal{B} \\ \bar U \arrow[ur, "\bar f"'] & \end{tikzcd}$$

[12] this direction also works for the regularity classes $\mathscr{F}^{\mathrm{str}}_\rho$ and $\mathscr{F}_\rho$;

That is, $f \in \mathscr{L}_\varphi(U)$ extends $\mathscr{L}_\varphi$-differentiably uniquely to a $\tilde{f} \in \mathscr{L}_\varphi(\widetilde{U})$. It follows that

$$\mathscr{L}_\varphi(U) \cong \mathscr{L}_\varphi(\widetilde{U}) \tag{1.2.38}$$

as differential $\mathcal{B}$-algebras.

Proof: We only prove (1) since (2) follows from (1) in the usual way. The proof is only a slight modification of the proof of Proposition 1.2.9 (3). Write $f = (f_1, \dots, f_m)$ with $f_k \colon U \to M_k$, $1 \leq k \leq m$. Since $f \in \mathscr{L}_\rho(U)$, we have $\forall\, 1 \leq k \leq m$: $f_k \in \mathscr{L}_{\rho_k}(U) \overset{\text{def}}{=} \mathscr{L}_\mathcal{A}(U, {}_\mathcal{A}M_k)$ by Proposition 1.2.9 (1)(i). Since ρ_k factors through the epimorphism $\mathrm{pr}^\mathcal{A}_{j(k)} \colon \mathcal{A} \twoheadrightarrow \mathcal{A}_{j(k)}$ as $\rho_k = \bar{\rho}_k \circ \mathrm{pr}^\mathcal{A}_{j(k)}$, we have by Proposition 1.2.1 that $f_k = \bar{f}_k \circ \mathrm{pr}^\mathcal{A}_{j(k)}$ for some unique $\bar{f}_k \in \mathscr{L}_{\bar{\rho}_k}(U_{j(k)}) \overset{\text{def}}{=} \mathscr{L}_{\mathcal{A}_{j(k)}}(U_{j(k)}, {}_{\mathcal{A}_{j(k)}}M_k)$, $1 \leq k \leq m$. Now put $\bar{f} := \times_{k=1}^m \bar{f}_k \colon \bar{U} \to M$. Then in fact $\bar{f} \in \mathscr{L}_{\bar{\rho}}(\bar{U}) \overset{\text{def}}{=} \mathscr{L}_{\bar{\mathcal{A}}}(\bar{U}, {}_{\bar{\mathcal{A}}}M)$ by Proposition 1.2.9 (2). Thus, schematically, we obtain the following implications of commutative diagrams

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\rho_k"] \arrow[d, "\mathrm{pr}^\mathcal{A}_{j(k)}"'] & \mathrm{End}_\mathbb{K}(M_k) \\ \mathcal{A}_{j(k)} \arrow[ur, dashed, "\bar{\rho}_k"'] & \end{tikzcd} \Longrightarrow \begin{tikzcd} U \arrow[r, "f_k"] \arrow[d, "\mathrm{pr}^\mathcal{A}_{j(k)}"'] & M_k \\ U_{j(k)} \arrow[ur, dashed, "\exists_1 \bar{f}_k"'] & \end{tikzcd} \Longrightarrow \begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\Pi_\rho"'] & M \\ \bar{U} \arrow[ur, dashed, "\exists_1 \bar{f}"'] & \end{tikzcd} \Longrightarrow \begin{tikzcd} \widetilde{U} \arrow[r, dashed, "\exists_1 \tilde{f}"] \arrow[d, "\Pi_\rho"'] & M \\ \bar{U} \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

where the third diagram is assembled together from the second one as $k \in [m]$. Indeed:

$$f(Z) = (f_k(Z))_{1 \leq k \leq m} = \big((\bar{f}_k \circ \mathrm{pr}^\mathcal{A}_{j(k)})(Z)\big)_{1 \leq k \leq m} = \big((\times_{k=1}^m \bar{f}_k) \circ \Pi_\rho\big)(Z) = (\bar{f} \circ \Pi_\rho)(Z).$$

The fourth diagram follows from the third one by the preceeding discussion, hence $\bar{f} \circ \Pi_\rho$ extends naturally to $\widetilde{U}$, and taking $\tilde{f}$ to be this extension therefore produces an extension of f itself to $\widetilde{U}$. Since $\bar{f} \in \mathscr{L}_{\bar{\rho}}(\bar{U})$ and tautologically $\Pi_\rho \in \mathscr{L}_{\Pi_\rho}(\widetilde{U})$, it follows from Proposition 1.1.23 (1) that $\tilde{f} = \bar{f} \circ \Pi_\rho \in \mathscr{L}_{\bar{\rho} \circ \Pi_\rho}(\widetilde{U}) = \mathscr{L}_\rho(\widetilde{U})$, i.e. $\tilde{f}$ is indeed a $\mathscr{L}_\rho$-differentiable extension. Moreover, it is easy to see that it is unique with that property by uniqueness of $\bar{f}_k$, $1 \leq k \leq m$. Finally, $\mathscr{L}_\rho(U) \to \mathscr{L}_\rho(\widetilde{U})$, $f \mapsto \tilde{f}$, and $\mathscr{L}_\rho(\widetilde{U}) \to \mathscr{L}_\rho(U)$, $g \mapsto g|_U$, give the desired isomorphism $\mathscr{L}_\rho(U) \cong \mathscr{L}_\rho(\widetilde{U})$. □

Remark: Since we can write $\rho = \iota_{\mathrm{cor}} \circ \bar{\rho} \circ \Pi_\rho = (\iota_{\mathrm{cor}} \circ \bar{\rho} \circ \Delta_{I_\rho}) \circ \mathrm{pr}_{I_\rho} =: \lambda \circ \mathrm{pr}_{I_\rho}$ with $\lambda \colon \mathcal{A}_{I_\rho} \to \mathrm{End}_\mathbb{K}(M)$, it follows from Proposition 1.2.1 that $f \in \mathscr{L}_\rho(U)$ also factorizes as $f = g \circ \mathrm{pr}_{I_\rho}$ for some $g \in \mathscr{L}_\lambda$ and extends to $f^{\mathrm{pr}_{I_\rho}} \colon U^{\mathrm{pr}_{I_\rho}} \to M$. By Lemma 0.1.46 we only have $U^{\mathrm{pr}_{I_\rho}} \subseteq \widetilde{U}$. Furthermore, Π_ρ need not be surjective. Therefore, Proposition 1.2.12 gives altogether a stronger a result than Proposition 1.2.1 on both fronts, however it is restricted to the regularity class $\mathscr{L}_\rho$. □

Theorem 1.2.13 (Automatic Extension)**:** Let $\mathcal{A} = \prod_{j=1}^n (\mathcal{A}_j, \mathfrak{m}_j, \mathbb{K}_j)$ and $U \subseteq \mathcal{A}$ open.

(1) Representation Setting: if $\rho \colon \mathcal{A} \to \mathrm{End}_\mathbb{K}(M)$ is a finite-dimensional representation and $f \in \mathscr{L}_\rho(U)$, then f extends $\mathscr{L}_\rho$-differentiably uniquely to $f := f^\rho \colon \widetilde{U^\rho} \to M$ (abuse of notation), and we have an isomorphism

$$\mathscr{L}_\rho(U) \cong \mathscr{L}_\rho(\widetilde{U^\rho}) \tag{1.2.39}$$

of $(\mathrm{End}_\mathcal{A}(M), \partial)$-modules.

(2) Morphism Setting: if $\varphi \colon \mathcal{A} \to \mathcal{B}$ is a morphism in $\mathsf{FdCAlg}_\mathbb{K}$ and $f \in \mathscr{L}_\varphi(U)$, then f extends $\mathscr{L}_\varphi$-differentiably uniquely to $f := f^\varphi \colon \widetilde{U^\varphi} \to M$ (abuse of notation), and we have an isomorphism

$$\mathscr{L}_\varphi(U) \cong \mathscr{L}_\varphi(\widetilde{U^\varphi}) \tag{1.2.40}$$

of differential $\mathcal{B}$-algebras.

Proof: This is immediate from Proposition 1.2.1 and Proposition 1.2.12. □

Remarks:

(1) By Lemma 0.1.46 we have $\widetilde{U^\rho} = (\widetilde{U})^\rho$, so it does not matter in what order we perform the

extension procedure.

(2) It follows from Lemma 0.1.46 that among the *automatic extensions* implied by various combinations of the preceeding propositions $\widetilde{U^\rho}$ is maximal *without a spectral argument*. Of course, it is conceivable that concrete functions could be ρ-differentiably extended beyond $\widetilde{U^\rho}$. Such is the case when $\mathcal{A}$ carries a complex structure: when $\mathbb{K}=\mathbb{C}$, we will show later that $f\in\mathcal{O}_\rho(U)$ extends ρ-holomorphically further to $\widehat{U}$ in a natural way, and in fact $\widehat{U}\overset{\text{def}}{=}\widehat{U}_\rho$ will turn out to be the *domains of ρ-holomorphy*. Of course, one can also deduce this directly from Proposition 1.1.16 (3). □

Lemma-Definition 1.2.14: Let $\rho\colon\mathcal{A}\to\operatorname{End}_{\mathbb{K}}(M)$.

(1) Kernel-Projection topology on $\mathcal{A}$ (with respect to ρ): $\mathcal{T}^{\text{kp}}_{\mathcal{A}}$ is the topology generated by the base

$$\mathscr{B}^{\text{kp}}_{\mathcal{A}}:=\{(\widetilde{U}_\rho)^\rho=\widetilde{(U^\rho)}_\rho : U\in\mathcal{T}^{\text{an}}_{\mathcal{A}}\}.$$

We will denote $\mathcal{A}^{\text{kp}}:=(\mathcal{A},\mathcal{T}^{\text{kp}}_{\mathcal{A}})$ for short.

(2) Spectral (polycylindrical) topology on $\mathcal{A}$ (with respect to ρ): $\mathcal{T}^{\text{sp}}_{\mathcal{A}}$ is the topology generated by the base

$$\mathscr{B}^{\text{sp}}_{\mathcal{A}}:=\{\widehat{U}_\rho\overset{\text{def}}{=}\widehat{U}_{(I_\rho)} : U\in\mathcal{T}^{\text{an}}_{\mathcal{A}}\}.$$

We will denote $\mathcal{A}^{\text{sp}}:=(\mathcal{A},\mathcal{T}^{\text{sp}}_{\mathcal{A}})$ for short.

(3) We also put $\mathcal{A}^{\text{an}}:=(\mathcal{A},\mathcal{T}^{\text{an}}_{\mathcal{A}})$ for $\mathcal{A}$ equipped with the usual analytic topology.

Proof: Since Cartesian product commutes with intersections, both $\mathscr{B}^{\text{kp}}_{\mathcal{A}}$ and $\mathscr{B}^{\text{sp}}_{\mathcal{A}}$ are indeed well-defined topological bases. □

Remarks:

(1) Amusingly, if $|I_\rho|=1$, then in fact $\mathcal{T}^{\text{kp}}_{\mathcal{A}}=\mathscr{B}^{\text{kp}}_{\mathcal{A}}$ and $\mathcal{T}^{\text{sp}}_{\mathcal{A}}=\mathscr{B}^{\text{sp}}_{\mathcal{A}}$, but otherwise not.

(2) As already alluded, $\mathcal{T}^{\text{sp}}_{\mathcal{A}}$ will tacitly appear for the case $\mathbb{K}=\mathbb{C}$ in Section 2.3 and onwards, more specifically see for example Theorem 2.3.5. □

Lemma 1.2.15 (Comparison of Extension Topologies)**:** We have $\mathcal{T}^{\text{sp}}_{\mathcal{A}}\subseteq\mathcal{T}^{\text{kp}}_{\mathcal{A}}\subseteq\mathcal{T}^{\text{an}}_{\mathcal{A}}$.

Proof: Clearly, if $U\in\mathcal{T}^{\text{an}}_{\mathcal{A}}$, then $\widehat{U}\in\mathcal{T}^{\text{an}}_{\mathcal{A}}$ too. Moreover, by Lemma 0.1.46 we have $\widehat{U}=(\widetilde{\widehat{U}})^\rho$ (i.e. of the desired form), hence in fact $\mathscr{B}^{\text{kp}}_{\mathcal{A}}\supseteq\mathscr{B}^{\text{sp}}_{\mathcal{A}}$. □

Remarks:

(1) Proposition 1.2.12 shows in particular that – like ρ – a function $f\in\mathscr{L}_\rho(U)$ is up to a projection selection homomorphism Π itself an external direct product of functions of the same regularity class between indecomposable objects.

(2) Successive application of Proposition 1.2.12 and Proposition 1.2.1 reduces study of $\mathscr{L}_\rho$-differentiable functions to faithful modules over local algebras, i.e. representations $\rho\colon(\mathcal{A},\mathfrak{m},\mathbb{K})\hookrightarrow\operatorname{End}_{\mathbb{K}}(M)$, where ${}_{\mathcal{A}}M$ is indecomposable. Similarly, the same combination reduces the study of $\mathscr{L}_\varphi$-differentiable functions to inclusions of local algebras $\varphi\colon(\mathcal{A},\mathfrak{m},\mathbb{K})\hookrightarrow(\mathcal{B},\mathfrak{n},\mathbb{L})$, $\mathbb{L}\in\{\mathbb{R},\mathbb{C}\}$. In other words, this suggests that the only morphisms of essence for our to-be-constructed category $\mathsf{FdCFkth}$ are inclusions of local algebras[13].

(3) The question of whether a φ-differentiable function $f\colon U\to\mathcal{B}$ is of the form $f=g\circ\varphi$ for some $\mathcal{B}$-differentiable function g thus amounts to whether f is always a dimension-lowering restriction of such g and, relatedly, whether every $\mathcal{A}$-differentiable function h admits an extension to a $\mathcal{B}$-differentiable function g along an inclusion $\mathcal{A}\subseteq\mathcal{B}$ of $\mathbb{K}$-algebras. This becomes clear once

[13] the eerie similarity in that regard with the category of field extensions is not lost on the author;

we establish (again) analyticity in φ (see Proposition 2.3.17). In other words, we will have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{A} & U \arrow[l, hook'] \arrow[r, "f"] \arrow[d, hook, "\varphi"'] & \mathcal{B} \\
\mathcal{B} \arrow[from=u, hook, "\varphi"'] & V \arrow[l, hook'] \arrow[ur, dashed, "\exists_1 g \in \mathcal{O}_{\mathcal{B}}(V)"']
\end{tikzcd}$$

Even though $\mathcal{A}$ can be a subspace of $\mathcal{B}$ of positive codimension, the φ-holomorphic function of several complex variables f extends locally in a unique fashion to a $\mathcal{B}$-holomorphic function g of several complex variables. Put differently, the presence of the additional algebra structures $\mathcal{A}$ and $\mathcal{B}$ provides extra rigidity for f to *extend uniquely throughout dimensions*.

(4) As previously mentioned, a version of the decomposition of an $\mathscr{L}_{\mathcal{A}}$-differentiable function f into $\mathscr{L}_{\mathcal{A}_j}$-differentiable components f_j, $1 \leq j \leq n$, where $\mathcal{A} = \prod_{j=1}^{n}(\mathcal{A}_j, \mathfrak{m}_j, \mathbb{K}_j)$, $\mathbb{K}_j \in \{\mathbb{R}, \mathbb{C}\}$, is given in [Wat92a], but we believe the proof presented there to be incomplete. The author proves this first in the case of products of copies of $\mathbb{R}$ and $\mathbb{C}$ by using the multiplicativity of the absolute value on $\mathbb{K}$ in what seems to us an essential way, and then later claims that the same proof extends to the case of products of local $\mathbb{K}$-algebras. However, algebra norms are in general only submultiplicative, which appears to break his original argument.

(5) The reduction of the general case to (local) inclusions $\mathcal{A} \subseteq \mathcal{B}$ by means of Proposition 1.2.12 and Proposition 1.2.1 raises the question as to why one should still bother with morphisms of $\mathbb{K}$-algebras at all (as opposed to, say, functional restriction). We believe we can provide here some conceptual reasons in favour of doing so:

   (i) Even the case when φ is just an isomorphism of $\mathbb{K}$-algebras, say interpreted as a $\mathbb{K}$-automorphism of an isomorphism type $[\mathcal{A}]$, is already interesting. Not only it encompasses ordinary anti-holomorphy (unless we want to remove anti-holomorphy from the mathematical dictionary altogether), but $\mathrm{Aut}_{\mathbb{K}}(\mathcal{A})$ is usually quite non-trivial. For the simplest example, one readily checks that $\mathrm{Aut}_{\mathbb{K}}(\mathbb{D}) \cong \mathbb{K}^{\times}$, where $\mathbb{D} := \mathbb{D}_{\mathbb{K}} := \mathbb{D}\mathbb{K} := \mathbb{K}[X]/(X^2) = \mathbb{K}[\epsilon]$ for $\epsilon \neq 0$, $\epsilon^2 = 0$. More generally, $\mathrm{Mor}_{\mathsf{FdCAlg}_{\mathbb{K}}}(\mathcal{A}, -)$ and $\mathrm{Mor}_{\mathsf{FdCAlg}_{\mathbb{K}}}(-, \mathcal{B})$ are themselves highly non-trivial and interesting.

   (ii) Continuing the previous point, much like complex conjugation itself is anti-holomorphic, it is natural to think of $\varphi \colon \mathcal{A} \to \mathcal{B}$ as it too being *some kind of holomorphic*. Indeed, somewhat tautologically, without acknowledging some form of compatibility between domain and codomain, we cannot speak of φ, let alone of some f, as being $\mathcal{A}$-*holomorphic with values in* $\mathcal{B}$ – even though it does intuitively seem so for φ – despite the fact that, ironically, φ is precisely the link providing said compatibility. This chicken-and-egg issue remains hidden when $\mathcal{B} = \mathcal{A}$ since $\mathcal{A}$ already acts on itself in the obvious way.

   (iii) By extension, a composition of a $\mathcal{B}$-holomorphic map f with a morphism φ should also be seen as "holomorphy-like" due to its niceness. After all, in general one cannot expect (and rightly so) that a holomorphic map when precomposed with some arbitrary map should still remain equally well-behaved.

   (iv) Since we have already moved on from *canon-law* $\mathbb{K}$ and into $\mathsf{FdCAlg}_{\mathbb{K}}$ anyway, it's only natural to take it one more step further and replace $\mathsf{FdCAlg}_{\mathbb{K}}$ by its fully relative version $\mathsf{FdCAlg}_{\mathcal{A}}$. Moreover, building φ into the definition allows at a later point to glue sheaves of $\mathcal{A}$-differentiable functions for varying $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{R}}$.

   (v) Finally, a rather mundane argument is that there is no reason not to do it, since all the theory automatically works for φ-holomorphic functions in place of $\mathcal{A}$-holomorphic ones. □

1.3. The Homogeneous Cauchy–Riemann–Scheffers Equations

Recall that we have an inclusion of ρ-differentiability classes $\mathscr{L}_\rho \subseteq \mathscr{F}_\rho^{\mathrm{str}} \subseteq \mathscr{F}_\rho$. That is, all three are in particular Fréchet (i.e. totally) differentiable with a $\mathcal{A}$-equivariant differential. Unsurprisingly, this imposes a type of a generalized Cauchy–Riemann PDE condition on the partial derivatives. Moreover, even though the complex setting is obviously a special case of the real setting, it's more practical to treat both cases simultaneously and on an equal footing. To this end, we will need at first to make a superficial distinction between $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$. This is most easily resolved by introducing the "mixed" operator

$$\mathrm{d}_{\mathbb{K}} := \begin{cases} \mathrm{d}, \text{ if } \mathbb{K} = \mathbb{R} \\ \partial, \text{ if } \mathbb{K} = \mathbb{C} \end{cases} \tag{1.3.1}$$

In reality, this is rather harmless since for $\mathbb{K} = \mathbb{C}$ we have $\mathrm{d}f = \partial f$ anyway (however, part of the general theme of the present work is that we want to avoid relying conceptually on SCVs when unnecessary). Now let $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$ and $M \in \mathsf{FdMod}(\mathcal{A})$ via a representation $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$. Furthermore, let $\{a_1, \dots, a_n\} \subset \mathcal{A}$ and $\{v_1, \dots, v_m\} \subset M$ be respective choices of bases and $(\alpha_{j\ell}^k)_{1 \le j,k,\ell, \le n}$ be the corresponding structure tensor of $\mathcal{A}$ and $(\gamma_{j\ell}^k)_{1 \le j \le n, 1 \le k,\ell \le m}$ the corresponding structure tensor of the action $\mathcal{A} \curvearrowright M$. For a general variable $Z \in \mathcal{A}$ we will write $Z =: z^1 a_1 + \cdots + z^n a_n$ (alluding to the case $\mathbb{K} = \mathbb{C}$). Observe that $\mathrm{d}_{\mathbb{K}} Z = \mathrm{d}Z$ is an $\mathcal{A}$-valued differential form acting on ${}_{\mathcal{A}}M$-valued functions (see also Section 1.4 for more details). Recall from Lemma A.1.3 that we also write $1_{\mathcal{A}} =: \epsilon_1 a_1 + \cdots + \epsilon_n a_n$ for the coordinates of the identity.

Proposition 1.3.1 (Generalized Cauchy–Riemann–Scheffers[14] Equations)**:** Fix $Z_0 \in \mathcal{A}$ and let $f\colon \mathcal{A} \to M$ be a function with $f := f^1 v_1 + \cdots + f^m v_m$ for some coordinate functions $f^k\colon U \to \mathbb{K}$, $1 \le k \le m$.

(1) General form of ρ-differentiability: f is ρ-differentiable at Z_0 if and only if f is totally $\mathbb{K}$-differentiable at Z_0 and there exists an M-valued function $g := g^1 v_1 + \cdots + v^m b_m$ defined at Z_0 such that

$$\forall\, 1 \le j \le n\ \forall\, 1 \le k \le m\colon \frac{\partial f^k}{\partial z^j} = \sum_{s=1}^m \gamma_{js}^k g^s \tag{1.3.2}$$

at Z_0. In this case, $f'(Z_0) = g(Z_0)$.[15]

(2) Generalized Unital Cauchy–Riemann Equations: f is ρ-differentiable at Z_0 if and only if f is totally $\mathbb{K}$-differentiable at Z_0 and

$$\forall\, 1 \le j \le n\ \forall\, 1 \le k \le m\colon \frac{\partial f^k}{\partial z^j} = \sum_{r=1}^n \sum_{s=1}^m \epsilon_r \gamma_{js}^k \frac{\partial f^s}{\partial z^r} \tag{1.3.3}$$

at Z_0. In this case:

$$f'(Z_0) = \sum_{r=1}^n \epsilon_r \frac{\partial f}{\partial z^r}(Z_0) \tag{1.3.4}$$

(3) Generalized Cauchy–Riemann Equations with Unit: Assuming $a_1 = 1_{\mathcal{A}}$, f is ρ-differentiable at Z_0 if and only if f is totally $\mathbb{K}$-differentiable at Z_0 and

$$\forall\, 2 \le j \le n\ \forall\, 1 \le k \le m\colon \frac{\partial f^k}{\partial z^j} = \sum_{s=1}^m \gamma_{js}^k \frac{\partial f^s}{\partial z^1} \tag{1.3.5}$$

[14] In honour of Georg Scheffers (1893), who was the first to generalize the usual Cauchy-Riemann equations to finite-dimensional unital commutative associative $\mathbb{C}$-algebras (see [Sch93a]);

[15] In other words, g^k, $1 \le k \le m$, are the coordinate functions of the to-be-derivative $g = f'$ when it exists, not to be confused with the derivatives of the coordinate functions f^i, $1 \le i \le n$.

in Z_0. In this case:

$$f'(Z_0) = \frac{\partial f}{\partial z^1}(Z_0) \tag{1.3.6}$$

In particular, if $\rho = \rho_{\mathcal{A}}$, then

$$\forall\, 2 \leq j \leq n\ \forall\, 1 \leq k \leq n\colon \frac{\partial f^k}{\partial z^j} = \frac{\partial}{\partial z^1} \sum_{r=1}^{n} \alpha^k_{jr} f^r \tag{1.3.7}$$

at Z_0.

(4) Generalized Cauchy–Riemann–Scheffers Equations: f is ρ-differentiable at Z_0 if and only if f is totally $\mathbb{K}$-differentiable at Z_0 and

$$\forall\, 1 \leq i, j \leq n\ \forall\, 1 \leq k \leq m\colon \left(\sum_{r=1}^{n} \alpha^r_{ij} \frac{\partial}{\partial z^r} \right) f^k = \frac{\partial}{\partial z^j} \left(\sum_{s=1}^{m} \gamma^k_{is} f^s \right) \tag{1.3.8}$$

in Z_0.

(5) Cauchy–Riemann–Scheffers Equations: in the special case $\rho = \rho_{\mathcal{A}}$, f is $\mathcal{A}$-differentiable at Z_0 if and only if f is totally $\mathbb{K}$-differentiable at Z_0 and

$$\forall\, 1 \leq i, j, k \leq n\colon \left(\sum_{r=1}^{n} \alpha^r_{ij} \frac{\partial}{\partial z^r} \right) f^k = \frac{\partial}{\partial z^j} \left(\sum_{r=1}^{n} \alpha^k_{ir} f^r \right) \tag{1.3.9}$$

in Z_0.

Proof: To (1): Recasted in *differential form*, Eq. (1.1.6) is equivalent to

$$\mathrm{d}_{\mathbb{K}} f = \mathrm{d}_{\mathbb{K}} Z \cdot g(Z) = \mathrm{d}Z \cdot g(Z)$$

(at Z_0) for some M-valued function $g(Z)$ defined at Z_0, and then we have $f'(Z_0) = g(Z_0)$ by definition. We calculate:

$$\mathrm{d}_{\mathbb{K}} f = \sum_{s=1}^{m} v_s \mathrm{d}_{\mathbb{K}} f^s = \sum_{s=1}^{m} \sum_{r=1}^{n} \frac{\partial f^s}{\partial z^r} v_s \mathrm{d}z^r$$

On the other hand:

$$\mathrm{d}Z \cdot g(Z) = \left(\sum_{r=1}^{n} a_r \mathrm{d}z^r \right) \cdot \left(\sum_{t=1}^{m} g^t v_t \right) = \sum_{t=1}^{m} \sum_{r=1}^{n} g^t (a_r \cdot v_t) \mathrm{d}z^r = \sum_{t=1}^{m} \sum_{r=1}^{n} g^t \sum_{s=1}^{m} \gamma^s_{rt} v_s \mathrm{d}z^r =$$
$$= \sum_{s=1}^{m} \sum_{r=1}^{n} \left(\sum_{t=1}^{m} \gamma^s_{rt} g^t \right) v_s \mathrm{d}z^r$$

A direct comparison of the coefficients now yields the claim.

To (2): By definition, we have

$$\forall\, 1 \leq i \leq n\colon \frac{\partial f}{\partial z^i} = a_i \cdot f'(Z)$$

Using Eq. (A.1.4), we get:

$$\sum_{r=1}^{n} \epsilon_r \frac{\partial f}{\partial z^r} = \sum_{r=1}^{n} \epsilon_r a_r \cdot f'(Z) = f'(Z) \stackrel{\text{def}}{=} g(Z).$$

Therefore taking the k-th component gives

$$\forall\, 1 \leq k \leq m\colon g^k(Z) = \sum_{r=1}^{n} \epsilon_r \frac{\partial f^k}{\partial z^r}.$$

Now substituting this in Eq. (1.3.2) proves both directions.

To (3): Special case of (2). Notice that for $j = 1$ the system is vacuous since $\gamma^i_{1r} = \delta^i_r$.

To (4): "⇒": Using Eq. (1.3.2) and Eq. (A.3.4), we get:

$$\sum_{r=1}^{n} \alpha_{ij}^{r} \frac{\partial f^k}{\partial z^r} = \sum_{r=1}^{n} \alpha_{ij}^{r} \sum_{t=1}^{m} \gamma_{rt}^{k} g^t = \sum_{t=1}^{m} g^t \sum_{r=1}^{n} \alpha_{ij}^{r} \gamma_{rt}^{k} = \sum_{t=1}^{m} g^t \sum_{s=1}^{m} \gamma_{is}^{k} \gamma_{jt}^{s} = \sum_{s=1}^{m} \gamma_{is}^{k} \sum_{t=1}^{m} \gamma_{jt}^{s} g^t =$$
$$= \sum_{s=1}^{m} \gamma_{is}^{k} \frac{\partial f^s}{\partial z^j}$$

"⇐": Since $\mathcal{A}$ is unital, we can use Eq. (A.1.5) to obtain:

$$\sum_{t=1}^{n} \sum_{s=1}^{m} \epsilon_t \gamma_{is}^{k} \frac{\partial f^s}{\partial z^t} = \sum_{t=1}^{n} \epsilon_t \frac{\partial}{\partial z^t} \left(\sum_{s=1}^{m} \gamma_{is}^{k} f^s \right) = \sum_{t=1}^{n} \epsilon_t \left(\sum_{r=1}^{n} \alpha_{it}^{r} \frac{\partial}{\partial z^r} \right) f^k = \left(\sum_{r=1}^{n} \delta_i^r \frac{\partial}{\partial z^r} \right) f^k = \frac{\partial f^k}{\partial z^i},$$

which implies the claim by (2).

To (5): Special case of (4) with $\rho = \rho_{\mathcal{A}}$. □

Remarks:

(1) For the case $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ morphism in $\mathsf{FdCAlg}_{\mathbb{R}}$ (hence $\rho = \rho_\varphi$) Eq. (1.3.3) and Eq. (1.3.8) are also stated in [Shu99, p. 32]. One can obtain the version of the PDE system in terms of the representation matrix of φ stated there by using Section A.2 (e.g. Lemma A.2.4).

(2) Regardless of $\mathbb{K}$, total $\mathbb{K}$-differentiability implies existence of the partial $\mathbb{K}$-derivatives, which is in any case sufficient. Depending on $\mathbb{K}$, however, the assumption of total differentiability can be weakened appropriately. If $\mathbb{K} = \mathbb{C}$, then by a standard theorem of Hartogs in SCVs partial (or separate) holomorphy implies total holomorphy. Writing $z^j = x^j + iy^j$, one can further impose weaker conditions on $\frac{\partial f}{\partial x^j}$ and $\frac{\partial f}{\partial y^j}$ à la Loomen-Menchoff to ensure (partial) holomorphy. For a nice overview of such weaker conditions the interested reader may consult [GM78].

(3) In the case $\mathbb{K} = \mathbb{C}$, the (generalized) CRS-equations are PDEs of holomorphic functions in several complex variables with complex coefficients – unlike the classical CR-equations which have strictly real coefficients. Thus, *implicitly*, f additionally satisfies the classical CR-equations for each of its complex variables.

(4) More generally, any morphism $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ in $\mathsf{FdCAlg}_{\mathbb{K}}$ induces an $\mathcal{A}$-module structure ${}_{\mathcal{A}}\mathcal{B}$ which need not be free. For instance, the (generalized) CRS-equations stated above correspond to ${}_{\mathbb{K}}\mathcal{B}$ (i.e. $\mathcal{B}$ as a $\mathbb{K}$-vector space). Thus, in a sense, $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ also gives rise to generalized CRS-equations in several, *not necessarily free* $\mathcal{A}$-variables. This will be treated in a different work as part of the more general setting $\mathscr{F}_{\mathcal{A}}({}_{\mathcal{A}}M, {}_{\mathcal{A}}N)$.

(5) A short comparison between the different forms of the generalized Cauchy–Riemann equations is in order. The PDE system in Eq. (1.3.5) gives an explicit expression for each partial derivative of each component as a linear combination of the "free parameters", played by the $\frac{\partial}{\partial z^1}$-derivatives, and thus consists of only $(n-1)m$ equations, which is optimal. In comparison, the generalization of Scheffers' PDE system (Eq. (1.3.8)) consists of n^2m equations, an order of magnitude more than Eq. (1.3.5), and thus inevitably contains some "garbage". Of course, this also applies to the special case of Scheffers' original PDE system (Eq. (1.3.9)), consisting of n^3 equations as opposed to $(n-1)n$ equations. This difference can be easily illustrated already in the case $\mathbb{K} = \mathbb{R}$, $n = 2$, $\mathcal{A} = \mathbb{C}$. Indeed, for the structure tensor of the extension $\mathbb{C}/\mathbb{R}$ one has

$$\alpha_{11}^1 = 1, \alpha_{11}^2 = 0, \alpha_{12}^1 = \alpha_{21}^1 = 0, \alpha_{12}^2 = \alpha_{21}^2 = 1, \alpha_{22}^1 = -1, \alpha_{22}^2 = 0.$$

Then the PDE system (1.3.5) gives only two equations

$$\frac{\partial f^1}{\partial x^2} = \alpha^1_{21}\frac{\partial f^1}{\partial x^1} + \alpha^1_{22}\frac{\partial f^2}{\partial x^1} = -\frac{\partial f^2}{\partial x^1},\ \frac{\partial f^2}{\partial x^2} = \alpha^2_{21}\frac{\partial f^1}{\partial x^1} + \alpha^2_{22}\frac{\partial f^2}{\partial x^1} = \frac{\partial f^1}{\partial x^1},$$

which, unsurprisingly, are exactly the classical Cauchy–Riemann equations for f, while PDE system (1.3.8) additionally yields several vacuous or repeated equations:

- $i = 1, j = 1, k = 1$: $0 = 0$;
- $i = 1, j = 1, k = 2$: $\frac{\partial f^1}{\partial x^2} = \frac{\partial f^1}{\partial x^2}$;
- $i = 1, j = 2, k = 1$: $\frac{\partial f^1}{\partial x^2} = -\frac{\partial f^2}{\partial x^1}$;
- $i = 1, j = 2, k = 2$: $-\frac{\partial f^1}{\partial x^1} = -\frac{\partial f^2}{\partial x^2}$;
- $i = 2, j = 1, k = 1$: $\frac{\partial f^2}{\partial x^1} = \frac{\partial f^2}{\partial x^1}$;
- $i = 2, j = 1, k = 2$: $0 = 0$;
- $i = 2, j = 2, k = 1$: $\frac{\partial f^2}{\partial x^2} = \frac{\partial f^1}{\partial x^1}$;
- $i = 2, j = 2, k = 2$: $-\frac{\partial f^1}{\partial x^1} = \frac{\partial f^1}{\partial x^2}$;

(6) The PDE system (1.3.5) can also be written in the form

$$\Gamma_j \frac{\partial}{\partial z^1}\begin{pmatrix} f^1 \\ \vdots \\ f^m \end{pmatrix} = \frac{\partial}{\partial z^j}\begin{pmatrix} f^1 \\ \vdots \\ f^m \end{pmatrix},\ 2 \le j \le n, \tag{1.3.10}$$

where $\Gamma_j \overset{\text{def}}{=} (\gamma^k_{j\ell})_{1\le k,\ell\le m} = \rho(a_j) \in \mathrm{M}_m(\mathbb{K}) \cong \mathrm{End}_{\mathbb{K}}(M)$, $1 \le j \le n$. In particular, if $\rho = \rho_{\mathcal{A}}$, then $\Gamma_j = A_j \overset{\text{def}}{=} \rho_{\mathcal{A}}(a_j) = (\alpha^k_{j\ell})_{1\le k,\ell\le n} \in \mathrm{M}_n(\mathbb{K})$, $1 \le j \le n$. Thus, if for instance $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{K})$ is equipped with a choice of basis $\{a_1, \ldots, a_n\}$ such that $a_1 := 1_{\mathcal{A}}$ and $a_2, \ldots, a_n \in \mathfrak{m}$, then necessarily $\forall\, 2 \le j \le n$: $\det(A_j) = 0$, so there is no a priori reason to expect the matrices Γ_j, $2 \le j \le n$, to be invertible, either. It follows that for some "natural" basis choices like the above it is generically not possible to express $f'(Z) = \frac{\partial f}{\partial z^1}$ via the remaining paartial derivatives $\frac{\partial f^i}{\partial z^j}$, $2 \le j \le n$ (see also examples below). Nevertheless, it might be possible to paste together different parts of the PDE systems for varying $2 \le j \le n$ to obtain a full rank linear system for the vector $\frac{\partial f}{\partial z^1}$. In any case, this is dependent on the choice of basis for $\mathcal{A}$.

(7) Continuing the previous point, since $\mathcal{A}^\times$ is open, it is always possible to pick a basis of units. If $a_1, \ldots, a_n \in \mathcal{A}^\times$, then $A_1, \ldots, A_n \in \mathrm{GL}_n(\mathbb{K})$ and hence

$$\forall\, 2 \le j \le n\colon f'(Z) = \frac{\partial f}{\partial z^1} = A_j^{-1}\frac{\partial f}{\partial z^j}.$$

Constructively, if $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{K})$ and we start with a $\mathbb{K}$-basis $a_1 := 1_{\mathcal{A}}$ and $a_2, \ldots, a_n \in \mathfrak{m}$, then we can obtain a new basis $a'_1 := 1_{\mathcal{A}}, a'_2, \ldots, a'_n$ with the property that $\exists\, 2 \le j \le n\colon a'_j \in \mathcal{A}^\times$ as follows: let $\epsilon_2, \ldots, \epsilon_n \in \{0, 1\}$, not all of which are 0, and set

$$\begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \\ \vdots \\ a'_n \end{pmatrix} := \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ \epsilon_2 & 1 & 0 & \ldots & 0 \\ \epsilon_3 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \epsilon_n & 0 & 0 & \ldots & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix} \tag{1.3.11}$$

This is clearly again a basis for $\mathcal{A}$ since the transition matrix has det $= 1$, and $a'_j \in \mathcal{A}^\times$ for all $2 \le j \le n$ with $\epsilon_j = 1$. Observe that even though taking the tranposed version of the above matrix produces a valid new choice of $\mathbb{K}$-basis, it doesn't resolve the issue. □

Problem 7: All of the PDE systems in Proposition 1.3.1 can be written down *ad hoc* (as opposed to deduced) for arbitrary structure tensors $(\alpha^k_{j\ell})$ and $(\gamma^k_{j\ell})$, but the different versions of the equations will no longer remain equivalent. It is an interesting question if these equations still enjoy a meaningful interpretation in the setting of finite-dimensional non-commutative and/or non-associative $\mathbb{K}$-algebras and their finite-dimensional representations. □

Examples:

(1) If $\mathcal{A} := \mathbb{C}[\epsilon]$, $\epsilon^2 = 0$, (dual numbers), then $\alpha_{11}^1 = 1$, $\alpha_{11}^2 = 0$, $\alpha_{12}^1 = \alpha_{21}^1 = 0$, $\alpha_{12}^2 = \alpha_{21}^2 = 1$, $\alpha_{22}^1 = \alpha_{22}^2 = 0$, hence

$$\frac{\partial f^1}{\partial z^2} = 0 \text{ and } \frac{\partial f^2}{\partial z^2} = \frac{\partial f^1}{\partial z^1}.$$

In particular, we have no expression for $\frac{\partial f^2}{\partial z^1}$. On the other hand, if we choose the basis $a_1 := 1_{\mathcal{A}}$ and $a_2 := 1+\epsilon$, then $\alpha_{11}^1 = 1$, $\alpha_{11}^2 = 0$, $\alpha_{12}^1 = \alpha_{21}^1 = 0$, $\alpha_{12}^2 = \alpha_{21}^2 = 1$, $\alpha_{22}^1 = -1$, $\alpha_{22}^2 = 2$, thus

$$\frac{\partial f^1}{\partial z^2} = 2\frac{\partial f^2}{\partial z^1} \text{ and } \frac{\partial f^2}{\partial z^2} = \frac{\partial f^1}{\partial z^1} + 2\frac{\partial f^2}{\partial z^1} = \frac{\partial f^1}{\partial z^1} + \frac{\partial f^1}{\partial z^2},$$

hence

$$f'(Z) = \frac{\partial f}{\partial z^1} = \left(\frac{\partial f^2}{\partial z^2} - \frac{\partial f^1}{\partial z^2}\right) + \frac{1+\epsilon}{2}\frac{\partial f^1}{\partial z^2}$$

(2) If $\mathcal{A} := \mathbb{C}[j]$, $j^2 = 1$, (bi-complex numbers), then $\alpha_{11}^1 = 1$, $\alpha_{11}^2 = 0$, $\alpha_{12}^1 = \alpha_{21}^1 = 0$, $\alpha_{12}^2 = \alpha_{21}^2 = 1$, $\alpha_{22}^1 = 1$, $\alpha_{22}^2 = 0$, hence

$$\frac{\partial f^1}{\partial z^2} = \frac{\partial f^2}{\partial z^1} \text{ and } \frac{\partial f^2}{\partial z^2} = \frac{\partial f^1}{\partial z^1}.$$

□

Definition 1.3.2 (Generalized Wirtinger Operators)**:** Let $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$, $M \in \mathsf{FdMod}(\mathcal{A})$, and $U \subseteq \mathcal{A}$ open. For $\{a_1, \dots, a_n\} \subset \mathcal{A}$ a $\mathbb{K}$-basis and $f\colon U \to M$ partially $\mathbb{K}$-differentiable with respect to it at $Z_0 \in U$, we define the linear differential operators

$$(\mathrm{d}_{ij} f)(Z_0) := a_i \cdot \frac{\partial f}{\partial z^j}(Z_0) - a_j \cdot \frac{\partial f}{\partial z^i},\ 1 \le i, j \le n. \tag{1.3.12}$$

Furthermore, if $a_1 = 1_{\mathcal{A}}$, we also put

$$(\mathrm{d}_j f)(Z_0) := (\mathrm{d}_{1j} f)(Z_0) \overset{\text{def}}{=} \frac{\partial f}{\partial z^j}(Z_0) - a_j \cdot \frac{\partial f}{\partial z^1}(Z_0),\ 2 \le j \le n. \tag{1.3.13}$$

Remarks:

(1) Antisymmetry: clearly, $\forall\, 1 \le i, j \le n\colon \mathrm{d}_{ij} = -\mathrm{d}_{ji}$.

(2) Basis change: suppose that $(a'_1, \dots, a'_n) = (a_1, \dots, a_n) \cdot (u^i_j)_{1 \le i,j \le n}$ is a change of basis for $\mathcal{A}$ and let $Z \overset{\text{def}}{=} z^1 a_1 + \dots + z^n a_n =: w^1 a'_1 + \dots + w^n a'_n$ be the components of Z in this new basis. If we define

$$\mathrm{d}'_{k\ell} := a'_k \frac{\partial}{\partial w^\ell} - a'_\ell \frac{\partial}{\partial w^k},$$

then one immediately checks that

$$\forall\, 1 \le k, \ell \le n\colon \mathrm{d}'_{k\ell} = \sum_{i,j=1}^n u^i_k u^j_\ell \mathrm{d}_{ij}. \tag{1.3.14}$$

In particular, even though d_{ij} themselves depend on the choice of $\mathbb{K}$-basis for $\mathcal{A}$, their kernels $\mathrm{Ker}(\mathrm{d}_{ij})$ do not, $1 \le i, j \le n$.

Proposition 1.3.3: Let $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$, let $\{a_1, \dots, a_n\}$ be any $\mathbb{K}$-basis for $\mathcal{A}$, $M \in \mathsf{FdMod}(\mathcal{A})$, $U \subseteq \mathcal{A}$ open, and $f\colon U \to M$ totally $\mathbb{K}$-differentiable. Then

$$f \in \mathscr{F}_\rho(U) \Leftrightarrow f \in \bigcap_{1 \le i < j \le n} \mathrm{Ker}\, \mathrm{d}_{ij}. \tag{1.3.15}$$

If additionally $a_1 = 1_{\mathcal{A}}$, then

$$f \in \mathscr{F}_\rho(U) \Leftrightarrow f \in \bigcap_{2\le j\le n} \operatorname{Ker} \mathrm{d}_j. \tag{1.3.16}$$

Proof: "$\Rightarrow$": Writing $Z = z^1 a_1 + \dots + z^n a_n$, we have $\forall\, 1 \le i, j \le n$:

$$\mathrm{d}_{ij}(f(Z)) \overset{\text{def}}{=} a_i \cdot \frac{\partial}{\partial z^j} f(Z) - a_j \cdot \frac{\partial}{\partial z^i} f(Z) \overset{\text{def}}{=} a_i \cdot (a_j \cdot f'(Z)) - a_j \cdot (a_i \cdot f'(Z)) = (a_i a_j - a_j a_i) f'(Z) = 0.$$

by definition of Fréchet ρ-differentiability and commutativity of $\mathcal{A}$.

"$\Leftarrow$": We show that f satisfies Eq. (1.3.5). Let $\{v_1, \dots, v_m\}$ be a $\mathbb{K}$-basis for M and write $f = f^1 v_1 + \dots + f^m v_m$. We have:

$$a_j \cdot \frac{\partial f}{\partial z^1} \overset{\text{def}}{=} a_j \cdot \sum_{s=1}^m \frac{\partial f^s}{\partial z^1} v_s = \sum_{s=1}^m \frac{\partial f^s}{\partial z^1} a_j \cdot v_s = \sum_{s=1}^m \frac{\partial f^s}{\partial z^1} \sum_{r=1}^m \gamma_{js}^r v_r = \sum_{r=1}^m \left(\sum_{s=1}^m \gamma_{js}^r \frac{\partial f^s}{\partial z^1} \right) v_r$$

Since $f \in \bigcap_{2\le j\le n} \operatorname{Ker} \mathrm{d}_j$, it follows that

$$\forall\, 2 \le j \le n\colon \frac{\partial f}{\partial z^j} = a_j \cdot \frac{\partial f}{\partial z^1} = \sum_{r=1}^m \left(\sum_{s=1}^m \gamma_{js}^r \frac{\partial f^s}{\partial z^1} \right) v_r.$$

Taking the r-th component on both sides yields

$$\forall\, 1 \le r \le m\ \forall\, 2 \le j \le n\colon \frac{\partial f^r}{\partial z^j} = \sum_{s=1}^m \gamma_{js}^r \frac{\partial f^s}{\partial z^1}$$

as desired. $\square$

Here is a direct way to derive the operators in Eq. (1.3.13). They are a simple consequence of a "change of variables" (more precisely, change of (co)frames). Let $\{a_1, \dots, a_n\}$ be a $\mathbb{K}$-basis for $\mathcal{A}$ and put $Z_1 := Z \overset{\text{def}}{=} z^1 a_1 + \dots + z^n a_n$. Our goal is to find "complementary" (or "conjugated") variables $Z_2, \dots, Z_n$ and corresponding derivations $\frac{\partial}{\partial Z_j}$, $1 \le j \le n$, such that

$$\forall\, 1 \le i, j \le n\colon \frac{\partial Z_i}{\partial Z_j} = \delta_{ij}. \tag{1.3.17}$$

Define

$$\begin{pmatrix} Z \\ Z_2 \\ Z_3 \\ \vdots \\ Z_n \end{pmatrix} := \begin{pmatrix} 1 & a_2 & a_3 & \dots & a_n \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} z^1 \\ z^2 \\ z^3 \\ \vdots \\ z^n \end{pmatrix} =: U \begin{pmatrix} z^1 \\ z^2 \\ z^3 \\ \vdots \\ z^n \end{pmatrix} \tag{1.3.18}$$

in $\mathcal{A}^n$, where clearly $U \in \mathrm{GL}_n(\mathcal{A})$. In $\Gamma(T^*\mathcal{A} \otimes_{\mathbb{K}} \mathcal{A}) \cong \Gamma(T^*\mathcal{A}) \otimes_{\mathbb{K}} \mathcal{A}$ this immediately yields

$$\begin{pmatrix} \mathrm{d}Z \\ \mathrm{d}Z_2 \\ \mathrm{d}Z_3 \\ \vdots \\ \mathrm{d}Z_n \end{pmatrix} := \begin{pmatrix} 1 & a_2 & a_3 & \dots & a_n \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}z^1 \\ \mathrm{d}z^2 \\ \mathrm{d}z^3 \\ \vdots \\ \mathrm{d}z^n \end{pmatrix}. \tag{1.3.19}$$

In $\Gamma(T\mathcal{A} \otimes_{\mathbb{K}} \mathcal{A}) \cong \Gamma(T\mathcal{A}) \otimes_{\mathbb{K}} \mathcal{A}$ we thus get:

$$\begin{pmatrix} \frac{\partial}{\partial Z} \\ \frac{\partial}{\partial Z_2} \\ \frac{\partial}{\partial Z_3} \\ \vdots \\ \frac{\partial}{\partial Z_n} \end{pmatrix} = (U^{-1})^T \begin{pmatrix} \frac{\partial}{\partial z^1} \\ \frac{\partial}{\partial z^2} \\ \frac{\partial}{\partial z^3} \\ \vdots \\ \frac{\partial}{\partial z^n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -a_2 & 1 & 0 & \dots & 0 \\ -a_3 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \\ -a_n & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial z^1} \\ \frac{\partial}{\partial z^2} \\ \frac{\partial}{\partial z^3} \\ \vdots \\ \frac{\partial}{\partial z^n} \end{pmatrix} \tag{1.3.20}$$

That is,

$$\frac{\partial}{\partial Z} = \frac{\partial}{\partial z^1} \text{ and } \forall\, 2 \le j \le n\colon \frac{\partial}{\partial Z_j} = -a_j \frac{\partial}{\partial z^1} + \frac{\partial}{\partial z^j} = \frac{\partial}{\partial z^j} - a_j \frac{\partial}{\partial z^1} \stackrel{\text{def}}{=} \mathrm{d}_j. \tag{1.3.21}$$

Then ρ-differentiability of f at a point $Z_0 \in U$ is simply equivalent to

$$\forall\, 2 \le j \le n\colon \frac{\partial f}{\partial Z_j}(Z_0) = 0.,$$

which gives another proof of Eq. (1.3.16) (assuming total $\mathbb{K}$-differentiability of f at Z_0).

1.4. The Associated 1-Form & Integrability

This section naturally bridges the current Chapter 1 with the next Chapter 2. Given $(\mathcal{A}, \|\cdot\|_{\mathcal{A}}) \in \mathsf{NrmFdCAlg}_{\mathbb{K}}$, $U \subseteq \mathcal{A}$ open, $(M, \|\cdot\|_M) \in \mathsf{NrmFdMod}(\mathcal{A})$ via $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$, where $\mathrm{End}_{\mathbb{K}}(M)$ is equipped with $\|\cdot\|_{\mathrm{End}} := \|\cdot\|_{\mathrm{End}_{\mathbb{K}}(M)} := \|\cdot\|_{M,M}$ as usual (cf. Section 0.3), and $f \in \mathscr{F}_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$, we discuss here the basic properties of the *associated* differential 1-form

$$\omega := \mathrm{d}Z \cdot f(Z) \overset{\mathrm{def}}{=} \rho(\mathrm{d}Z) f(Z) \in \Omega^1(U, M).$$

Moreover, observe that since $\mathcal{A}$ is commutative, M is naturally an $\mathcal{A}$-bimodule with both actions coinciding, hence we can also write our differential form in the more standard way

$$\omega = f(Z) \cdot \mathrm{d}Z$$

with $\cdot$ also standing for the right $\mathcal{A}$-action. This notation has the advantage of lending itself rather well to exterior products & derivatives. Nevertheless, by convention older than mathematics itself, $\mathrm{End}_{\mathbb{K}}(M)$ acts on M from the left, hence in the cases when we need to mention ρ explicitly, $\rho(\mathrm{d}Z)$ will too act on $f(Z)$ only from the left. This also encompasses the morphism case $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ for $\mathcal{B} \in \mathsf{NrmFdCAlg}(\mathcal{A})$ by passing to $\rho_\varphi := \varphi^{\mathrm{rep}}$, which is the more natural approach for this section than treating the representation setting as a special case of the morphism setting via ρ^{mor} since we won't be making any use of the algebra structure of $\mathcal{B}$. We remark that this part of the theory too does not really rely on the presence of a complex structure in any essential way. Furthermore, while continuous functions can in general be integrated over rectifiable paths, in the following we will impose the somewhat stronger regularity assumption of $\mathscr{C}^1_{\mathrm{pw}}$-differentiability of the integration paths for the sake of the occasional technical convenience:

Definition 1.4.1: Let $U \subseteq \mathcal{A}$. We put for short:
(1) Piece-wise Differentiable Path Space: $\mathcal{P}U := \mathscr{C}^1_{\mathrm{pw}}([0,1], U)$;
(2) Piece-wise Differentiable Free Loop Space: $\mathcal{L}U := \mathscr{C}^1_{\mathrm{pw}}(\mathbb{S}^1, U)$.

This is mostly to fix more convenient notations as we won't have any need to endow $\mathcal{P}U$ and $\mathcal{L}U$ with any further topological structure (as is usually done). Moreover, if $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ or $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$, then we have induced maps $\varphi_\# := \mathcal{P}(\varphi)\colon \mathcal{P}\mathcal{A} \to \mathcal{P}\mathcal{B}$, $\varphi_\# := \mathcal{L}(\varphi)\colon \mathcal{L}\mathcal{A} \to \mathcal{L}\mathcal{B}$, $\gamma \mapsto \varphi \circ \gamma$, and $\rho_\# := \mathcal{P}(\rho)\colon \mathcal{P}\mathcal{A} \to \mathcal{P}\,\mathrm{End}_{\mathbb{K}}(M)$, $\rho_\# := \mathcal{L}(\rho)\colon \mathcal{L}\mathcal{A} \to \mathcal{L}\,\mathrm{End}_{\mathbb{K}}(M)$, $\gamma \mapsto \rho \circ \gamma$. Analogously for topological subspaces $U \subseteq \mathcal{A}$. It is clear that $\mathcal{P}(-)$ and $\mathcal{L}(-)$ behave functorially.

Lemma-Definition 1.4.2: Let $\gamma \in \mathcal{P}\mathcal{A}$.
(1) $L_\rho(\gamma) := L_{\mathrm{End}}(\rho_\#\gamma) = \int_0^1 \|\rho(\gamma'(t))\|_{\mathrm{End}}\,\mathrm{d}t$ will denote the *relative* (in the case $\rho = \varphi^{\mathrm{rep}}$) or *represented* length of γ.
(2) We also put for short $L_\varphi := L_{\varphi^{\mathrm{rep}}}$ and $L_{\mathcal{A}} := L_{\mathrm{id}_{\mathcal{A}}}$.
(3) If γ is a scalar curve, then $L_\rho(\gamma) = L_{\mathbb{K}}(\gamma)$.

Proof: To (1): Since ρ is automatically continuous and always $\mathbb{R}$-linear, we have $(\rho(\gamma(t)))' = \rho(\gamma'(t))$.

To (3): Since γ is a curve in $\mathbb{K} \hookrightarrow \mathcal{A}$ and ρ is $\mathbb{K}$-linear, we have $\|\rho(\gamma'(t))\|_{\mathrm{op}} = |\gamma'(t)|\,\|\rho(1_{\mathcal{A}})\|_{\mathrm{End}} = |\gamma'(t)|$ since $\|\cdot\|_{\mathrm{End}}$ is unital. ☐

Definition 1.4.3: If $\Gamma := \sum_{\ell=1}^n m_\ell \gamma_\ell \in Z_1(U, \mathbb{Z})$, then $L_\rho(\Gamma) := \sum_{\ell=1}^n |m_\ell|\, L_\rho(\gamma_\ell)$ will be called the total *relative* or *represented* length of Γ. We also put $L_\varphi := L_{\varphi^{\mathrm{rep}}}$.

Remarks:
(1) $L_\rho(\Gamma)$ should not be confused with the (relative / represented) length of the *support* of Γ.
(2) As remarked previously, we have $L_\varphi(\Gamma) \overset{\mathrm{def}}{=} L_{\rho_\varphi}(\Gamma) = L_{\mathcal{B}}(\varphi_\#\Gamma)$.

(3) We have

$$L_\rho(\gamma) = \int_0^1 \left\|\rho(\gamma'(t))\right\|_{\mathrm{End}} \mathrm{d}t \le \int_0^1 \|\rho\|_{\mathrm{op}} \left\|\gamma'(t)\right\|_{\mathcal{A}} \mathrm{d}t = L_{\mathcal{A}}(\gamma),$$

so $L_\rho(\gamma)$ yields sharper upper estimates.

(4) Since we are working with paths in $\mathrm{End}_{\mathbb{K}}(M)$, equipping it with the non-unital norm $\|\cdot\|_{\mathrm{F}}$ would have given the Euclidean length of $\rho_{\#}\gamma$. On the other hand, when γ is a *scalar* curve, the proof shows that it is more practical to choose a unital norm. □

Let $U \subseteq \mathbb{R}^n$ be open, $V_1, V_2 \in \mathsf{FdVec}_{\mathbb{R}}$, and $\Phi \in \mathrm{Hom}_{\mathbb{R}}(V_1, V_2)$. Then Φ induces an obvious map of vector-valued differential 1-forms (also denoted by) $\Phi\colon \Omega^1(U, V_1) \to \Omega^1(U, V_2)$, $\omega \mapsto \Phi(\omega)$. Moreover, since Φ is automatically continuous, we also have

$$\int_\gamma \Phi(\omega) = \Phi\Big(\int_\gamma \omega\Big). \tag{1.4.1}$$

In particular, if $U \subseteq \mathcal{A}$ is open, we get maps $\rho\colon \Omega^1(U, \mathcal{A}) \to \Omega^1(U, \mathrm{End}_{\mathbb{K}}(M))$ and $\varphi\colon \Omega^1(U, \mathcal{A}) \to \Omega^1(U, \mathcal{B})$.

Remark: Even if $f \in \mathscr{C}^\omega_\varphi(U)$, then $\omega := \varphi(\mathrm{d}Z)f(Z) \in \Omega^1(U, \mathcal{B})$ is not necessarily of the form $\varphi(\eta)$ for some $\eta \in \Omega^1(U, \mathcal{A})$, since f need not be of the form $\varphi \circ g$ for $g \in \mathscr{C}^\omega_{\mathcal{A}}(U)$ (see Section 1.1).

Definition 1.4.4: Let $U \subseteq \mathcal{A}$ be open, $f\colon U \to M$ a function, and $k \in \mathbb{N} \cup \{\infty\}$.

(1) f is called *$\mathscr{L}_\rho$-integrable* (on U) if $\exists\, F \in \mathscr{L}_\rho(U)\colon F' = f$. In this case F will be called a *$\mathscr{L}_\rho$-primitive or -anti-derivative* of f (on U). Then

$$\mathscr{L}_\rho^{-1}(U) := \{f\colon U \to M \mid f \text{ is } \mathscr{L}_\rho\text{-integrable}\}$$

will denote the space of $\mathscr{L}_\rho$-integrable functions on U. More generally, we put

$$\mathscr{L}_\rho^{-k}(U) := \{f\colon U \to M \mid f \text{ is } k\text{-times } \mathscr{L}_\rho\text{-integrable}\}.$$

(2) f is called *$\mathscr{F}_\rho^{\mathrm{str}}$-integrable* (on U) if $\exists\, F \in \mathscr{F}_\rho^{\mathrm{str}}(U)\colon F' = f$. In this case F will be called a *$\mathscr{F}_\rho^{\mathrm{str}}$-primitive or -anti-derivative* of f on U. Then

$$\mathscr{F}_\rho^{\mathrm{str},-1}(U) := \{f\colon U \to M \mid f \text{ is } \mathscr{F}_\rho^{\mathrm{str}}\text{-integrable}\}$$

will denote the space of $\mathscr{F}_\rho^{\mathrm{str}}$-integrable functions on U. More generally, we put

$$\mathscr{F}_\rho^{\mathrm{str},-k}(U) := \{f\colon U \to M \mid f \text{ is } k\text{-times } \mathscr{F}_\rho^{\mathrm{str}}\text{-integrable}\}$$

(3) f is called *$\mathscr{F}_\rho$-integrable* (on U) if $\exists\, F \in \mathscr{F}_\rho(U)\colon F' = f$. In this case F will be called a *$\mathscr{F}_\rho$-primitive* of f (on U). Then

$$\mathscr{F}_\rho^{-1}(U) := \{f\colon U \to M \mid f \text{ is } \mathscr{F}_\rho\text{-integrable}\}$$

will denote the space of $\mathscr{F}_\rho$-integrable functions on U. More generally, we put

$$\mathscr{F}_\rho^{-k}(U) := \{f\colon U \to M \mid f \text{ is } k\text{-times } \mathscr{F}_\rho\text{-integrable}\}.$$

(4) f is called *$\mathscr{C}^1_\rho$-integrable* (on U) if $\exists\, F \in \mathscr{C}^1_\rho(U)\colon F' = f$. In this case F will be called a *$\mathscr{C}^1_\rho$-primitive* of f (on U). Then

$$\mathscr{C}_\rho^{-1}(U) := \{f \in \mathscr{C}^0(U, M) \mid f \text{ is } \rho\text{-integrable}\}$$

will denote the space of $\mathscr{C}^1_\rho$-integrable functions on U. More generally, we put

$$\mathscr{C}_\rho^{-k}(U) := \{f \in \mathscr{C}^0(U, M) \mid f \text{ is } k\text{-times } \rho\text{-integrable}\}.$$

Analogously for a morphism $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$, or equivalently, for the special case $\rho = \rho_\varphi$. □

Remarks:

(1) When the regularity class is understood, we will just speak of *ρ-integrable*, *ρ-primitive* or *ρ-anti-derivative* like we already do occasionally for *ρ-differentiable*. Similarly, we will use the terms *$\mathcal{A}$-integrable*, *$\mathcal{A}$-primitive* or *$\mathcal{A}$-anti-derivative* synonymously to their ρ-counterparts (i.e. module POV vs. representation POV) like we already do for *$\mathcal{A}$-differentiability*. Analogously for a morphism $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ in place of a representation $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$, or equivalently, in the special case $\rho = \rho_\varphi$.

(2) Recall that by Proposition 1.1.14 we have $\mathscr{L}_\rho(U) \cap \mathscr{C}^1(U, M) = \mathscr{F}_\rho^{\mathrm{str}}(U) \cap \mathscr{C}^1(U, M) = \mathscr{F}_\rho(U) \cap \mathscr{C}^1(U, M) \overset{\mathrm{def}}{=} \mathscr{C}^1_\rho(U)$. Hence, if we replace $f\colon U \to M$ by the additional requirement $f \in \mathscr{C}^0(U, M)$, then for the ρ-primitive we would have automatically $F \in \mathscr{C}^1_\rho(U)$ as well as $\forall\, k \in \mathbb{N} \cup \{\infty\}\colon \mathscr{L}_\rho^{-k}(U) = \mathscr{F}_\rho^{\mathrm{str},-k}(U) = \mathscr{F}_\rho^{-k}(U)$. In this case we can speak of *ρ-integrability* and *ρ-primitives* without any ambiguity left.

(3) To be completely rigorous, we have $\mathscr{L}_\rho^{-\infty}(U) \overset{\mathrm{def}}{=} \bigcap_{k\in\mathbb{N}} \mathscr{L}_\rho^{-k}(U)$ and likewise for $\mathscr{F}_\rho^{\mathrm{str}}$ and $\mathscr{F}_\rho$. □

Lemma 1.4.5: Let $U \subseteq \mathcal{A}$ be open, $f \in \mathscr{C}^0(U, M)$ ρ-integrable, and $\gamma \in \mathcal{L}U$. Then

$$\oint_\gamma \rho(\mathrm{d}Z) f(Z) = 0.$$

Proof: Let $F \in \mathscr{C}^1_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$ be a ρ-primitive of f. Then

$$\oint_\gamma \rho(\mathrm{d}Z) f(Z) = \oint_\gamma \rho(\mathrm{d}Z) F'(Z) = \int_0^1 \rho(\gamma'(t)) F'(\gamma(t)) \mathrm{d}t = \int_0^1 \mathrm{d}(F \circ \gamma) = F(\gamma(1)) - F(\gamma(0)) = 0$$

where we have used that $\mathrm{d}F$ is $\mathcal{A}$-linear (and possible subdivision of γ pieces). □

Lemma 1.4.6: Let $\gamma \in \mathcal{L}\mathcal{A}$ and $f \in \mathscr{C}^0(\gamma, M)$. Then:

$$\left\| \oint_\gamma \rho(\mathrm{d}Z) f(Z) \right\|_M \le \|f\|_{M,\gamma} L_\rho(\gamma) \le \|f\|_{M,\gamma} L_{\mathcal{A}}(\gamma).$$

Proof: We have

$$\left\| \oint_\gamma \rho(\mathrm{d}Z) f(Z) \right\|_M = \left\| \int_0^1 \rho(\gamma'(t)) f(\gamma(t)) \mathrm{d}t \right\|_M \le \int_0^1 \|\rho(\gamma'(t))\|_{\mathrm{End}} \|f(\gamma(t))\|_M \,\mathrm{d}t \le$$
$$\le \sup_{Z\in\gamma} \|f(Z)\|_M \int_0^1 \|\rho(\gamma'(t))\|_{\mathrm{End}} \,\mathrm{d}t \overset{\mathrm{def}}{=} \|f\|_{M,\gamma} L_\rho(\gamma)$$

as desired. □

Lemma 1.4.7 (Pre-Morera: ρ-integrability of continuous functions)**:**

(1) Let $U \subseteq \mathcal{A}$ be open with $\pi_0(U) = 1$ and let $f \in \mathscr{C}^0(U, M)$ be such that

$$\forall\, \gamma \in \mathcal{L}U\colon \oint_\gamma \rho(\mathrm{d}Z) f(Z) = 0.$$

Then f is (globally) ρ-integrable.

(2) Let $☆ \subseteq \mathcal{A}$ be open and star-convex with center $* \in ☆$ and let $f \in \mathscr{C}^0(☆, M)$ be such that for any triangle $\triangle \subseteq ☆$ with vertex at $*$

$$\oint_{\partial\triangle} \rho(\mathrm{d}Z) f(Z) = 0.$$

Then f is ρ-integrable in $☆$ with a ρ-primitive given by

$$F(Z) \coloneqq \int_*^Z \rho(\mathrm{d}W) f(W).$$

In particular, it follows that

$$\forall\, \gamma \in \mathcal{L}☆\colon \oint_\gamma \rho(\mathrm{d}Z) f(Z) = 0.$$

Proof: To (1): Fix $* \in U$. By assumption, the function

$$F(Z) := \int_*^Z \rho(\mathrm{d}W) f(W),$$

where the integral is taken over an arbitrary simple $\mathscr{C}^1_{\mathrm{pw}}$-path from $*$ to Z, is well-defined as it is independent of the choice of the integration path. We are going to show that F is a ρ-primitive of f. For a fixed Z we have:

$$F(Z+H) - F(Z) \overset{\mathrm{def}}{=} \int_Z^{Z+H} \rho(\mathrm{d}W) f(W) = f(Z) \int_Z^{Z+H} \rho(\mathrm{d}W) + \int_Z^{Z+H} \rho(\mathrm{d}W) g(W) =$$
$$= \rho(H) f(Z) + \int_Z^{Z+H} \rho(\mathrm{d}W) g(W),$$

where $g(W) := f(W) - f(Z) \xrightarrow{W \to Z} 0$ by continuity of f. Since $\mathcal{A}$ is automatically locally convex, we have for all sufficiently small H that $U \supseteq [Z, Z+H]$, which we now also take to be our integration path. Using Lemma 1.4.6, we estimate the last term as

$$\left\| \int_Z^{Z+H} \rho(\mathrm{d}W) g(W) \right\|_M \leq \|g\|_{M,[Z,Z+H]}\, L_\rho([Z, Z+H]),$$

where $\|g\|_{M,[Z,Z+H]} \xrightarrow{H \to 0} 0$ because $g(W) \xrightarrow{W \to Z} 0$. Since ρ is in any case $\mathbb{R}$-linear,

$$L_\rho([Z, Z+H]) \overset{\mathrm{def}}{=} L_{\mathrm{End}}(\rho([Z, Z+H])) = L_{\mathrm{End}}([\rho(Z), \rho(Z) + \rho(H)]) = \|\rho(H)\|_{\mathrm{End}}\,.$$

Therefore $F(Z+H) - F(Z) = \rho(H) f(Z) + o(\|\rho(H)\|_{\mathrm{End}})$ (as $H \to 0$), i.e. $F \in \mathscr{F}^{\mathrm{str}}_\rho(U)$.

To (2): For H small enough we have $☆ \supseteq \triangle$ defined by $\partial\triangle := [*, Z+H, Z, *] = [*, Z+H] + [Z+H, Z] + [Z, *]$. Since $\oint_{\partial\triangle} f(W)\rho(\mathrm{d}W) = 0$, we get $F(Z+H) - F(Z) = \int_{[Z,Z+H]} f(W)\rho(\mathrm{d}W)$, from where one proceeds as in (1). □

Remarks:

(1) In (1) and its proof $\mathscr{C}^1_{\mathrm{pw}}$-regularity of γ can be replaced by rectifiability without adjustment.

(2) In the proof of (1) we estimated the remaining expression in terms of the sharper L_ρ instead of $L_\mathcal{A}$, which in turn produced strict ρ-differentiability ($o(\|\rho(H)\|_{\mathrm{End}})$) for F instead of plain ρ-differentiability ($o(\|H\|_\mathcal{A})$). Of course, since f is assumed continuous, both are equivalent, as previously remarked. □

The following observation for the case $\varphi = \mathrm{id}_\mathcal{A}$ is contained in [Wat92b, Thm. 2.2], but we remark that it must have been well-known already much earlier to the Kazan school of $\mathcal{A}$-geometry, though we haven't been able to pinpoint a precise reference for this. Waterhouse does not offer an explicit proof, but instead defers to a (somewhat tedious) computation via the generalized Cauchy–Riemann equations (see for instance Proposition 1.3.1). For these reasons we include here a short and sweet coordinate-free[16] proof. We follow the notations introduced in Section 1.3.

Proposition 1.4.8 (Closedness of ω)**:** Let $U \subseteq \mathcal{A}$ be open, $f\colon U \to M$ a function, and $\omega := f(Z)\cdot\mathrm{d}Z$. Then:

$$f \in \mathscr{F}_\mathcal{A}(U, {}_\mathcal{A}M) \Leftrightarrow f \in \mathscr{F}_\mathbb{K}(U, M) \text{ and } \mathrm{d}_\mathbb{K}\omega = 0.$$

[16] we are a little baffled by the fact that even in most books on Riemann Surfaces the verification of this key observation is done by means of the Cauchy-Riemann equations;

Proof: To (1): First observe that, since ρ commutes with $\mathrm{d}_{\mathbb{K}}$, we have

$$\mathrm{d}_{\mathbb{K}}\omega \overset{\text{def}}{=} \mathrm{d}_{\mathbb{K}}(f(Z)\cdot \mathrm{d}Z) = (\mathrm{d}_{\mathbb{K}}f)\wedge(\mathrm{d}Z) + f(Z)\cdot \mathrm{d}^2 Z = (\mathrm{d}_{\mathbb{K}}f)\wedge(\mathrm{d}Z). \tag{1.4.2}$$

in any case.

"$\Rightarrow$": If $f \in \mathscr{F}_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$, then $f \in \mathscr{F}_{\mathbb{K}}(U, M)$, hence ω is $\mathbb{K}$-differentiable[17], and $\mathrm{d}_{\mathbb{K}}f \overset{\text{def}}{=} \rho(\mathrm{d}Z)f'(Z) \overset{\text{def}}{=} f'(Z)\cdot \mathrm{d}Z$. Since ρ respects $\wedge$, it follows from Eq. (1.4.2) that $\mathrm{d}_{\mathbb{K}}\omega = (f'(Z)\cdot \mathrm{d}Z)\wedge \mathrm{d}Z = f'(Z)\cdot(\mathrm{d}Z\wedge \mathrm{d}Z) = 0$.

"$\Leftarrow$": Since $f \in \mathscr{F}_{\mathbb{R}}(U, M)$ by hypothesis, ω is $\mathbb{K}$-differentiable. Let $\{a_1, \dots, a_n\}$ be an $\mathbb{K}$-basis for $\mathcal{A}$ and write $Z = z^1 a_1 + \dots + z^n a_n$. Using $\mathrm{d}z^i \wedge \mathrm{d}z^j = -\mathrm{d}z^j \wedge \mathrm{d}z^i$, we can rewrite $\mathrm{d}_{\mathbb{K}}\omega$ in terms of the operators d_{ij}, $1 \le i < j \le n$, from Definition 1.3.2 as follows:

$$\begin{aligned}\mathrm{d}_{\mathbb{K}}\omega &= \sum_{i=1}^n \frac{\partial f}{\partial z^i}\mathrm{d}z^i \wedge \mathrm{d}Z = \sum_{i=1}^n \frac{\partial f}{\partial z^i}\mathrm{d}z^i \wedge \left(\sum_{j=1}^n a_j \mathrm{d}z^j\right) = \sum_{i,j=1}^n \left(\frac{\partial f}{\partial z^i}\cdot a_j\right)\mathrm{d}z^i\wedge \mathrm{d}z^j = \\ &= \sum_{1\le i<j\le n}\left(\frac{\partial f}{\partial z^i}\cdot a_j - \frac{\partial f}{\partial z^j}\cdot a_i\right)\mathrm{d}z^i\wedge \mathrm{d}z^j = \sum_{1\le i<j\le n}(\mathrm{d}_{ij}f)\mathrm{d}z^i\wedge\mathrm{d}z^j.\end{aligned}$$

The claim now follows from Proposition 1.3.3 (in fact, both directions do so now, but whatever). □

Lemma 1.4.9: Let $\mathfrak{m}$ be a finite-dimensional commutative associative nilpotent $\mathbb{K}$-algebra of height ν. Then $\forall\, 1 \le k \le \nu - 1$:

$$\mathrm{d}(X^k) = kX^{k-1}\mathrm{d}X \in \Omega^1(\mathfrak{m}, \mathfrak{m}). \tag{1.4.3}$$

In particular, the 1-forms $X^k \mathrm{d}X \in \Omega^1(\mathfrak{m}, \mathfrak{m})$, $1 \le k \le \nu - 2$, are d-closed.

Proof: Let $e_1, \dots, e_n$ be a $\mathbb{K}$-basis of $\mathfrak{m}$ and write $X = x_1 e_1 + \dots + x_n e_n$. Since $\frac{\partial}{\partial x_i}$ is a derivation, it is easy to see by induction that $\forall\, 1 \le k \le \nu - 1\ \forall\, 1 \le i \le n$:

$$\frac{\partial(X^k)}{\partial x_i} = kX^{k-1}e_i.$$

Hence $\forall\, 1 \le k \le \nu - 1$:

$$\mathrm{d}(X^k) = \sum_{i=1}^n \frac{\partial(X^k)}{\partial x_i}\mathrm{d}x_i = \sum_{i=1}^n kX^{k-1}e_i \mathrm{d}x_i = kX^{k-1}\sum_{i=1}^n e_i\mathrm{d}x_i = kX^{k-1}\mathrm{d}X$$

as claimed. □

Remarks:

(1) The identity $\mathrm{d}\omega = (\mathrm{d}f)\wedge(\mathrm{d}Z)$ from Eq. (1.4.2) for $f\colon U \to \mathcal{A}$ is actually the starting point of Habetha's highly insightful observation ([Hab76]) that most of the *associative function theories* at the time (incl. Quaternionic Analysis and Clifford Analysis) arise from the single condition $(\mathrm{d}f)\wedge\Omega = 0$ or $\Omega \wedge (\mathrm{d}f) = 0$, where Ω is some continuously differentiable closed p-form. In fact, a similar unifying attempt appears a few years earlier in [DR73].

(2) The direction "$\Leftrightarrow$" in Proposition 1.4.8 can also be characterized as a "differential" version Morera's theorem.

(3) We will give a *topological*, basis-free proof of Lemma 1.4.9 in Section 2.2.

(4) Even though $\mathfrak{m}$-differentiability is well-defined only (mod $\mathfrak{m}^{\nu-1}$), Lemma 1.4.9 shows that every $f(X) \in \mathfrak{m}[X]$ is at least *morally* $\mathfrak{m}$-differentiable. Moreover, we remark that showing d-closedness of $X^{k-1}\mathrm{d}X$ without having to guess a primitive is equally straightforward as it basically rests on the same inductive observation.

[17] has $\mathbb{K}$-differentiable coefficients;

(5) Much of what follows hinges upon the fact that $\omega = \rho(\mathrm{d}Z)f(Z)$ is d-closed. Another byproduct of the proof is the interpretation of $\mathrm{d}_{ij}f$ as the coordinate functions of $\mathrm{d}(\rho(\mathrm{d}Z)f(Z))$.

(6) Recall that if ω is a d-closed 1-form, then it is locally exact by Poincaré's lemma. This is one way to show that ρ-differentiable functions admit local ρ-primitives. We will give another proof of this based on a generalized Cauchy–Goursat theorem below.

(7) The existence of local ρ-primitives in turn enables the integration of ω along *continuous* paths, see for instance [BG91, Ch. 1, §7]. For the same reason, homotopy arguments about $\int \omega$ work in the continuous category, so technically we don't have to always take extra care of path regularity. Consequently, the Proposition 1.4.11 (4) can be stated for continuous paths, but for the purpose of obtaining another proof of the existence of local ρ-primitives without appealing to Poincare's lemma one should assume the paths to be rectifiable or $\mathscr{C}^1_{\mathrm{pw}}$-regular instead.

Lemma 1.4.10 (Triangles in Normed Spaces)**:** Let $(V, \|\cdot\|_V) \in \mathsf{NrmVec}_{\mathbb{K}}$ and let $\triangle \subseteq V$ be a (non-degenerate) solid triangle with (oriented) boundary given by $\partial\triangle = [v_1, v_2, v_3, v_1]$. Furthermore, let $w_1, w_2, w_3 \in \partial\triangle$ be the middle points of $[v_1, v_2]$, $[v_2, v_3]$, and $[v_3, v_1]$, respectively, and let $\triangle'$ be any of the four solid triangles obtained by additionally connecting w_1, w_2, and w_3 together. Then:

(1) $\mathrm{diam}_V(\triangle) = \max\left\{ \|[v_1, v_2]\|_V, \|[v_2, v_3]\|_V, \|[v_3, v_1]\|_V \right\}$;

(2) $\mathrm{diam}_V(\triangle) < \frac{1}{2} L_V(\partial\triangle)$;

(3) $L_V(\partial\triangle') = \frac{1}{2} L_V(\partial\triangle)$.

Proof: To (1): Put $D := \max\left\{ \|[v_1, v_2]\|_V, \|[v_2, v_3]\|_V, \|[v_3, v_1]\|_V \right\}$. Then clearly $\mathrm{diam}_V(\triangle) \geq D$ (as attained by one of the sides of $\triangle$). Thus we only need to show that $\mathrm{diam}_V(\triangle) \leq D$. Let $x, y \in \triangle$ be arbitrary. Since $\triangle$ is a triangle, x lies on a line segment between one of the vertices and the opposite side, i.e. without loss of generality $\exists\, w \in [v_2, v_3]\ \exists\, t \in [0, 1]\colon x = (1-t)v_1 + tw$. Hence

$$\begin{aligned} \|x - y\|_V &= \|(1-t)(v_1 - y) + t(w - y)\|_V \leq (1-t)\|v_1 - y\|_V + t\|w - y\|_V \leq \\ &\leq \max\left\{ \|v_1 - y\|_V, \|w - y\|_V \right\}. \end{aligned}$$

Since $w \in [v_2, v_3]$, we have that $\exists\, s \in [0, 1]\colon w = (1-s)v_2 + sv_3$, hence similarly

$$\|w - y\|_V = \|(1-s)(v_2 - y) + s(v_3 - y)\|_V \leq \max\left\{ \|v_2 - y\|_V, \|v_3 - y\|_V \right\}.$$

It follows that

$$\|x - y\|_V \leq \max\left\{ \|v_1 - y\|_V, \|v_2 - y\|_V, \|v_3 - y\|_V \right\}.$$

Applying this inequality to the special cases of the inside norms with the roles of x and y reversed (by symmetry), we get

$$\forall\, 1 \leq i \leq 3\colon\ \|v_i - y\|_V = \|y - v_i\|_V \leq \max_{\substack{1 \leq j \leq 3 \\ j \neq i}} \|v_j - v_i\|_V,$$

hence

$$\|x - y\|_V \leq \max\left\{ \|v_2 - v_1\|_V, \|v_3 - v_2\|_V, \|v_1 - v_3\| \right\} \overset{\mathrm{def}}{=} D,$$

i.e. $\mathrm{diam}_V(\triangle) \overset{\mathrm{def}}{=} \sup\limits_{x, y \in \triangle} \|x - y\|_V \leq D$ as desired.

To (2): Using (1), suppose without loss of generality that $\mathrm{diam}_V(\triangle) = \|[v_1, v_2]\|_V$. By the *triangle* inequality we have

$$\begin{aligned} \mathrm{diam}_V(\triangle) &= \|v_2 - v_1\|_V = \|v_2 - v_3 + v_3 - v_1\|_V \leq \|v_3 - v_2\|_V + \|v_1 - v_3\|_V \overset{\mathrm{def}}{=} \\ &\overset{\mathrm{def}}{=} \|[v_2, v_3]\|_V + \|[v_3, v_1]\|_V. \end{aligned}$$

It follows that

$$\begin{aligned}L_V(\partial\triangle) &\stackrel{\text{def}}{=} \|[v_1,v_2]\|_V + \|[v_2,v_3]\|_V + \|[v_3,v_1]\|_V = \operatorname{diam}_V(\triangle) + \|[v_2,v_3]\|_V + \|[v_3,v_1]\|_V \geq \\ &\geq \operatorname{diam}_V(\triangle) + \operatorname{diam}_V(\triangle) = 2\operatorname{diam}_V(\triangle),\end{aligned}$$

which implies the claim.

To (3): We prove the claim only for $\triangle'$ with $\partial\triangle' = [w_1,w_2,w_3,w_1]$. We have:

$$\begin{aligned}L_V(\partial\triangle') &= \|w_2-w_1\|_V + \|w_3-w_2\|_V + \|w_1-w_3\|_V \stackrel{\text{def}}{=} \\ &\stackrel{\text{def}}{=} \left\|\frac{v_2+v_3}{2} - \frac{v_1+v_2}{2}\right\|_V + \left\|\frac{v_3+v_1}{2} - \frac{v_2+v_3}{2}\right\|_V + \left\|\frac{v_1+v_2}{2} - \frac{v_3+v_1}{2}\right\|_V = \\ &= \left\|\frac{v_3-v_1}{2}\right\|_V + \left\|\frac{v_1-v_2}{2}\right\|_V + \left\|\frac{v_2-v_3}{2}\right\| = \frac{1}{2}(\|v_3-v_1\|_V + \|v_1-v_2\|_V + \|v_2-v_3\|_V) \stackrel{\text{def}}{=} \\ &\stackrel{\text{def}}{=} \frac{L_V(\partial\triangle)}{2}\end{aligned}$$

as desired. The proofs for the remaining three triangles are completely analogous. □

Proposition 1.4.11 (Cauchy Integral Theorem over $\mathcal{A}$)**:** Let $U \subseteq \mathcal{A}$ be open with $\pi_0(U) = 1$ and let $f \in \mathscr{C}^0(U,M)$.

(1) Cauchy–Goursat over $\mathcal{A}$ minus a point: if $f \in \mathscr{F}_{\mathcal{A}}(U \setminus \mathrm{pt}, {}_{\mathcal{A}}M)$, then:

$$\forall \triangle \subseteq U\colon \oint_{\partial\triangle} \rho(\mathrm{d}Z) f(Z) = 0. \tag{1.4.4}$$

(2) Cauchy–Goursat over $\mathcal{A}$ minus a finite number of points: if $f \in \mathscr{F}_{\mathcal{A}}(U \setminus \{p_1,\dots,p_k\}, {}_{\mathcal{A}}M)$, then:

$$\forall \triangle \subseteq U\colon \oint_{\partial\triangle} \rho(\mathrm{d}Z) f(Z) = 0. \tag{1.4.5}$$

(3) Homological Cauchy Integral Theorem over $\mathcal{A}$: if $f \in \mathscr{C}^1_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$ and $\Gamma \in Z_1(U,\mathbb{Z})$ is a 1-cycle with $[\Gamma] = 0$ in $H_1(U,\mathbb{Z})$, then

$$\int_\Gamma \rho(\mathrm{d}Z) f(Z) = 0. \tag{1.4.6}$$

(4) Homotopical Cauchy Integral Theorem over $\mathcal{A}$: in particular, if $f \in \mathscr{C}^1_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$ and $\pi_1(U) = 1$, then:

$$\forall \gamma \in \mathcal{L}U\colon \oint_\gamma \rho(\mathrm{d}Z) f(Z) = 0. \tag{1.4.7}$$

Proof: To (1): We adapt Pringsheim's approach following Rudin's exposition, which is applicable to $\mathbb{K}$ without the need to impose additional $\mathscr{C}^1$-regularity on ω. First suppose that $p \coloneqq \mathrm{pt} \notin \triangle$ and let $\partial\triangle = [a,b,c,a]$. Let a',b',c' be the mid points on the opposite sides and let $\triangle'_j$, $1 \leq j \leq 4$, be the respective newly formed triangles by connecting a',b',c' with each other. Then

$$J \coloneqq \oint_{\partial\triangle} \omega = \sum_{j=1}^4 \oint_{\partial\triangle'_j} \omega$$

and therefore

$$\exists\, 1 \leq j \leq 4\colon\ \|J\|_M \stackrel{\text{def}}{=} \left\|\oint_{\partial\triangle} \omega\right\|_M \leq 4 \left\|\oint_{\partial\triangle'_j} \omega\right\|_M.$$

Repeating the same procedure for $\triangle'_j$ in place of $\triangle$, we get a nested sequence of triangles $\triangle \supset \triangle_1 \supset \triangle_2 \supset \dots$ with $L_{\mathcal{A}}(\partial\triangle_k) = 2^{-k} L_{\mathcal{A}}(\partial\triangle)$ by Lemma 1.4.10 (3) and

$$\|J\|_M \leq 4^k \left\|\oint_{\partial\triangle_k} \omega\right\|_M$$

for all $k \in \mathbb{N}$. It follows from Cantor's Intersection Theorem that there exists $Z_0 \in \bigcap_{k\in\mathbb{N}} \triangle_k \subseteq \triangle$, and observe that f is ρ-differentiable at Z_0 by assumption. Therefore,

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0\ \forall\, \|Z - Z_0\|_{\mathcal{A}} < \delta\colon\ \big\|f(Z) - f(Z_0) - \rho(Z - Z_0)f'(Z_0)\big\|_M \le \varepsilon\, \|Z - Z_0\|_{\mathcal{A}}\,.$$

Furthermore, there exists $k \in \mathbb{N}$ such that $\forall\, Z \in \triangle_k\colon\ \|Z - Z_0\|_{\mathcal{A}} < \delta$. Moreover, by Lemma 1.4.10 (2) we also have $\forall\, Z \in \triangle_k\colon\ \|Z - Z_0\|_{\mathcal{A}} \le \operatorname{diam}_{\mathcal{A}} \triangle_k < \frac{1}{2}L_{\mathcal{A}}(\partial\triangle_k) = 2^{-(k+1)}L_{\mathcal{A}}(\partial\triangle)$. On the other hand,

$$\oint_{\partial\triangle_k} \rho(\mathrm{d}Z)f(Z_0) = \rho\bigg(\oint_{\partial\triangle_k} \mathrm{d}Z\bigg)f(Z_0) = 0$$

as well as

$$\oint_{\partial\triangle_k} \rho(\mathrm{d}Z)\rho(Z - Z_0)f'(Z_0) = \oint_{\partial\triangle_k} \rho((Z - Z_0)\mathrm{d}Z)f'(Z_0) = \rho\bigg(\oint_{\partial\triangle_k} (Z - Z_0)\mathrm{d}Z\bigg)f'(Z_0) = 0$$

since $(Z - Z_0)$ has the obvious $\mathcal{A}$-primitive $\frac{1}{2}(Z - Z_0)^2$. It follows that

$$\oint_{\partial\triangle_k} \omega = \oint_{\partial\triangle_k} \rho(\mathrm{d}Z)\big(f(Z) - f(Z_0) - \rho(Z - Z_0)f'(Z_0)\big),$$

whence using Lemma 1.4.6 in combination with the above estimates we obtain

$$\begin{aligned}
\|J\|_M &\le 4^k \left\|\oint_{\partial\triangle_k} \rho(\mathrm{d}Z)\big(f(Z) - f(Z_0) - \rho(Z - Z_0)f'(Z_0)\big)\right\|_M \le \\
&\le 4^k \sup_{Z\in\partial\triangle_k} \big\|f(Z) - f(Z_0) - \rho(Z - Z_0)f'(Z_0)\big\|_M L_{\mathcal{A}}(\partial\triangle_k) \le \\
&\le 4^k \sup_{Z\in\partial\triangle_k} \varepsilon\, \|Z - Z_0\|_{\mathcal{A}} L_{\mathcal{A}}(\partial\triangle_k) \le 4^k \varepsilon 2^{-(k+1)} L_{\mathcal{A}}(\partial\triangle)2^{-k}L_{\mathcal{A}}(\partial\triangle) = \\
&= \frac{\varepsilon}{2}L_{\mathcal{A}}(\partial\triangle)^2
\end{aligned}$$

for all $\varepsilon > 0$. It follows that $J = 0$.

Now suppose without loss of generality that $p = a$ is a vertex of $\triangle$. If a, b, c are colinear, then the claim is trivial. If not, choose $x \in [a, b]$ and $y \in [a, c]$ both close to $p = a$. Then

$$\oint_{\partial\triangle} \omega = \bigg(\oint_{[a,x,y,a]} + \oint_{[x,b,y,x]} + \oint_{[b,c,y,b]}\bigg)\,\omega,$$

the last two being 0, since they do not contain $p = a$. Therefore,

$$\oint_{\partial\triangle} \omega = \bigg(\int_a^x + \int_x^y + \int_y^a\bigg)\,\omega.$$

Since f is bounded on $\triangle$, letting $x, y \to p = a$ we get the desired result. Finally, if $p \in \triangle$ arbitrary, then apply the above to the triangles $[a, b, p, a]$, $[b, c, p, b]$ and $[c, a, p, c]$.

To (2): follows from (1) (or its proof).

To (3): Apply Stokes' theorem to $\omega \coloneqq \rho(\mathrm{d}Z)f(Z)$: since $\Gamma \in B_1(U, \mathbb{Z})$, there exists $\Sigma \in Z_2(U, \mathbb{Z})$ such that $\Gamma = \partial\Sigma$. But ω is closed by Proposition 1.4.8, therefore

$$\int_\Gamma \omega = \int_{\partial\Sigma} \omega = \int_\Sigma \mathrm{d}\omega = 0.$$

To (4): Since $\pi_1(U) = 1$, any $\gamma \in \mathcal{L}U$ is homotopical to a point, hence null-homologous in particular. Alternatively, use homotopy-invariance of closed 1-forms. □

Thus, Proposition 1.4.11 in conjunction with Lemma 1.4.7 establishes (global) ρ-integrability of ρ-differentiable functions of different regularity on various types of domains.

Remarks:

(1) If $U \subseteq \mathcal{A}$ is open and $f \in \mathscr{F}_{\mathcal{A}}(U \setminus \{p_1, \ldots, p_s\}, {}_{\mathcal{A}}M)$, then also $\forall \square \subseteq U\colon \oint_{\partial\square} \rho(\mathrm{d}Z) f(Z) = 0$ since $\partial\square = \partial\triangle + \partial\nabla$, where $\partial\square$ denotes an (oriented) simple 4-gon in U.

(2) If $f \in \mathscr{C}^1_{\mathcal{A}}(U, {}_{\mathcal{A}}M)$, then $\omega := \rho(\mathrm{d}Z) f(Z) \in \Omega^1(U, M)$ is in particular of regularity $\mathscr{C}^1$ and d-closed by Proposition 1.4.8. Thus, if $\pi_0(U) = \pi_1(U) = 1$ (e.g. $\mathbb{B}^n \setminus \mathrm{pt}$, or more generally, $\star \setminus \{p_1, \ldots, p_k\} \subseteq \mathbb{R}^n$ for $n \geq 3$), then the existence of a ρ-primitive of $f(Z)$, i.e. $F(Z)$ such that $\mathrm{d}F = \omega$, is also guaranteed by the *simply-connected* version[18] of Poincaré's Lemma (rather than the usual, *contractible* version of the lemma). Nevertheless, the wombo combo Proposition 1.4.11 (3) & Lemma 1.4.7 is still more general than that since *(co)homological* 1-connectedness need not imply *homotopical* 1-connectedness (for instance consider the complement of Alexander's Horned Sphere). Finally, unlike Poincaré's lemma, Proposition 1.4.11 (1) & (2) only require $\mathscr{F}_{\mathcal{A}}$-regularity. The latter is perhaps a little surprising when $\mathcal{A}$ lacks a compatible complex structure since in that case $\mathscr{F}_{\mathcal{A}}$-regularity does not seem to imply $\mathscr{C}^1$-regularity (see also Section 1.1 and Section 1.3).

(3) Continuing (2), despite all the similarities to single-variable complex analysis so far, ρ-integrability in higher dimensions ($\dim_{\mathbb{R}} \mathcal{A} \geq 3$) is in stark contrast to the complex plane, where removing even a single point breaks global integrability (e.g. $1/z$ on $\mathbb{C} \setminus \{0\}$). When $\mathcal{A}$ carries a compatible complex structure, this is of course completely expected from Hartog's extension theorem. But more suprisingly, it otherwise suggests the existence of a *real* Hartog's phenomenon for $\mathcal{A}$-differentiable functions even in the non-real-analytic setting, including the lower-dimensional case of $\mathcal{A} = \mathbb{R}[t]/(t^2)$.

(4) As is well-known, Goursat's version of Cauchy's Integral Theorem (CIT) removes the $\mathscr{C}^1$-condition from CIT by only requiring $f(z)$ to be holomorphic (i.e. complex-differentiable in a neighbourhood) with no continuity assumption on $f'(z)$. Observe that under continuity of $f'(z)$, CIT is of course just a simple corollary of Stokes' Theorem and therefore holds not just for loops (as often stated in the literature), but more generally for (continuous) 1-boundaries since $\omega := f(z)\mathrm{d}z$ is then not only d-closed, but also $\mathscr{C}^1$-regular (obviously, this is the approach we took in Proposition 1.4.11 (3)). Of course, continuity of $f'(z)$ turns out to be automatic as it follows from the (complex) analyticity of $f(z)$, which in turn is a consequence of Cauchy's reproducing kernel (Cauchy's Integral *Formula*, or CIF for short) and thus actually comes after CIT. Therefore, it is conceptually very desirable to find an *integral-free* proof of the $\mathscr{C}^1$-regularity of holomorphic functions – as pointed out by Ahlfors in the 1st edition of his book from 1953 – and/or find an *index-free* proof of Cauchy–Goursat Integral Theorem for *1-chains*. The former was addressed by the development of *Topological Analysis* (see also the discussion in [MO07]), whereas a Cauchy-Goursat Theorem for 1-chains (that is, CIT for 1-chains, but without continuity of $f'(z)$) is indeed realized by the roundabout way of Artin's theorem on the correspondence between homology and the analytic index (winding numbers) in the plane (hence the desire for an *index-free* proof of that). For the latter we also refer the reader to the discussion in [MO05] for a more direct topological proof[19] without invoking winding numbers. In brief, it was realized by S. Stoilow in the '30s that holomorphic functions in the complex plane are completely characterized by two core *topological* properties, namely *openness* (as we know it today) and *lightness* (=totally disconnected fibers) (cf. [Sto37]). This then allowed for the basic theory of single-variable complex analysis to be carried out in a completely *integral-free* manner by appealing to the integral-free formulation of *topological degree* (analytic index, winding number) in the plane. A concise treatment of *Topological*

[18] we have $H^1_{\mathrm{dR}}(U) \cong H^1(U, \mathbb{R}) \cong H_1(U, \mathbb{R})^* = 0$ with the last equality following from the Hurewicz theorem;
[19] due to Sam Nead;

Analysis is given in Whyburn's book [Why64], but we also recommend the interested reader to check out the lovely papers [EU52], [TY52], [Plu59], [PC61], [Con61], [Rea61], [Why62b], [Con65], [MW67] (in that order) for a more direct exposure (or the short survey [Why62a] for the state of the subject up to 1962). *However, in view of the present work we argue that Topological Analysis doesn't provide a completely satisfactory answer, either, at least not in that form, since $\mathcal{A}$-holomorphic functions enjoy for the most part a completely analogous theory to that of the functions of a single complex variable, yet they are manifestly neither open (being SCV functions), nor necessarily light.* Circling back to the Cauchy–Goursat theorem, given that it also holds without the $\mathscr{C}^1$-condition over $\mathcal{A}$-s without complex structure, we speculate that continuity of the derivative is in fact a red herring (see also the discussion in [MO06]) and that therefore there must exist a version of Stokes' Theorem for 1-chains that replaces $\mathscr{C}^1$-regularity of ω by continuity of $\mathrm{d}\omega$.

Problem 8: Does there exist a version of Stokes' Theorem for *continuous* (1-)boundaries that replaces $\mathscr{C}^1$-regularity of ω by $\mathscr{F}_{\mathbb{R}}$-regularity (=mere differentiability) of ω in combination with $\mathscr{C}^0$-regularity of $\mathrm{d}\omega$? A version of Stokes' Theorem for integrands of this regularity is indeed stated in [Nev73, p. 131], but for domains of integration that are closed simplices (=closed convex hulls of finitely many points)[20], which are inherently piece-wise smooth (in fact PL). In other words, one replaces the *total function derivative* in the $\mathscr{C}^1$-condition (for the form coefficients) by the *exterior derivative of the form itself.* This seems rather natural because we are not merely differentiating the coefficients of ω as separate unrelated entities, but we are actually differentiating ω itself. Note that generalizations of (the generalized) Stokes' theorem are typically a balancing act between regularity of the integrand and regularity of the domains of integration, trading higher regularity of the former for lower regularity of the latter or vice versa, which is somewhat orthogonal to what we propose here. □

Proposition 1.4.12 (Infinite Integrability)**:** Let $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{K}}$, $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$ a representation.

(1) If $☆ \subseteq \mathcal{A}$ is open and star-convex, then $\mathscr{F}_\rho(☆) = \mathscr{F}_\rho^{-\infty}(☆)$.

(2) If $U \subseteq \mathcal{A}$ is open and *homologically 1-connected*, then $\mathscr{C}^1_\rho(U) = \mathscr{C}_\rho^{-\infty}(U)$.

Proof: To (1): This is immediate from Proposition 1.4.11 (1) & Lemma 1.4.7 (2) by simple induction on $\mathscr{F}_\rho^{-k}$, $k \in \mathbb{N}$.

To (2): Since U is a manifold, we have that $H^1_{\mathscr{C}^0}(U,\mathbb{Z}) \cong H^1_{\mathscr{C}^\infty}(U,\mathbb{Z})$ (with the self-explanatory notation), hence the claim follows from Proposition 1.4.11 (3) & Lemma 1.4.7 (1) by induction on $\mathscr{C}_\rho^{-k}$, $k \in \mathbb{N}$. □

Remarks:

(1) By using standard arguments of polygonal approximations of rectifiable 1-chains, Proposition 1.4.11 (1), Lemma 1.4.7 (1), and $H^1_{\mathscr{C}^0}(U,\mathbb{Z}) \cong H^1_{\mathscr{C}^\infty}(U,\mathbb{Z})$ as in the proof of (2), one can relax the hypothesis of (1) from star-convex domains to *homologically 1-connected* domains as well. We will omit this proof here.

(2) We conjecture that (2) (and (1)) hold under the much weaker topological hypothesis of being *spectrally homologically 1-connected*, see the later Definition 2.1.6.

(3) Even though $\mathcal{A}$-differentiability in a neighbourhood of a point does not imply $\mathcal{A}$-analyticity if $\mathcal{A}$ has no complex structure, it is similar to holomorphic functions in the plane in that it still implies "reverse analyticity" (=infinite integrability) on appropriate domains. □

Ending on this note, we can now seamlessly transition to the next chapter.

[20] not to be confused with *singular simplices*, which are only *continuous* images of closed simplices;

2. Integration over $\mathcal{A}$

In Section 2.1 we give a careful treatment of the sets of points and singular 1-cycles that are *admissible* for our to-be-defined generalized Cauchy integration kernel (associated to a finite-dimensional representation $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$) so that the corresponding integral is well-defined. Such a discussion is merited by the fact that the *relative* singular locus, $\mathrm{Sing}_\rho(\mathcal{A})$, of $\mathcal{A}$ is somewhat more complicated than that of $\mathbb{C}$ (the origin), which in turn then translates into a singularity for the corresponding differential form. These sets are intimately related to the closure and complement notions introduced in Section 0.1.7 and Section 0.1.9 (or see Section 0.1.13 for summary). As an organic part of this discussion, Section 2.1 also serves as a preparation for our *spectral homotopy argument* later on, which not only enables the explicit calculation of the generalized (relative) index of a loop as done in Section 2.2 but also allows for the strongest form of a generalized homological Cauchy's Integral Formula presented in Section 2.3. In this vein, at the end of Section 2.1 we also introduce the elementary algebro-topological invariants that govern the corresponding topology.

In Section 2.2 we introduce the generalized Cauchy integration kernel in the disguise of the generalized analytic index $\mathrm{Ind}_\rho(\gamma, Z_0)$ of a loop γ relative to a point $Z_0 \in \mathcal{A}$ and the finite-dimensional representation ρ. We give a complete characterization of the generalized index by reducing it to the analytic index (winding number) in the complex plane, in particular showing that $\mathrm{Ind}_\rho(\gamma, Z_0)$ is valued in a canonical lattice in $\mathrm{End}_{\mathbb{C}}(M)$.

In Section 2.3 we prove various versions of the generalized Cauchy's Integral Formula, including a homotopical and a homological version as well as a Cauchy transform, and then study various consequences thereof. These include the automatic ρ-holomorphic extension of ρ-holomorphic functions to spectral polycylinders, the canonical global form of ρ-holomorphic functions (separation of spectral and nilpotent variable), generalized Cauchy's inequality and generalized Liouville theorem, and generalized Weierstraß Convergence Theorems. A self-contained exposition for the special case $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C})$ up to the generalized homological version of Cauchy's Integral Formula without the additional load of representations or *other local factors* is given in our paper [Gen26].

In Section 2.4 we prove a generalized Cauchy–Pompeiu integral formula over sufficiently regular orientable surfaces $\Sigma \subset \mathcal{A}$ in a general position relative to the radical, yielding uncountably infinitely many low-dimensional Bochner–Martinelli formulas for $\mathbb{C}$-*holomorphic mappings*, but with *holomorphic kernels* and at the expense of a non-vanishing surface integral term.

2.1. Admissible Points & 1-Cycles

We have reached the point where the real and complex theory start to diverge from each other. The main reason for this is that we will need $\mathcal{A}^\times$ to stay path-connected. By Proposition 0.2.3 this is the case if and only if $\mathcal{A}$ carries a complex structure extending the underlying $\mathbb{R}$-algebra structure. Here we follow closely the notations and conventions in Section 0.1.13. We thus fix $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{C}}$ and a *complete* decomposition $\mathcal{A} = \prod_{j=1}^n (\mathcal{A}_j, \mathfrak{m}_j, \mathbb{C})$ as well as $M \in \mathsf{FgMod}(\mathcal{A})$ given by $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{C}}(M)$ with a *complete* decomposition ${}_{\mathcal{A}}M = \bigoplus_{k=1}^m {}_{\mathcal{A}}M_k$ in $\mathsf{FdMod}(\mathcal{A})$. In particular, ρ determines unique subsets $I_\rho \subseteq [n]$ of "relevant" indices and $\mathcal{X}_\rho(\mathcal{A}) = \{\chi_j : j \in I_\rho\}$ of "relevant" characters. Using the non-unital embeddings $\mathcal{A}_j \overset{\varepsilon_j}{\hookrightarrow} \mathcal{A}$ (hence $\mathcal{A}$ being an $\mathcal{A}_j$-algebra), $1 \le j \le n$, every $Z \in \mathcal{A}$ can be uniquely written as $Z = (Z_1, \ldots, Z_n) = \sum_{j=1}^n e_j Z_j$ with $Z_j \coloneqq \mathrm{pr}_j(Z) \in \mathcal{A}_j$, where $e_j = \varepsilon_j(1_{\mathcal{A}_j}) = \iota_j(1_{\mathbb{C}})$, $1 \le j \le n$, are the corresponding idempotents, and further $Z_j = z_j \oplus X_j$ with $z_j \coloneqq \chi_j(Z) \in \mathbb{C}$ and $X_j \in \mathfrak{m}_j$, $1 \le j \le n$.

Now let $U \subseteq \mathcal{A}$ be *open and connected*, hence also $\pi_0(U) = 1$. Then so are $U_j \overset{\text{def}}{=} \mathrm{pr}_j(U) \subseteq \mathcal{A}_j$, $U_j^{\mathrm{sp}} \overset{\text{def}}{=} \chi_j(U) \subseteq \mathbb{C}$, $U^{\mathrm{sp}} \overset{\text{def}}{=} \prod_{j\in I_\rho} U_j^{\mathrm{sp}} \subseteq \mathcal{A}_{I_\rho} \overset{\text{def}}{=} \prod_{j\in I_\rho} \mathcal{A}_j$, and $\widehat{U} \supseteq \widetilde{U} \supseteq U$, since pr_j and χ_j are continuous and open by virtue of being projections, $1 \leq j \leq n$. To fix notations, for $j \in I_\rho$ we have topological retraction diagrams

$$\begin{tikzcd} U \arrow[r, two heads, "\mathrm{pr}_j"] & U_j \arrow[r, hook, "\varepsilon_j"] \arrow[dr, "\mathrm{id}"'] & \widetilde{U} \arrow[d, "\mathrm{pr}_j"] \\ & & U_j \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} U \arrow[r, two heads, "\chi_j"] & U_j^{\mathrm{sp}} \arrow[r, hook, "\iota_j"] \arrow[dr, "\mathrm{id}"'] & \widehat{U} \arrow[d, "\chi_j"] \\ & & U_j^{\mathrm{sp}} \end{tikzcd} \tag{2.1.1}$$

where note that in general neither $\varepsilon_j|_{U_j}$, nor $\iota_j|_{U_j^{\mathrm{sp}}}$ maps into U, but rather into $\widetilde{U}$ and $\widehat{U} \supseteq \widetilde{U}$, respectively. In turn, these functorially induce various push-forwards such as

$$\begin{tikzcd} \mathcal{L}U \arrow[r, "(\mathrm{pr}_j)_\#"] & \mathcal{L}U_j \arrow[r, hook, "(\varepsilon_j)_\#"] \arrow[dr, "\mathrm{id}"'] & \mathcal{L}\widetilde{U} \arrow[d, "(\mathrm{pr}_j)_\#"] \\ & & \mathcal{L}U_j \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{L}U \arrow[r, "(\chi_j)_\#"] & \mathcal{L}U_j^{\mathrm{sp}} \arrow[r, hook, "(\iota_j)_\#"] \arrow[dr, "\mathrm{id}"'] & \mathcal{L}\widehat{U} \arrow[d, "(\chi_j)_\#"] \\ & & \mathcal{L}U_j^{\mathrm{sp}} \end{tikzcd}$$

where we also put $\gamma_j \coloneqq (\mathrm{pr}_j)_\#(\gamma) \overset{\text{def}}{=} \mathrm{pr}_j \circ \gamma$ and $\gamma_j^{\mathrm{sp}} \coloneqq (\chi_j)_\#(\gamma) \overset{\text{def}}{=} \chi_j \circ \gamma$, $1 \leq j \leq n$. Taking homotopy classes, we obtain corresponding retraction diagrams in Grp

$$\begin{tikzcd} \pi_1(U) \arrow[r, "(\mathrm{pr}_j)_*"] & \pi_1(U_j) \arrow[r, hook, "(\varepsilon_j)_*"] \arrow[dr, "\mathrm{id}"'] & \pi_1(\widetilde{U}) \arrow[d, "(\mathrm{pr}_j)_*"] \\ & & \pi_1(U_j) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \pi_1(U) \arrow[r, "(\chi_j)_*"] & \pi_1(U_j^{\mathrm{sp}}) \arrow[r, hook, "(\iota_j)_*"] \arrow[dr, "\mathrm{id}"'] & \pi_1(\widehat{U}) \arrow[d, "(\chi_j)_*"] \\ & & \pi_1(U_j^{\mathrm{sp}}) \end{tikzcd}$$

where in particular $(\mathrm{pr}_j)_*[\gamma] \overset{\text{def}}{=} [\gamma_j]$ and $(\chi_j)_*[\gamma] \overset{\text{def}}{=} [\gamma_j^{\mathrm{sp}}]$, $1 \leq j \leq n$. Similarly, if $G \in \mathsf{Ab}$ and $\Gamma \coloneqq \sum_{\ell=1}^p g_\ell \gamma^\ell \in Z_1(U, G)$ is a (singular) 1-cycle[1], we get corresponding retraction diagrams in Ab of push-forwards of 1-cycles

$$\begin{tikzcd} Z_1(U,G) \arrow[r, "(\mathrm{pr}_j)_\#"] & Z_1(U_j,G) \arrow[r, hook, "(\varepsilon_j)_\#"] \arrow[dr, "\mathrm{id}"'] & Z_1(\widetilde{U},G) \arrow[d, "(\mathrm{pr}_j)_\#"] \\ & & Z_1(U_j,G) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Z_1(U,G) \arrow[r, "(\chi_j)_\#"] & Z_1(U_j^{\mathrm{sp}},G) \arrow[r, hook, "(\iota_j)_\#"] \arrow[dr, "\mathrm{id}"'] & Z_1(\widehat{U},G) \arrow[d, "(\chi_j)_\#"] \\ & & Z_1(U_j^{\mathrm{sp}},G) \end{tikzcd}$$

where we put likewise $\Gamma_j \coloneqq (\mathrm{pr}_j)_\#(\Gamma) \overset{\text{def}}{=} \sum_{\ell=1}^p g_\ell \gamma_j^\ell$ and $\Gamma_j^{\mathrm{sp}} \coloneqq (\chi_j)_\#(\Gamma) \overset{\text{def}}{=} \sum_{\ell=1}^p g_\ell (\gamma^\ell)_j^{\mathrm{sp}}$. Accordingly, we get a second set of retraction diagrams in Ab in homology

$$\begin{tikzcd} H_1(U,G) \arrow[r, "(\mathrm{pr}_j)_*"] & H_1(U_j,G) \arrow[r, hook, "(\varepsilon_j)_*"] \arrow[dr, "\mathrm{id}"'] & H_1(\widetilde{U},G) \arrow[d, "(\mathrm{pr}_j)_*"] \\ & & H_1(U_j,G) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} H_1(U,G) \arrow[r, "(\chi_j)_*"] & H_1(U_j^{\mathrm{sp}},G) \arrow[r, hook, "(\iota_j)_*"] \arrow[dr, "\mathrm{id}"'] & H_1(\widehat{U},G) \arrow[d, "(\chi_j)_*"] \\ & & H_1(U_j^{\mathrm{sp}},G) \end{tikzcd}$$

where $(\mathrm{pr}_j)_*[\Gamma] \overset{\text{def}}{=} [\Gamma_j]$ and $(\chi_j)_*[\Gamma] \overset{\text{def}}{=} [\Gamma_j^{\mathrm{sp}}]$, $1 \leq j \leq n$. More generally, the individual topological retractions neatly consolidate into the topological retractions

$$\begin{tikzcd} U \arrow[r, "\mathrm{pr}"] & U_{I_\rho} \arrow[r, hook, "\varepsilon"] \arrow[dr, "\mathrm{id}"'] & \widetilde{U} \arrow[d, "\mathrm{pr}"] \\ & & U_{I_\rho} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} U \arrow[r, "\chi"] & U^{\mathrm{sp}} \arrow[r, hook, "\iota"] \arrow[dr, "\mathrm{id}"'] & \widehat{U} \arrow[d, "\chi"] \\ & & U^{\mathrm{sp}} \end{tikzcd} \tag{2.1.2}$$

[1] we will be using upper indices for the components of singular 1-cycles in order to avoid notational overload with the previous conventions;

respectively, where we have put $\mathrm{pr} := \mathrm{pr}_\rho \overset{\mathrm{def}}{=} \mathrm{pr}_{I_\rho}$, $\varepsilon := \varepsilon_\rho \overset{\mathrm{def}}{=} \varepsilon_{I_\rho}$, $\iota := \iota_\rho \overset{\mathrm{def}}{=} \iota_{I_\rho}$, and $\chi := \chi_\rho \overset{\mathrm{def}}{=} \chi_{I_\rho}$ for short. As before, these in turn induce the following four sets of respective retractions:

$$\begin{tikzcd} \mathcal{L}U \arrow[r, "\mathrm{pr}_\#"] & \mathcal{L}U_{I_\rho} \arrow[r, hook, "\varepsilon_\#"] \arrow[dr, "\mathrm{id}"'] & \mathcal{L}\widetilde{U} \arrow[d, "\mathrm{pr}_\#"] \\ & & \mathcal{L}U_{I_\rho} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{L}U \arrow[r, "\chi_\#"] & \mathcal{L}U^{\mathrm{sp}} \arrow[r, hook, "\iota_\#"] \arrow[dr, "\mathrm{id}"'] & \mathcal{L}\widehat{U} \arrow[d, "\chi_\#"] \\ & & \mathcal{L}U^{\mathrm{sp}}_j \end{tikzcd}$$

$$\begin{tikzcd} \pi_1(U) \arrow[r, "\mathrm{pr}_*"] & \pi_1(U_{I_\rho}) \arrow[r, hook, "\varepsilon_*"] \arrow[dr, "\mathrm{id}"'] & \pi_1(\widetilde{U}) \arrow[d, "\mathrm{pr}_*"] \\ & & \pi_1(U_{I_\rho}) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \pi_1(U) \arrow[r, "\chi_*"] & \pi_1(U^{\mathrm{sp}}) \arrow[r, hook, "\iota_*"] \arrow[dr, "\mathrm{id}"'] & \pi_1(\widehat{U}) \arrow[d, "\chi_*"] \\ & & \pi_1(U^{\mathrm{sp}}_j) \end{tikzcd}$$

$$\begin{tikzcd} Z_1(U,G) \arrow[r, "\mathrm{pr}_\#"] & Z_1(U_{I_\rho},G) \arrow[r, hook, "\varepsilon_\#"] \arrow[dr, "\mathrm{id}"'] & Z_1(\widetilde{U},G) \arrow[d, "\mathrm{pr}_\#"] \\ & & Z_1(U_{I_\rho},G) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Z_1(U,G) \arrow[r, "\chi_\#"] & Z_1(U^{\mathrm{sp}},G) \arrow[r, hook, "\iota_\#"] \arrow[dr, "\mathrm{id}"'] & Z_1(\widehat{U},G) \arrow[d, "\chi_\#"] \\ & & Z_1(U^{\mathrm{sp}}_j,G) \end{tikzcd}$$

$$\begin{tikzcd} H_1(U,G) \arrow[r, "\mathrm{pr}_*"] & H_1(U_{I_\rho},G) \arrow[r, hook, "\varepsilon_*"] \arrow[dr, "\mathrm{id}"'] & H_1(\widetilde{U},G) \arrow[d, "\mathrm{pr}_*"] \\ & & H_1(U_{I_\rho},G) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} H_1(U,G) \arrow[r, "\chi_*"] & H_1(U^{\mathrm{sp}},G) \arrow[r, hook, "\iota_*"] \arrow[dr, "\mathrm{id}"'] & H_1(\widehat{U},G) \arrow[d, "\chi_*"] \\ & & H_1(U^{\mathrm{sp}}_j,G) \end{tikzcd}$$

Definition 2.1.1: Let $G \in \mathsf{Ab}$ and $\Gamma \in Z_1(U,G)$. We put $\Gamma^{\mathrm{sp}} := (\iota \circ \chi)_\# \Gamma \in Z_1(\widehat{U},G)$.

Observe that it follows from the definition of $\widehat{U}$ that $\pi_1(\widehat{U}) \cong \prod_{j \in I_\rho} \pi_1(U^{\mathrm{sp}}_j) \cong \pi_1(U^{\mathrm{sp}})$. Similarly, it follows from Lemma-Definition 0.1.86 that $\pi_0(\mathcal{A}^\times_\rho) = 1$ and $\pi_1(\mathcal{A}^\times_\rho) \cong \pi_1(\prod_{j\in I_\rho} \mathbb{C}^\times) \cong \mathbb{Z}^{|I_\rho|}$. Upon identifying $U^{\mathrm{sp}} \cong \iota(U^{\mathrm{sp}})$ we now not only have a retraction $\widehat{U} \twoheadrightarrow U^{\mathrm{sp}}$ but actually a strong deformation retraction $\widehat{U} \simeq U^{\mathrm{sp}}$ given explicitly by the homotopy $H \colon [0,1] \times \widehat{U} \to \widehat{U}$, $H(t,-) := (1-t)\chi \circ \iota + t\,\mathrm{id}_{\widehat{U}}$. Hence, in particular, the isomorphism $\pi_1(\widehat{U}) \cong \pi_1(U^{\mathrm{sp}})$ can also be given by a loop homotopy $\mathcal{L}\widehat{U} \ni \gamma \simeq \gamma^{\mathrm{sp}} \in \mathcal{L}U^{\mathrm{sp}}$, and furthermore $H_1(\widehat{U},G) \cong H_1(U^{\mathrm{sp}},G)$ with $\Gamma \simeq \Gamma^{\mathrm{sp}}$ in $\widehat{U}$, that is, $[\Gamma] = [\Gamma^{\mathrm{sp}}]$ in $H_1(\widehat{U},G)$. In fact, we also have:

Lemma 2.1.2: $H_1(\widehat{U},\mathbb{Z}) \cong H_1(U^{\mathrm{sp}},\mathbb{Z}) \cong \bigoplus_{j \in I_\rho} H_1(U^{\mathrm{sp}}_j,\mathbb{Z})$.

Proof: We prove the statement in the case $|I_\rho| = 2$, without loss of generality, say, $I_\rho = \{1,2\}$. Since $\mathbb{Z}$ is a PID, we have by Künneth's theorem the SES

$$0 \to \bigoplus_{k+\ell=1} H_k(U^{\mathrm{sp}}_1,\mathbb{Z}) \otimes_{\mathbb{Z}} H_\ell(U^{\mathrm{sp}}_2,\mathbb{Z}) \hookrightarrow H_1(U^{\mathrm{sp}}_1 \times U^{\mathrm{sp}}_2,\mathbb{Z}) \twoheadrightarrow \bigoplus_{k=\ell} \mathrm{Tor}^{\mathbb{Z}}_1(H_k(U^{\mathrm{sp}}_1,\mathbb{Z}), H_\ell(U^{\mathrm{sp}}_2,\mathbb{Z})) \to 0.$$

Moreover, $H_0(U^{\mathrm{sp}}_j,\mathbb{Z}) \cong \mathbb{Z}$ as $\pi_0(U) = 1$ and $H_{\geq 2}(U^{\mathrm{sp}}_j,\mathbb{Z}) = 0$ by [Hat02, Prop. 3.29]. Since U^{sp}_j is a non-compact connected surface, it follows from [Joh31] that $H_1(U^{\mathrm{sp}}_j,\mathbb{Z})$ is free (and countably generated)[2]. Therefore, all the $\mathrm{Tor}^{\mathbb{Z}}_1$-terms vanish, and we obtain $H_1(U^{\mathrm{sp}}_1 \times U^{\mathrm{sp}}_2,\mathbb{Z}) \cong H_1(U^{\mathrm{sp}}_1,\mathbb{Z}) \oplus H_1(U^{\mathrm{sp}}_2,\mathbb{Z})$. The general claim follows similarly by induction. $\square$

Recall that $\mathrm{Sing}_\rho(\mathcal{A}) \overset{\mathrm{def}}{=} \mathcal{A} \setminus \mathcal{A}^\times_\rho = \mathfrak{m}_\rho = \bigcup_{j\in I_\rho} \mathfrak{M}_j$ is the singular cone of $\mathcal{A}$ with respect to ρ, where $\mathrm{Spm}(\mathcal{A}) = \{\mathfrak{M}_1, \cdots, \mathfrak{M}_n\}$ and $\mathfrak{M}_j = \mathcal{A}_1 \times \cdots \times \mathcal{A}_{j-1} \times \mathfrak{m}_j \times \mathcal{A}_{j+1} \times \cdots \times \mathcal{A}_n$, $1 \leq j \leq n$.

[2] see also Andy Putman's answer in [MO04]

Tautologically, $\mathrm{Sing}_\rho(\mathcal{A})$ is by definition the singularity of the $\rho(\mathcal{A})$-valued holomorphic 1-form

$$\omega := \frac{\rho(\mathrm{d}Z)}{\rho(Z)} \overset{\text{def}}{=} \frac{\rho^{\mathrm{mor}}(\mathrm{d}Z)}{\rho^{\mathrm{mor}}(Z)} \overset{\text{def}}{=} \frac{\rho^{\mathrm{im}}(\mathrm{d}Z)}{\rho^{\mathrm{im}}(Z)} \in \Omega^1_{\mathrm{hol}}(\mathcal{A}^\times_\rho, \rho(\mathcal{A})) \subseteq \Omega^1_{\mathrm{hol}}(\mathcal{A}^\times_\rho, \mathcal{B}) \subseteq \Omega^1_{\mathrm{hol}}(\mathcal{A}^\times_\rho, \mathrm{End}_{\mathbb{C}}(M)),$$

where $\mathcal{B} := \prod_{k=1}^m \mathcal{B}_k := \mathrm{Codom}(\rho^{\mathrm{mor}})$. Observe that passing to the corestrictions ρ^{im} or ρ^{mor} allows us to detach ourselves from the inherently non-commutative object $\mathrm{End}_{\mathbb{K}}(M)$ and thus to only consider commutatively valued 1-forms. Since $\pi_0(\mathrm{Sing}_\rho(\mathcal{A})^c) = \pi_0(\mathcal{A}^\times_\rho) = 1$, certain $\int_\Gamma \omega$ can be interpreted as integrating ω *around* $\mathrm{Sing}_\rho(\mathcal{A})$. More generally, for fixed $Z_0 \in \mathcal{A}$ we will consider integrals of the *form*

$$\omega := \frac{\rho(\mathrm{d}Z)}{\rho(Z - Z_0)} \overset{\text{def}}{=} \frac{\rho^{\mathrm{mor}}(\mathrm{d}Z)}{\rho^{\mathrm{mor}}(Z - Z_0)} \in \Omega^1_{\mathrm{hol}}([Z_0]^\times_\rho, \mathcal{B}).$$

Thus, given some $\Gamma \in Z_1(U, \mathbb{Z})$, our first order of business is to characterize the set of all $Z_0 \in \mathcal{A}$ such that the above $\int_\Gamma \omega$ is well-defined.

Definition 2.1.3 (Admissible Points)**:** Fix $I \subseteq [n]$.

(1) I-forbidden zone of points for a loop: if $\gamma \in \mathcal{L}U$, then:

$$\mathfrak{F}(\gamma) := \mathfrak{F}_I(\gamma) := \bigcup_{j \in I} \chi_j^{-1}(\gamma_j^{\mathrm{sp}}) \overset{\text{def}}{=} \{Z \in \mathcal{A} \mid \exists\, j \in I \colon z_j \in \gamma_j^{\mathrm{sp}}\}.$$

(2) I-admissible zone of points for a loop: if $\gamma \in \mathcal{L}U$, then

$$\mathrm{Adm}(\gamma) := \mathrm{Adm}_I(\gamma) := \mathfrak{F}_I(\gamma)^c \overset{\text{def}}{=} \check{\gamma} \overset{\text{def}}{=} \widecheck{\mathrm{supp}(\gamma)} \overset{\text{def}}{=} \{Z \in \mathcal{A} \mid \forall\, j \in I \colon z_j \notin \gamma_j^{\mathrm{sp}}\} \overset{\text{def}}{=}$$
$$\overset{\text{def}}{=} \prod_{j=1}^n \begin{cases} \left(\mathbb{C} \setminus \gamma_j^{\mathrm{sp}}\right) \times \mathfrak{m}_j, \text{if } j \in I \\ \mathcal{A}_j, \text{otherwise.} \end{cases}$$

(3) I-forbidden zone of points for a singular 1-cycle: if $\Gamma := \sum_{\ell=1}^p g_\ell \gamma^\ell \in Z_1(U, G)$, then

$$\mathfrak{F}(\Gamma) := \mathfrak{F}_I(\Gamma) := \bigcup_{\ell=1}^p \mathfrak{F}_I(\gamma^\ell) \overset{\text{def}}{=} \{Z \in \mathcal{A} \mid \exists\, j \in I \colon z_j \in \Gamma_j^{\mathrm{sp}}\}.$$

(4) I-admissible zone of points for a singular 1-cycle: if $\Gamma := \sum_{\ell=1}^p g_\ell \gamma^\ell \in Z_1(U, G)$, then

$$\mathrm{Adm}(\Gamma) := \mathrm{Adm}_I(\Gamma) := \mathfrak{F}_I(\Gamma)^c \overset{\text{def}}{=} \check{\Gamma} \overset{\text{def}}{=} \widecheck{\mathrm{supp}(\Gamma)} \overset{\text{def}}{=} \bigcap_{\ell=1}^p \mathrm{Adm}_I(\gamma^\ell) \overset{\text{def}}{=}$$
$$\overset{\text{def}}{=} \prod_{j=1}^n \begin{cases} (\mathbb{C} \setminus \Gamma_j^{\mathrm{sp}}) \times \mathfrak{m}_j, \text{if } j \in I \\ \mathcal{A}_j, \text{otherwise} \end{cases} \overset{\text{def}}{=} \prod_{j=1}^n \begin{cases} \left(\mathbb{C} \setminus \bigcup_{\ell=1}^p (\gamma^\ell)_j^{\mathrm{sp}}\right) \times \mathfrak{m}_j, \text{if } j \in I \\ \mathcal{A}_j, \text{otherwise} \end{cases}$$

(5) A subset $S \subseteq \mathcal{A}$ is called I-admissible for $\Gamma \in Z_1(U, G)$ iff $S \subseteq \mathrm{Adm}_I(\Gamma)$, and I-forbidden otherwise. Symmetrically, given a subset $S \subseteq \mathcal{A}$, a 1-cycle $\Gamma \in Z_1(U, G)$ is called I-admissible (I-forbidden) for S iff S is an I-admissible (I-forbidden) set for Γ.

Remarks:

(1) Dependence on the ambient space: the definitions of $\mathfrak{F}(\Gamma)$ and $\mathrm{Adm}(\Gamma)$ are independent of the topological subspace $\mathcal{A} \supseteq U \supset \Gamma$, but depend implicitly on the algebra *structure* $\mathcal{A}$ on the underlying ambient vector space, more precisely on the ensemble of "spectral" planes $\{\mathbb{C}e_j : j \in I\}$ (for fixed isomorphism type $[\mathcal{A}] \in \mathsf{FdCAlg}_{\mathbb{C}}$).
(2) In other words, $\mathfrak{F}(\Gamma)$ is the *double complement* $(\check{\Gamma})^c$ (see also Lemma 0.1.43 (7)).
(3) Relation between admissible zones and spectral closures: in the special case when $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C})$ (i.e. $n = 1$), we have $\mathfrak{F}(\Gamma) = \widehat{\Gamma}$ and $\mathrm{Adm}(\Gamma) = (\widehat{\Gamma})^c \subseteq \widehat{\Gamma^c}$. In general, if $\mathcal{A}$ is *not* local, we only have $\mathfrak{F}(\Gamma) \supseteq \widehat{\Gamma}$, i.e. only $\mathrm{Adm}(\Gamma) \subseteq (\widehat{\Gamma})^c$ (cf. Lemma 0.1.43 (6)).
(4) Inclusion reversal: if $I \subseteq J \subseteq [n]$, then clearly $\mathfrak{F}_I(\Gamma) \subseteq \mathfrak{F}_J(\Gamma)$, or equivalently, $\mathrm{Adm}_I(\Gamma) \supseteq$

$\mathrm{Adm}_J(\Gamma)$ (cf. Lemma 0.1.43 (5)).

(5) Individual winding numbers and $\mathcal{A}_I^\times$: we can unpack Definition 2.1.4 further as follows. First notice that if $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C})$ (i.e. $n = 1$), we can characterize $\mathrm{Adm}(\Gamma)$ in two different ways:

$$Z_0 \in \mathrm{Adm}(\Gamma) \Leftrightarrow z_0 \notin \Gamma^{\mathrm{sp}} \begin{cases} \Leftrightarrow \mathrm{ind}(\Gamma^{\mathrm{sp}}, z_0) \text{ well-defined in } \mathbb{C} \\ \Leftrightarrow \Gamma - Z_0 \subset \mathcal{A}^\times \end{cases}$$

Thus, slightly more generally, if $\mathcal{A} = \prod_{j=1}^n (\mathcal{A}_j, \mathfrak{m}_j, \mathbb{C})$ and $I \subseteq [n]$ is any subset, then:

$$Z \in \mathrm{Adm}_I(\Gamma) \Leftrightarrow \forall\, j \in I \colon Z_j \in \mathrm{Adm}(\Gamma_j) \begin{cases} \Leftrightarrow \forall\, j \in I \colon \mathrm{ind}(\Gamma_j^{\mathrm{sp}}, z_j) \text{ well-defined in } \mathbb{C} \\ \Leftrightarrow \forall\, j \in I \colon \Gamma_j - Z_j \subset \mathcal{A}_j^\times \end{cases}$$

In other words:

$$Z_0 \in \mathrm{Adm}_I(\Gamma) \Leftrightarrow \Gamma - Z_0 \subset \mathcal{A}_{(I)}^\times. \tag{2.1.3}$$

In particular, $Z_0 \in \mathrm{Adm}_{[n]}(\Gamma) \Leftrightarrow \Gamma - Z_0 \subseteq \mathcal{A}_{([n])}^\times \overset{\text{def}}{=} \mathcal{A}^\times$.

(6) Existence of admissible points: $\mathfrak{F}_I(\Gamma) \subseteq \mathcal{A}$ is a closed subset for any $I \subseteq [n]$ as it is a finite union of continuous preimages of closed sets, hence $\mathrm{Adm}_I(\Gamma)$ is open in $\mathcal{A}$. If Γ consists of sufficiently nice loops such that each $\mathbb{C} \setminus \gamma_j^{\ell,\mathrm{sp}}$ is dense in $\mathbb{C}$ for all $j \in I$ and all $1 \le \ell \le p$, then each $\mathrm{Adm}_I(\gamma^\ell)$, $1 \le \ell \le p$, is a finite product of dense open subsets, hence $\mathrm{Adm}_I(\Gamma)$ too, being their intersection, is dense open in $\mathcal{A}$. Thus, if $U \subseteq \mathcal{A}$ is open and Γ is not too pathological in the above sense, then $U \cap \mathrm{Adm}_I(\Gamma) \neq \varnothing$.

(7) Invariance under translations by the "indexed" radical: we have $\mathfrak{F}_I(\Gamma) + \mathrm{nil}_{(I)}(\mathcal{A}) = \mathfrak{F}_I(\Gamma)$ and, equivalently, $\mathrm{Adm}_I(\Gamma) + \mathrm{nil}_{(I)}(\mathcal{A}) = \mathrm{Adm}_I(\Gamma)$ for any $I \subseteq [n]$. This follows immediately from the fact that $\forall\, I \subseteq [n] \colon \mathcal{A}_{(I)}^\times + \mathrm{nil}_{(I)}(A) = \mathcal{A}_{(I)}^\times$.

(8) We have $\mathrm{Adm}_{[n]}(\Gamma) \overset{\text{def}}{=} \mathrm{Adm}_{\mathcal{A}}(\Gamma)$ and

$$\mathrm{Adm}_I(\Gamma) \overset{\text{def}}{=} \prod_{j=1}^n \begin{cases} \mathrm{Adm}_{\mathcal{A}_j}(\Gamma_j), & \text{if } j \in I \\ \mathcal{A}_j, & \text{otherwise} \end{cases} \overset{\text{def}}{=} \prod_{j=1}^n \begin{cases} \mathrm{Adm}_{\mathbb{C}}(\Gamma_j^{\mathrm{sp}}) \times \mathfrak{m}_j, & \text{if } j \in I \\ \mathcal{A}_j, & \text{otherwise.} \end{cases}$$

Moreover, $\forall\, I \subseteq [n]\ \forall\, j \in [n] \colon \mathrm{Adm}_I((\iota_j)_\# \Gamma_j^{\mathrm{sp}}) = \mathrm{Adm}_I((\varepsilon_j)_\# \Gamma_j) = \mathrm{Adm}_{\{j\}}(\Gamma)$.

Definition 2.1.4 (Admissiblity induced by ρ and φ)**:** Let $\Gamma \in Z_1(U, G)$, let $\rho \colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$ be a representation and $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ a morphism. We put:

(1) $\mathfrak{F}(\Gamma) := \mathfrak{F}_\rho(\Gamma) := \mathfrak{F}_{I_\rho}(\Gamma)$ or $\mathfrak{F}(\Gamma) := \mathfrak{F}_\varphi(\Gamma) := \mathfrak{F}_{I_\varphi}(\Gamma)$ will denote respectively the set of ρ-forbidden or φ-forbidden points for Γ.

(2) $\mathrm{Adm}(\Gamma) := \mathrm{Adm}_\rho(\Gamma) := \mathrm{Adm}_{I_\rho}(\Gamma)$ or $\mathrm{Adm}(\Gamma) := \mathrm{Adm}_\varphi(\Gamma) := \mathrm{Adm}_{I_\varphi}(\Gamma)$ will denote respectively the set of ρ-admissible or φ-admissible points for Γ.

(3) $\mathfrak{F}_{\mathcal{A}}(\Gamma) := \mathfrak{F}_{\mathrm{id}_{\mathcal{A}}}(\Gamma)$ and $\mathrm{Adm}_{\mathcal{A}}(\Gamma) := \mathrm{Adm}_{\mathrm{id}_{\mathcal{A}}}(\Gamma)$.

Remark (Spaces of admissible loops & 1-cycles)**:** Conversely, if $I \subseteq [n]$ and $Z_0 \in U$, then the sets of I-admissible loops and 1-cycles for Z_0 are simply given by $\mathcal{L}(U \setminus [Z_0]_{(I)}) \overset{\text{def}}{=} \mathcal{L}(U \cap [Z_0]_{(I)}^\times)$ and $Z_1(U \setminus [Z_0]_{(I)}, G) \overset{\text{def}}{=} Z_1(U \cap [Z_0]_{(I)}^\times, G)$, respectively. Accordingly, if $I = I_\rho$ or $I = I_\varphi$, then the sets of ρ-admissible or φ-admissible loops and 1-cycles for Z_0 are given by $\mathcal{L}(U \setminus [Z_0]_\rho) \overset{\text{def}}{=} \mathcal{L}(U \cap [Z_0]_\rho^\times)$ and $Z_1(U \setminus [Z_0]_\rho, G) \overset{\text{def}}{=} Z_1(U \cap [Z_0]_\rho^\times, G)$, respectively, or $\mathcal{L}(U \setminus [Z_0]_\varphi) \overset{\text{def}}{=} \mathcal{L}(U \cap [Z_0]_\varphi^\times)$ and $Z_1(U \setminus [Z_0]_\varphi, G) \overset{\text{def}}{=} Z_1(U \cap [Z_0]_\varphi^\times, G)$, respectively. $\square$

Remarks:

(1) Given representations $\rho_1 \neq \rho_2$ of $\mathcal{A}$, we have $\mathrm{Adm}_{\rho_1}(\Gamma) = \mathrm{Adm}_{\rho_2}(\Gamma) \Leftrightarrow I_{\rho_1} = I_{\rho_2}$.

(2) If $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C})$, then trivially $\mathrm{Adm}_\rho(\Gamma) \overset{\text{def}}{=} \mathrm{Adm}_{\mathcal{A}}(\Gamma)$.

(3) If $I \subseteq [n]$, then $\mathrm{Adm}_I(\Gamma) \overset{\text{def}}{=} \mathrm{Adm}_{\mathrm{pr}_I}(\Gamma)$ (cf. Section 0.1.5).

(4) If φ is injective or ρ faithful, then $\mathrm{Adm}_\varphi(\Gamma) = \mathrm{Adm}_\rho(\Gamma) = \mathrm{Adm}_{\mathcal{A}}(\Gamma)$.
(5) We have $\mathrm{Adm}_{\chi_j}(\Gamma) \stackrel{\text{def}}{=} \mathrm{Adm}_{\{j\}}(\Gamma)$ and more generally $\mathrm{Adm}_{\chi_\rho}(\Gamma) = \mathrm{Adm}_\rho(\Gamma)$.
(6) By inclusion reversal $\mathrm{Adm}_\rho(\Gamma) \stackrel{\text{def}}{=} \mathrm{Adm}_{I_\rho}(\Gamma) \supseteq \mathrm{Adm}_{[n]}(\Gamma) \stackrel{\text{def}}{=} \mathrm{Adm}_{\mathcal{A}}(\Gamma)$.
(7) If $\psi\colon \mathcal{B} \to \mathcal{C}$ is a further non-zero morphism in $\mathsf{FdCAlg}_{\mathbb{C}}$, then $\mathrm{Adm}_{\psi\circ\varphi}(\Gamma) \supseteq \mathrm{Adm}_\varphi(\Gamma)$.

It remains to see that Definition 2.1.4 is indeed the right one. Observe that ρ, by way of ρ^{mor}, induces $\rho_\#^{\mathrm{mor}}\colon Z_1(U,G) \to Z_1(\rho^{\mathrm{mor}}(U), G)$, where $\rho^{\mathrm{mor}}(U) \subseteq \mathcal{B} \coloneqq \mathrm{Codom}(\rho^{\mathrm{mor}})$. Now combining Eq. (2.1.3) and Lemma-Definition 0.1.86, one obtains

$$\rho(Z_0) \in \mathrm{Adm}_{\mathcal{B}}(\rho_\#\Gamma) \Leftrightarrow \rho_\#\Gamma - \rho(Z_0) \subset \mathcal{B}^\times \Leftrightarrow \Gamma - Z_0 \subseteq \mathcal{A}_\rho^\times = \mathcal{A}_{I_\rho}^\times \Leftrightarrow Z_0 \in \mathrm{Adm}_\rho(\Gamma),$$

In other words, we have precisely $\rho^{-1}(\mathrm{Adm}_{\mathcal{B}}(\rho_\#\Gamma)) = \mathrm{Adm}_\rho(\Gamma)$.

Definition 2.1.5 ("Spectral" Homotopy)**:** Let $U \subseteq \mathcal{A}$ with $\pi_0(U) = 1$.

(1) Spectral Homotopy Equivalence: $\gamma, \delta \in \mathcal{L}U$ are called spectrally homotopic, written $\gamma \simeq_{\mathrm{sp}} \delta$, if $\forall\, j \in I_\rho\colon \gamma_j^{\mathrm{sp}} \simeq \delta_j^{\mathrm{sp}}$ in $\mathcal{L}U_j^{\mathrm{sp}}$.
(2) "Spectral" Fundamental Group:

$$\pi_1^{\mathrm{sp}}(U) \coloneqq \pi_1(U) / \bigcap_{j\in I_\rho} \mathrm{Ker}\left(\pi_1(U) \xrightarrow{(\chi_j)_*} \pi_1(U_j^{\mathrm{sp}})\right) = \mathcal{L}U/\simeq_{\mathrm{sp}} . \tag{2.1.4}$$

Remark: If $\pi_0(U) \neq 1$, one can instead repeat the above construction for $(U, Z \in U) \in \mathsf{Top}_*$ and the corresponding *based* loop space ΩU as follows:

$$\pi_1^{\mathrm{sp}}(U,Z) \coloneqq \pi_1(U,Z) / \bigcap_{j\in I_\rho} \mathrm{Ker}\left(\pi_1(U,Z) \xrightarrow{(\chi_j)_*} \pi_1(U_j^{\mathrm{sp}}, z_j)\right) = \Omega U/\simeq_{\mathrm{sp}} . \tag{2.1.5}$$

Definition 2.1.6 ("Spectral" Homology)**:** Let $G \in \mathsf{Ab}$.

(1) "Spectral" 1-Boundaries:

$$B_1^{\mathrm{sp}}(U,G) \coloneqq \bigcap_{j\in I_\rho} (\chi_j)_\#^{-1}(B_1(U_j^{\mathrm{sp}}, G)) \leq B_1(U,G).$$

(2) 1^{st} "Spectral" Homology Group:

$$H_1^{\mathrm{sp}}(U,G) \coloneqq Z_1(U,G)/B_1^{\mathrm{sp}}(U,G).$$

(3) If $\Gamma_1, \Gamma_2 \in Z_1(U,G)$ are spectrally homologous, we will denote this by $\Gamma_1 \approx_{\mathrm{sp}} \Gamma_2$.

Remark: $H_1(U,G)$ is a quotient of $H_1^{\mathrm{sp}}(U,G)$ by the subgroup $B_1(U,G)/B_1^{\mathrm{sp}}(U,G)$. □

2.2. The Generalized Index

In this section we will be using the representation and the morphism perspectives interchangeably since the full characterization of the generalized index is treated almost better in the morphism setting. We will be occasionally using the notation $\varphi_\rho \overset{\text{def}}{=} \rho^{\text{mor}}$ and $\rho_\varphi \overset{\text{def}}{=} \varphi^{\text{rep}}$.

Definition 2.2.1 (Relative Index)**:** Let $\Gamma = \sum_{\ell=1}^{p} g_\ell \gamma^\ell \in Z_1(U, \mathbb{Z})$.

(1) Given a finite-dimensional representation $\rho\colon \mathcal{A} \to \operatorname{End}_{\mathbb{C}}(M)$ and $Z_0 \in \operatorname{Adm}_\rho(\Gamma)$, we define

$$\operatorname{Ind}(\Gamma, Z_0) := \operatorname{Ind}_\rho(\Gamma, Z_0) := \operatorname{Ind}_\rho^{\mathbb{Z}}(\Gamma, Z_0) := \frac{1}{2\pi i} \int_\Gamma \frac{\rho(\mathrm{d}Z)}{\rho(Z - Z_0)} \overset{\text{def}}{=} \sum_{\ell=1}^{p} g_\ell \frac{1}{2\pi i} \oint_{\gamma^\ell} \frac{\rho(\mathrm{d}Z)}{\rho(Z - Z_0)} \overset{\text{def}}{=}$$
$$\overset{\text{def}}{=} \sum_{\ell=1}^{p} g_\ell \operatorname{Ind}_\rho(\gamma^\ell, Z_0),$$

where $\operatorname{Ind}_\rho^{\mathbb{Z}}$ signifies that Γ has coefficients in $\mathbb{Z} \in \mathsf{Ab}$.

(2) Given a morphism $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ in $\mathsf{FdCAlg}_{\mathbb{C}}$ and $Z_0 \in \operatorname{Adm}_\varphi(\Gamma)$, we define:

$$\operatorname{Ind}_\varphi(\Gamma, Z_0) := \operatorname{Ind}_\varphi^{\mathbb{Z}}(\Gamma, Z_0) := \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(\mathrm{d}Z)}{\varphi(Z - Z_0)} \overset{\text{def}}{=} \sum_{\ell=1}^{p} g_\ell \frac{1}{2\pi i} \oint_{\gamma^\ell} \frac{\varphi(\mathrm{d}Z)}{\varphi(Z - Z_0)} \overset{\text{def}}{=}$$
$$\overset{\text{def}}{=} \sum_{\ell=1}^{p} g_\ell \operatorname{Ind}_\varphi(\gamma^\ell, Z_0)$$

(3) We also put $\operatorname{Ind}_{\mathcal{A}} := \operatorname{Ind}_{\mathrm{id}_{\mathcal{A}}}$ for short.

Remarks:

(1) As in Section 2.1, we have indeed $\operatorname{Ind}_\rho^{\mathbb{Z}} \overset{\text{def}}{=} \operatorname{Ind}_{\rho^{\text{mor}}}^{\mathbb{Z}}$.

(2) For example, if $\gamma \in \mathcal{L}\mathbb{C}$ and $z_0 \notin \gamma$, then $\operatorname{Ind}_{\mathbb{C}}^{\mathbb{Z}}(\gamma, z_0) \overset{\text{def}}{=} \operatorname{ind}(\gamma, z_0) \in \mathbb{Z}$ is the usual index (winding number, topological degree) in the complex plane.

(3) Generically, $\operatorname{Ind}_{\mathcal{A}}(\Gamma, Z_0) \in \mathcal{A}$, $\operatorname{Ind}_\varphi(\Gamma, Z_0) \in \mathcal{B}$, and $\operatorname{Ind}_\rho(\Gamma, Z_0) \in \operatorname{End}_{\mathbb{C}}(M)$. In particular $\operatorname{Ind}_\rho(\Gamma, Z_0)$ acts on ${}_{\mathcal{B}}M$.

(4) We will shortly see that $\operatorname{Ind}_\rho^G(\Gamma, Z_0)$ also makes sense for any $G \in \mathsf{Ab}$ and $\Gamma \in Z_1(U, G)$.

(5) Since $f(Z) := \rho(Z - Z_0)^{-1} \overset{\text{def}}{=} \varphi_\rho(Z - Z_0)^{-1}$ is φ_ρ-holomorphic with $f'(Z) = -\varphi_\rho(Z - Z_0)^{-2} \overset{\text{def}}{=} -\rho(Z - Z_0)^{-2}$, the associated holomorphic 1-form

$$\omega := \frac{\rho(\mathrm{d}Z)}{\rho(Z - Z_0)} \overset{\text{def}}{=} \frac{\varphi_\rho(\mathrm{d}Z)}{\varphi_\rho(Z - Z_0)} \in \Omega^1_{\text{hol}}([Z_0]_\rho^\times, \rho(\mathcal{A}))$$

is d-closed by Proposition 1.4.8. Thus, at least in Definition 2.2.1 it is not necessary to assume any stronger regularity of Γ than continuity (see [BG91, Ch. 1.7]).

(6) Since $\mathcal{A}_\rho^\times \supseteq \mathcal{A}^\times$ only (see also Lemma 0.1.85), the integral kernel $\frac{\rho(\mathrm{d}Z)}{\rho(Z - Z_0)}$ may be well-defined even if $\rho\big(\frac{\mathrm{d}Z}{Z - Z_0}\big)$ is not. More precisely, the former is well-defined for all $Z_0 \in \operatorname{Adm}_\rho(\Gamma)$, whereas the latter is well-defined only for $Z_0 \in \operatorname{Adm}_{\mathcal{A}}(\Gamma) \subseteq \operatorname{Adm}_\rho(\Gamma)$.

Lemma 2.2.2 (Homotopy Invariance of $\operatorname{Ind}_\varphi$)**:** Fix $Z_0 \in \mathcal{A}$ and let $\Gamma_1, \Gamma_2 \in Z_1([Z_0]_\rho^\times, \mathbb{Z})$ with $\Gamma_1 \simeq \Gamma_2$. Then $\operatorname{Ind}_\rho(\Gamma_1, Z_0) = \operatorname{Ind}_\rho(\Gamma_2, Z_0)$.

Proof: As pointed out above, the 1-form

$$\omega := \frac{\rho(\mathrm{d}Z)}{\rho(Z - Z_0)} = \frac{\varphi_\rho(\mathrm{d}Z)}{\varphi_\rho(Z - Z_0)} \in \Omega^1_{\text{hol}}([Z_0]_\rho^\times, \mathcal{B}) \subset \Omega^1_{\text{hol}}([Z_0]_\rho^\times, \operatorname{End}_{\mathbb{C}}(M))$$

is d-closed since $f(Z) := \frac{1}{\varphi_\rho(Z - Z_0)} \in \mathcal{O}_{\varphi_\rho}([Z_0]_\rho^\times)$. Hence $\int \omega$ is invariant under $\simeq$ within $[Z_0]_\rho^\times$. $\square$

Now let $V_j \subseteq \mathcal{A}_j$, $1 \leq j \leq n$, be open. Recall that the embedding $\varepsilon_j\colon \mathcal{A}_j \hookrightarrow \mathcal{A}$ also induces maps

$\varepsilon_j\colon \Omega^1(V_j,\mathcal{A}_j) \to \Omega^1(V_j,\mathcal{A})$ and $\varepsilon_j^*\colon \Omega^1(U,\mathcal{A}) \to \Omega^1(\varepsilon_j^{-1}(U),\mathcal{A})$, $1 \le j \le n$. Likewise, $\mathrm{pr}_j\colon \mathcal{A} \twoheadrightarrow \mathcal{A}_j$ also induces maps $\mathrm{pr}_j\colon \Omega^1(U,\mathcal{A}) \to \Omega^1(U,\mathcal{A}_j)$ and $\mathrm{pr}_j^*\colon \Omega^1(V_j,\mathcal{A}_j) \to \Omega^1(\mathrm{pr}_j^{-1}(V_j),\mathcal{A}_j)$, $1 \le j \le n$. Finally, since $\mathrm{id}_{\mathcal{A}} = \sum_{j=1}^n \varepsilon_j \circ \mathrm{pr}_j$, we have $\mathrm{id}_{\mathcal{A}}^* = \sum_{j=1}^n \mathrm{pr}_j^*\,\varepsilon_j^*$. We now give a complete characterization of Ind_φ, building our way successively, almost axiomatic-like, to the most general case.

Lemma 2.2.3 (Properties of Ind)**:** Let $\Gamma \in Z_1(U,\mathbb{Z})$ be $\mathscr{C}^1_{\mathrm{pw}}$-regular and let $\mathcal{B} \xrightarrow{\psi} \mathcal{C}$ be a further morphism.

(1) Compositional index as an index of a pushed-forward cycle:
$$\forall\, Z_0 \in \mathrm{Adm}_{\psi\circ\varphi}(\Gamma)\colon\ \mathrm{Ind}_{\psi\circ\varphi}(\Gamma,Z_0) = \mathrm{Ind}_\psi(\varphi_\#\Gamma,\varphi(Z_0)). \tag{2.2.1}$$
In particular:
$$\forall\, Z_0 \in \mathrm{Adm}_{\varphi}(\Gamma)\colon\ \mathrm{Ind}_{\varphi}(\Gamma,Z_0) = \mathrm{Ind}_{\mathcal{B}}(\varphi_\#\Gamma,\varphi(Z_0)). \tag{2.2.2}$$

(2) Compositional index as a pushed-forward index:
$$\forall\, Z_0 \in \mathrm{Adm}_{\varphi}(\Gamma)\colon\ \mathrm{Ind}_{\psi\circ\varphi}(\Gamma,Z_0) = \psi(\mathrm{Ind}_\varphi(\Gamma,Z_0)). \tag{2.2.3}$$
In particular:
$$\forall\, Z_0 \in \mathrm{Adm}_{\mathcal{A}}(\Gamma)\colon\ \mathrm{Ind}_{\varphi}(\Gamma,Z_0) = \varphi(\mathrm{Ind}_{\mathcal{A}}(\Gamma,Z_0)). \tag{2.2.4}$$
In particular, if $\Delta\colon \mathcal{A} \hookrightarrow \mathcal{A}^m$ denotes the obvious diagonal embedding, then
$$\forall\, Z_0 \in \mathrm{Adm}_{\Delta}(\Gamma)\colon\ \mathrm{Ind}_{\Delta}(\Gamma,Z_0) = \mathrm{Ind}_{\mathcal{A}}(\Gamma,Z_0)^{\times m} \in \mathcal{A}^m. \tag{2.2.5}$$

(3) Index with respect to external products of morphisms: if $\varphi = \times_{j=1}^n \varphi_j$ with $\varphi_j\colon \mathcal{A}_j \to \mathcal{B}_j$, $1 \le j \le n$, then
$$\forall\, Z \in \prod_{j=1}^n \mathrm{Adm}_{\varphi_j}(\Gamma_j)\colon\ \mathrm{Ind}_\varphi(\Gamma,Z) = \sum_{j=1}^n e_j^{\mathcal{B}}\,\mathrm{Ind}_{\varphi_j}(\Gamma_j,Z_j) \in \prod_{j=1}^n \mathcal{B}_j = \mathcal{B}, \tag{2.2.6}$$
where $\{e_1^{\mathcal{B}},\dots,e_n^{\mathcal{B}}\}$ is the corresponding complete system of idempotents.

(4) Index with respect to projections: if $\varnothing \ne I \subseteq [n]$, then
$$\forall\, Z \in \mathrm{Adm}_I(\Gamma)\colon\ \mathrm{Ind}_{\mathrm{pr}_I}(\Gamma,Z) = \sum_{j\in I} e_j^{\mathcal{A}}\,\mathrm{Ind}_{\mathcal{A}_j}(\Gamma_j,Z_j) \in \prod_{j\in I}\mathcal{A}_j \xhookrightarrow{\varepsilon_I} \mathcal{A}, \tag{2.2.7}$$
where $\{e_1^{\mathcal{A}},\dots,e_n^{\mathcal{A}}\}$ is the corresponding complete system of idempotents.

(5) Periods of Cauchy's Reproducing Kernel over a local algebra: if $\mathcal{A} = (\mathcal{A},\mathfrak{m},\mathbb{C})$, then
$$\forall\, Z_0 \in \mathrm{Adm}_{\varphi}(\Gamma)\colon\ \mathrm{Ind}_{\varphi}(\Gamma,Z_0) = \mathrm{ind}(\Gamma^{\mathrm{sp}},z_0) \in \mathbb{Z}. \tag{2.2.8}$$
In particular:
$$\forall\, Z_0 \in \mathrm{Adm}_{\mathcal{A}}(\Gamma)\colon\ \mathrm{Ind}_{\mathcal{A}}(\Gamma,Z_0) = \mathrm{ind}(\Gamma^{\mathrm{sp}},z_0) \in \mathbb{Z}. \tag{2.2.9}$$

(6) Local Target: if $\mathcal{B} = (\mathcal{B},\mathfrak{n},\mathbb{C})$, then
$$\forall\, Z_0 \in \mathrm{Adm}_{\varphi}(\Gamma)\colon\ \mathrm{Ind}_{\varphi}(\Gamma,Z_0) \in \mathbb{Z} \subset \mathbb{C} \xhookrightarrow{\iota_{\mathcal{B}}} \mathcal{B}. \tag{2.2.10}$$

(7) General Products:
$$\forall\, Z \in \mathrm{Adm}_{\varphi}(\Gamma)\colon\ \mathrm{Ind}_{\varphi}(\Gamma,Z) = \sum_{k=1}^m e_k^{\mathcal{B}}\,\mathrm{Ind}_{\bar\varphi_k}(\Gamma_{j(k)},Z_{j(k)}) \in \mathbb{Z}^m \subset \mathbb{C}^m \xhookrightarrow{\iota_{[m]}} \prod_{k=1}^m \mathcal{B}_k \overset{\mathrm{def}}{=} \mathcal{B}, \tag{2.2.11}$$
where $\{e_1^{\mathcal{B}},\dots,e_m^{\mathcal{B}}\}$ is the corresponding system of idempotents.

(8) Periods of Cauchy's Reproducing Kernel for a general morphism:

$$\forall\, Z \in \mathrm{Adm}_\varphi(\Gamma)\colon\ \mathrm{Ind}_\varphi(\Gamma, Z) = \sum_{k=1}^{m} e_k^{\mathcal{B}} \operatorname{ind}(\Gamma^{\mathrm{sp}}_{j(k)}, z_{j(k)}) \in \mathbb{Z}^m \subset \mathbb{C}^m \xhookrightarrow{\iota_{[m]}} \prod_{k=1}^{m} \mathcal{B}_k \overset{\text{def}}{=} \mathcal{B}, \tag{2.2.12}$$

where $\{e_1^{\mathcal{B}}, \dots, e_m^{\mathcal{B}}\}$ is the corresponding system of idempotents.

Proof: By $\mathbb{Z}$-linearity of $\mathrm{Ind}_\varphi(-, Z_0)$ it suffices to verify the claims for a single $\gamma \in \mathcal{L}U$.

To (1): Let $\gamma_\ell \in \mathscr{C}^1([0,1], U)$, $1 \le \ell \le q$, denote the smooth pieces of γ, i.e. $\gamma = \gamma_1 \cdot \gamma_2 \cdots \gamma_q$ (composition of paths), and observe that $\varphi \circ \gamma = (\varphi \circ \gamma_1) \cdot (\varphi \circ \gamma_2) \cdots (\varphi \circ \gamma_q)$. Since φ and ψ are in particular $\mathbb{R}$-linear, we have

$$\begin{aligned}\mathrm{Ind}_{\psi\circ\varphi}(\gamma, Z_0) &\overset{\text{def}}{=} \frac{1}{2\pi i} \sum_{\ell=1}^{q} \int_{\gamma_\ell} \frac{(\psi\circ\varphi)(\mathrm{d}Z)}{(\psi\circ\varphi)(Z - Z_0)} = \frac{1}{2\pi i} \sum_{\ell=1}^{q} \int_0^1 \frac{\psi(\varphi(\gamma_\ell'(t)))\mathrm{d}t}{\psi(\varphi(\gamma_\ell(t)) - \varphi(Z_0))} = \\ &= \frac{1}{2\pi i} \sum_{\ell=1}^{q} \int_0^1 \frac{\psi((\varphi\circ\gamma_\ell)'(t))\mathrm{d}t}{\psi(\varphi(\gamma_\ell(t)) - \varphi(Z_0))} = \frac{1}{2\pi i} \sum_{\ell=1}^{q} \int_{\varphi\circ\gamma_\ell} \frac{\psi(\mathrm{d}W)}{\psi(W - \varphi(Z_0))} = \\ &= \frac{1}{2\pi i} \oint_{\varphi\circ\gamma} \frac{\psi(\mathrm{d}W)}{\psi(W - \varphi(Z_0))} = \mathrm{Ind}_\psi(\varphi\circ\gamma, \varphi(Z_0))\end{aligned}$$

as desired.

To (2): Since $Z_0 \in \mathrm{Adm}_\varphi(\gamma)$ and ψ is $\mathbb{C}$-linear and continuous, it follows that

$$\begin{aligned}\mathrm{Ind}_{\psi\circ\varphi}(\gamma, Z_0) &\overset{\text{def}}{=} \frac{1}{2\pi i} \oint_\gamma \frac{(\psi\circ\varphi)(\mathrm{d}Z)}{(\psi\circ\varphi)(Z - Z_0)} = \frac{1}{2\pi i} \oint_\gamma \psi\left(\frac{\varphi(\mathrm{d}Z)}{\varphi(Z - Z_0)}\right) = \frac{1}{2\pi i} \psi\left(\oint_\gamma \frac{\varphi(\mathrm{d}Z)}{\varphi(Z - Z_0)}\right) = \\ &= \psi(\mathrm{Ind}_\varphi(\gamma, Z_0)).\end{aligned}$$

Finally, since $\Delta \overset{\text{def}}{=} (\mathrm{id}_{\mathcal{A}})_{1\le k\le m}$ (internal product) and $\mathrm{Adm}_\Delta(\Gamma) = \mathrm{Adm}_{\mathcal{A}}(\Gamma)$, we have $\mathrm{Ind}_\Delta(\Gamma, Z_0) = \Delta(\mathrm{Ind}_{\mathcal{A}}(\Gamma, Z_0)) = \mathrm{Ind}_{\mathcal{A}}(\Gamma, Z_0)^{\times m} \overset{\text{def}}{=} (\mathrm{Ind}_{\mathcal{A}}(\Gamma, Z_0))_{1\le k\le m}$.

To (3): This is obvious by component-wise integration, but we give two explicit proofs to illustrate the role of the algebra structure more closely. First we prove the special case $\varphi = \times_{j=1}^{n} \mathrm{id}_{\mathcal{A}_j}$ cleanly, by explicitly using the previously discussed maps induced by pr_j and ε_j, $1 \le j \le n$. Then we prove the general case by using the aforementioned identifications implicitly.

In the first case observe that

$$\varepsilon_j^*\left(\frac{\mathrm{d}W}{W - Z}\right) = \varepsilon_j\left(\frac{\mathrm{d}W_j}{W_j - Z_j}\right),\ 1 \le j \le n.$$

Hence

$$\frac{\mathrm{d}W}{W - Z} = \sum_{j=1}^{n} (\mathrm{pr}_j^* \varepsilon_j^*)\left(\frac{\mathrm{d}W}{W - Z}\right) = \sum_{j=1}^{n} \mathrm{pr}_j^* \varepsilon_j\left(\frac{\mathrm{d}W_j}{W_j - Z_j}\right) = \sum_{j=1}^{n} \varepsilon_j\left(\mathrm{pr}_j^*\left(\frac{\mathrm{d}W_j}{W_j - Z_j}\right)\right)$$

since the induced maps ε_j and pr_j^* commute, $1 \le j \le n$. Therefore,

$$\begin{aligned}\mathrm{Ind}_{\mathcal{A}}(\gamma, Z) &\overset{\text{def}}{=} \frac{1}{2\pi i} \oint_\gamma \frac{\mathrm{d}W}{W - Z} = \sum_{j=1}^{n} \frac{1}{2\pi i} \oint_\gamma \varepsilon_j\left(\mathrm{pr}_j^*\left(\frac{\mathrm{d}W_j}{W_j - Z_j}\right)\right) = \sum_{j=1}^{n} \frac{1}{2\pi i} \varepsilon_j\left(\oint_\gamma \mathrm{pr}_j^*\left(\frac{\mathrm{d}W_j}{W_j - Z_j}\right)\right) = \\ &= \sum_{j=1}^{n} \varepsilon_j\left(\frac{1}{2\pi i} \oint_{\mathrm{pr}_j\circ\gamma} \frac{\mathrm{d}W_j}{W_j - Z_j}\right) = \sum_{j=1}^{n} \varepsilon_j\left(\frac{1}{2\pi i} \oint_{\gamma_j} \frac{\mathrm{d}W_j}{W_j - Z_j}\right) = \sum_{j=1}^{n} \varepsilon_j(\mathrm{Ind}_{\mathcal{A}_j}(\gamma_j, Z_j))\end{aligned}$$

since $(\mathrm{d}\,\mathrm{pr}_j)(\gamma'(t)) = \mathrm{d}(\mathrm{pr}_j \circ \gamma)|_t$ and ε_j is clearly $\mathbb{C}$-linear and continuous, $1 \le j \le n$, which settles the special case. For the general case write $W = \sum_{j=1}^{n} e_j^{\mathcal{A}} W_j$ and $Z = \sum_{j=1}^{n} e_j^{\mathcal{A}} Z_j$ with $W_j, Z_j \in \mathcal{A}_j$, $1 \le j \le n$, where $\{e_1^{\mathcal{A}}, \dots, e_n^{\mathcal{A}}\}$ is the corresponding complete system of distinct mutually orthogonal idempotents of $\mathcal{A} = \prod_{j=1}^n \mathcal{A}_j$. Then:

$$\mathrm{Ind}_\varphi(\gamma, Z) \overset{\text{def}}{=} \frac{1}{2\pi i}\oint_\gamma \frac{\varphi(\mathrm{d}W)}{\varphi(W-Z)} = \frac{1}{2\pi i}\oint_\gamma \frac{\varphi(\mathrm{d}(\sum_{j=1}^n e_j^{\mathcal{A}} W_j))}{\varphi(\sum_{j=1}^n e_j^{\mathcal{A}}(W_j - Z_j))} = \frac{1}{2\pi i}\oint_\gamma \frac{\sum_{j=1}^n \varphi(e_j^{\mathcal{A}}\mathrm{d}W_j)}{\sum_{j=1}^n \varphi(e_j^{\mathcal{A}}(W_j - Z_j))} =$$
$$= \frac{1}{2\pi i}\oint_\gamma \frac{\sum_{j=1}^n \varphi(e_j^{\mathcal{A}})\varphi(\mathrm{d}W_j)}{\sum_{j=1}^n \varphi(e_j^{\mathcal{A}})\varphi(W_j - Z_j)} = \frac{1}{2\pi i}\oint_\gamma \frac{\sum_{j=1}^n e_j^{\mathcal{B}}\varphi_j(\mathrm{d}W_j)}{\sum_{j=1}^n e_j^{\mathcal{B}}\varphi_j(W_j - Z_j)} =$$
$$= \frac{1}{2\pi i}\oint_\gamma \sum_{j=1}^n e_j^{\mathcal{B}} \frac{\varphi_j(\mathrm{d}W_j)}{\varphi_j(W_j - Z_j)} = \sum_{j=1}^n \frac{1}{2\pi i}\oint_\gamma e_j^{\mathcal{B}} \frac{\varphi_j(\mathrm{d}W_j)}{\varphi_j(W_j - Z_j)} =$$
$$= \sum_{j=1}^n \frac{1}{2\pi i}\oint_{\gamma_j} e_j^{\mathcal{B}} \frac{\varphi_j(\mathrm{d}W_j)}{\varphi_j(W_j - Z_j)} = \sum_{j=1}^n e_j^{\mathcal{B}} \frac{1}{2\pi i}\oint_{\gamma_j} \frac{\varphi_j(\mathrm{d}W_j)}{\varphi_j(W_j - Z_j)} = \sum_{j=1}^n e_j^{\mathcal{B}}\,\mathrm{Ind}_{\varphi_j}(\gamma_j, Z_j).$$

To (4): Since $Z \in \mathrm{Adm}_I(\Gamma) \overset{\text{def}}{\Leftrightarrow} \mathrm{pr}_I(Z) \in \mathrm{Adm}_{\mathcal{A}_I}((\mathrm{pr}_I)_\#\Gamma)$, we have

$$\mathrm{Ind}_{\mathrm{pr}_I}(\gamma, Z) \overset{(2.2.2)}{=} \mathrm{Ind}_{\times_{j\in I}\mathrm{id}_{\mathcal{A}_j}}\big((\mathrm{pr}_I)_\#\Gamma, \mathrm{pr}_I(Z)\big) \overset{(2.2.6)}{=} \sum_{j\in I} e_j^{\mathcal{A}}\,\mathrm{Ind}_{\mathcal{A}_j}(\Gamma_j, Z_j) \in \mathcal{A}_I \subseteq \mathcal{A}$$

via the "consolidated" embedding ε_I.

To (5): We first prove this for $\varphi = \mathrm{id}_{\mathcal{A}}$ and $Z_0 = 0 \in \mathrm{Adm}_{\mathcal{A}}(\gamma)$. Since $\gamma \simeq \gamma^{\mathrm{sp}}$ in $\mathcal{A}^\times$, we have by Lemma 2.2.2

$$\mathrm{Ind}_{\mathcal{A}}(\gamma, 0) = \mathrm{Ind}_{\mathcal{A}}(\gamma^{\mathrm{sp}}, 0) = \frac{1}{2\pi i}\oint_{\gamma^{\mathrm{sp}}} \frac{\mathrm{d}z}{z} \overset{\text{def}}{=} \mathrm{ind}(\gamma^{\mathrm{sp}}, 0).$$

Next, substituting $W := Z - Z_0$ and $\delta := \gamma - Z_0$ (shifted curve), we get similarly

$$\mathrm{Ind}_{\mathcal{A}}(\gamma, Z_0) \overset{\text{def}}{=} \frac{1}{2\pi i}\oint_\gamma \frac{\mathrm{d}Z}{Z - Z_0} = \frac{1}{2\pi i}\oint_\delta \frac{\mathrm{d}W}{W} = \mathrm{ind}(\delta^{\mathrm{sp}}, 0) = \mathrm{ind}(\gamma^{\mathrm{sp}} - z_0, 0) = \mathrm{ind}(\gamma^{\mathrm{sp}}, z_0).$$

Finally, since $Z_0 \in \mathrm{Adm}_\varphi(\gamma) = \mathrm{Adm}_{\mathcal{A}}(\gamma)$, we have $\mathrm{Ind}_\varphi(\gamma, Z_0) \overset{(2.2.4)}{=} \varphi(\mathrm{Ind}_{\mathcal{A}}(\gamma, Z_0)) = \varphi(\mathrm{ind}(\gamma^{\mathrm{sp}}, z_0)) = \mathrm{ind}(\gamma^{\mathrm{sp}}, z_0)$.

To (6): We have $\forall\, Z_0 \in \mathrm{Adm}_\varphi(\Gamma, Z_0)$:

$$\mathrm{Ind}_\varphi(\Gamma, Z_0) \overset{(2.2.1)}{=} \mathrm{Ind}_{\mathcal{B}}(\varphi_\#\Gamma, \varphi(Z_0)) \overset{(2.2.8)}{=} \mathrm{ind}_{\mathbb{C}}((\varphi_\#\Gamma)^{\mathrm{sp}}, (\varphi(Z_0))^{\mathrm{sp}}) \in \mathbb{Z} \overset{\iota_{\mathcal{B}}}{\hookrightarrow} \mathcal{B}.$$

To (7): We have $Z \in \mathrm{Adm}_\varphi(\Gamma) \overset{\text{def}}{=} \mathrm{Adm}_{\mathrm{pr}_I}(\Gamma) \Leftrightarrow \forall\, j \in I\colon Z_j \in \mathrm{Adm}_{\mathcal{A}_j}(\Gamma_j) \Leftrightarrow \forall\, 1 \le k \le m\colon Z_{j(k)} \in \mathrm{Adm}_{\mathcal{A}_{j(k)}}(\Gamma_{j(k)}) \subseteq \mathrm{Adm}_{\bar\varphi_k}(\Gamma_{j(k)})$. Thus:

$$\mathrm{Ind}_\varphi(\Gamma, Z) \overset{(2.2.3)}{=} (\bar\varphi \circ \Delta_I)(\mathrm{Ind}_{\mathrm{pr}_I}(\Gamma, Z)) \overset{(2.2.6)}{=} \bar\varphi\Big((\times_{j\in I}\Delta_j)\big((\mathrm{Ind}_{\mathcal{A}_j}(\Gamma_j, Z_j))_{j\in I}\big)\Big) =$$
$$= \bar\varphi\Big(\big(\Delta_j(\mathrm{Ind}_{\mathcal{A}_j}(\Gamma_j, Z_j))\big)_{j\in I}\Big) \overset{(2.2.5)}{=} \bar\varphi\Big(\big(\mathrm{Ind}_{\mathcal{A}_j}(\Gamma_j, Z_j)^{\oplus m_j}\big)_{j\in I}\Big) \overset{\text{def}}{=}$$
$$\overset{\text{def}}{=} (\times_{k=1}^m \bar\varphi_k)\Big(\big(\mathrm{Ind}_{\mathcal{A}_{j(k)}}(\Gamma_{j(k)}, Z_{j(k)})\big)_{1\le k\le m}\Big) = \Big(\bar\varphi_k\big(\mathrm{Ind}_{\mathcal{A}_{j(k)}}(\Gamma_{j(k)}, Z_{j(k)})\big)\Big)_{1\le k\le m} \overset{(2.2.4)}{=}$$
$$\overset{(2.2.4)}{=} \big(\mathrm{Ind}_{\bar\varphi_k}(\Gamma_{j(k)}, Z_{j(k)})\big)_{1\le k\le m} \overset{(2.2.10)}{\in} \mathbb{Z}^m \subset \mathcal{B}.$$

To (8): This follows immediately from (7) and (5). □

Remarks:

(1) Since $\frac{\varphi(\mathrm{d}Z)}{\varphi(Z)}$ and $\frac{\mathrm{d}z}{z}$ are both d-closed, Lemma 2.2.3 should technically be valid even for *continuous* Γ in place of $\mathscr{C}^1_{\mathrm{pw}}$-regularity. Alternatively, Ind_φ can also be *defined* ad hoc entirely topologically for merely continuous Γ by a combination of Eq. (2.2.12) with the fact that the analytic index in the complex plane corresponds to the topological degree.

(2) Even though in the local case $\mathrm{Ind}_\varphi(\Gamma, Z_0)$ is $\mathbb{Z}$-valued, the precise proofs of Eq. (2.2.6),

Eq. (2.2.7), and Eq. (2.2.12) suggest that $\mathrm{Ind}_\varphi(\Gamma, Z)$ is more accurately seen as $\mathcal{B}$-valued (rather than just valued in the obvious lattice $\mathbb{Z}^m \subset \mathcal{B}$) since the definition of Ind_φ actively involves division in the algebra. This perspective is more pronounced if we consider 1-cycles with general coefficients G that can interact with $\mathcal{B}$ or $\mathrm{End}_\mathbb{C}(M)$ in more interesting ways, for example homology with Artinian coefficients.

(3) It follows from Eq. (2.2.12) that $\mathrm{Ind}_\varphi(\Gamma, Z) = 0$ if and only if $\forall\, j \in I_\varphi\colon \mathrm{ind}(\Gamma_j^{\mathrm{sp}}, z_j) = 0$. Similarly, $\mathrm{Ind}_\varphi(\Gamma, Z) \in \mathcal{B}^\times$ if and only if $\forall\, j \in I_\varphi\colon \mathrm{ind}(\Gamma_j^{\mathrm{sp}}, z_j) \neq 0$.

(4) Let $\mathcal{A} := (\mathcal{A}_1, \mathfrak{m}_1, \mathbb{C}) \times (\mathcal{A}_2, \mathfrak{m}_2, \mathbb{C})$ with $\mathfrak{m}_1, \mathfrak{m}_2 \neq 0$, in particular $\mathrm{nil}(\mathcal{A}) = \mathfrak{m}_1 \times \mathfrak{m}_2 \neq 0$, and put $Z_0 := 0$, hence $[Z_0]_{(\{1,2\})} = \mathrm{nil}(\mathcal{A})$. We consider $\mathrm{Ind}_\mathcal{A}(-, \mathrm{nil}(\mathcal{A}))\colon \mathcal{L}\mathcal{A}^\times \to \mathbb{Z}^2$, $\gamma \mapsto \mathrm{Ind}_\mathcal{A}(\gamma, 0)$. Then this produces only a "one-way" homotopy invariant in $\mathcal{A}^\times$. Indeed, if $\gamma, \delta \in \mathcal{L}\mathcal{A}^\times$ with $\gamma \simeq \delta$ in $\mathcal{A}^\times$, then $\gamma^{\mathrm{sp}} \simeq \delta^{\mathrm{sp}}$ in $\mathbb{C}^\times$ and hence $\mathrm{Ind}_\mathcal{A}(\gamma, 0) = \mathrm{Ind}_\mathcal{A}(\delta, 0)$. But clearly the converse need not hold.

(5) Let V be a $\mathbb{C}$-vector space and let $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C})$ be a choice of a local $\mathbb{C}$-algebra structure on V. Then Eq. (2.2.9) shows that, unsurprisingly, $\mathrm{Ind}_\mathcal{A}(\Gamma, Z_0)$ depends on this choice as it is determined by the one-dimensional complex subspace of V spanned by $1_\mathcal{A}$ (or equivalently, on the choice of a functional $\chi_\mathcal{A} \in V^*$ as the unique character / spectral projection of $\mathcal{A}$). On the bright side, $\mathrm{Ind}_\mathcal{A}$ is at least invariant under algebra endomorphisms by Corollary 2.2.6 below.

(6) Some partial examples of $\mathrm{Ind}_\mathcal{A}$ have been calculated in [Ede76b] and [Shp15], but no full description or calculation has been obtained by these authors. The crux of the matter is that this is not doable by elementary algebraic manipulations, but eventually one needs to invoke a homotopy argument *one way or another* (see also Corollary 2.2.4 below), for which one needs to know first that $\frac{\mathrm{d}Z}{Z-W}$ is a closed.

We can now derive the conclusion of Lemma 1.4.9 *directly* from Proposition 1.4.8 without the tedious d-calculation via a topological argument as follows.

Corollary 2.2.4 (Homotopy proof of d-closedness of $X^k\mathrm{d}X$)**:** The 1-form $X^k \mathrm{d}X \in \Omega^1_{\mathrm{hol}}(\mathfrak{m}, \mathfrak{m})$ is d-closed for all $1 \leq k \leq \nu - 2$.

Proof: Let $\gamma \in \mathcal{L}\mathcal{A}^\times$ be arbitrary and write $Z = z \oplus X$ and $\gamma = \gamma^{\mathrm{sp}} \oplus \gamma^{\mathfrak{m}}$ accordingly. Using Lemma 2.2.3, we obtain:

$$\begin{aligned}
\mathrm{ind}(\gamma^{\mathrm{sp}}, 0) = \mathrm{Ind}_\mathcal{A}(\gamma, 0) &= \frac{1}{2\pi i}\oint_\gamma \frac{\mathrm{d}Z}{Z} = \frac{1}{2\pi i}\oint_\gamma \frac{\mathrm{d}z \oplus \mathrm{d}X}{z \oplus X} = \frac{1}{2\pi i}\oint_\gamma \left(\frac{1}{z} \oplus \sum_{k=1}^{\nu-1}(-1)^k \frac{X^k}{z^{k+1}}\right)(\mathrm{d}z \oplus \mathrm{d}X) = \\
&= \left(\frac{1}{2\pi i}\oint_\gamma \frac{\mathrm{d}z}{z}\right) \oplus \left(\sum_{k=1}^{\nu-1}(-1)^k X^k \frac{1}{2\pi i}\oint_\gamma \frac{\mathrm{d}z}{z^{k+1}} + \sum_{k=0}^{\nu-1}\frac{(-1)^k}{z^{k+1}}\frac{1}{2\pi i}\oint_\gamma X^k \mathrm{d}X\right) = \\
&= \left(\frac{1}{2\pi i}\oint_{\gamma^{\mathrm{sp}}} \frac{\mathrm{d}z}{z}\right) \oplus \left(\sum_{k=1}^{\nu-1}(-1)^k X^k \frac{1}{2\pi i}\underbrace{\oint_{\gamma^{\mathrm{sp}}} \frac{\mathrm{d}z}{z^{k+1}}}_{=0 \text{ as } k\geq 1} + \sum_{k=0}^{\nu-1}\frac{(-1)^k}{z^{k+1}}\frac{1}{2\pi i}\oint_{\gamma^{\mathfrak{m}}} X^k \mathrm{d}X\right) = \\
&= \mathrm{ind}(\gamma^{\mathrm{sp}}, 0) \oplus \left(\sum_{k=0}^{\nu-1}\frac{(-1)^k}{z^{k+1}}\frac{1}{2\pi i}\oint_{\gamma^{\mathfrak{m}}} X^k \mathrm{d}X\right),
\end{aligned}$$

hence in particular $\forall\, 1 \leq k \leq \nu - 2\colon \oint_{\gamma^{\mathfrak{m}}} X^k \mathrm{d}X = 0$ by linear independence of $1/z^{k+1}$, $1 \leq k \leq \nu - 1$. But $\gamma^{\mathfrak{m}} \subseteq \mathfrak{m}$ was arbitrary (smooth suffices). This implies the claim □

Remarks:

(1) Observe that the above proof is ($\mathfrak{m}$-)basis-free.

(2) The above calculation also shows that Eq. (2.2.9) is in fact equivalent to all $X^k\mathrm{d}X$ being d-closed, $1 \leq k \leq \nu - 2$.

Lemma-Definition 2.2.5: Let $G \in \mathsf{Ab}$ and $\Gamma = \sum_{\ell=1}^{p} g_\ell \gamma^\ell \in Z_1(U,G)$. Then

$$\forall\, Z_0 \in \mathrm{Adm}_\varphi(\Gamma)\colon\ \mathrm{Ind}_\rho^G(\Gamma, Z_0) := \sum_{\ell=1}^{p} g_\ell \,\mathrm{Ind}_\rho^{\mathbb{Z}}(\gamma^\ell, Z_0) \in G^m \tag{2.2.13}$$

is well-defined.

Proof: By Eq. (2.2.12) we have $\forall\, 1 \le \ell \le p$: $\mathrm{Ind}_\varphi(\gamma^\ell, Z_0) \in \mathbb{Z}^m \subset \mathcal{B}$. Since G is a $\mathbb{Z}$-module, the quantity $g_\ell \,\mathrm{Ind}_\varphi(\gamma^\ell, Z_0)$, $1 \le \ell \le p$, is a well-defined element of G^m. The claim now follows by linearity. □

Corollary 2.2.6 (Invariance Properties of Ind)**:** Let $G \in \mathsf{Ab}$ and $\Gamma \in Z_1(U,G)$.

(1) Invariance under nilpotent translations: if $Z_0 \in \mathrm{Adm}_\varphi(\Gamma)$ and $X_0 \in \mathrm{nil}_\rho(\mathcal{A})$, then $\mathrm{Ind}_\varphi^G(\Gamma, Z_0 + X_0) = \mathrm{Ind}_\varphi^G(Z_0)$. Hence we have a well-defined map

$$\begin{aligned} \mathrm{Ind}_\varphi^G(\Gamma, -)\colon\ \mathrm{Adm}_\rho(\Gamma)/\,\mathrm{nil}_\rho(\mathcal{A}) &\to G^m \\ [Z_0] &\mapsto \mathrm{Ind}_\varphi^G(\Gamma, [Z_0]), \end{aligned} \tag{2.2.14}$$

where $\mathrm{Adm}_\varphi(\Gamma)/\,\mathrm{nil}_\rho(\mathcal{A})$ simply denotes the image of $\mathrm{Adm}_\varphi(\Gamma)$ under the quotient map $\mathcal{A} \twoheadrightarrow \mathcal{A}/\,\mathrm{nil}_\rho(\mathcal{A})$ and can also be identified with a subset of $\mathbb{C}^{|I_\rho|} \subseteq \mathbb{C}^n$. In particular, if $0 \in \mathrm{Adm}_\varphi(\Gamma)$, then $\mathrm{Ind}_\mathcal{A}^G(\Gamma, X_0) = \mathrm{Ind}_\mathcal{A}^G(\Gamma, 0)$.

(2) Bi-invariance under local morphisms: if $\varphi, \psi\colon (\mathcal{A}, \mathfrak{m}, \mathbb{C}) \to (\mathcal{B}, \mathfrak{n}, \mathbb{C})$ are two morphisms, then

$$\forall\, Z_0 \in \mathrm{Adm}_\mathcal{A}(\Gamma)\colon\ \mathrm{Ind}_\mathcal{B}^G(\varphi_\#\Gamma, \psi(Z_0)) = \mathrm{Ind}_\mathcal{A}^G(\Gamma, Z_0) = \mathrm{ind}_\mathbb{C}^G(\Gamma^{\mathrm{sp}}, z_0).$$

In particular, $\mathrm{Ind}_\mathcal{A}^G(-,-)$ is bi-invariant under $\mathrm{End}_{\mathsf{Alg}_\mathbb{C}}(\mathcal{A})$.

Proof: To (1): Since $\mathrm{nil}_\rho(\mathcal{A}) \overset{\text{def}}{=} \mathfrak{m}_\rho \overset{\text{def}}{=} \mathfrak{m}_{I_\rho}$, the claim follows from Eq. (2.2.12).

To (2): Let $\gamma \in \mathcal{L}U$. Since $\chi_\mathcal{B} \circ \varphi = \chi_\mathcal{A}$ for any $(\mathcal{A}, \mathfrak{m}, \mathbb{C}) \xrightarrow{\varphi} (\mathcal{B}, \mathfrak{n}, \mathbb{C})$, we have $\forall\, \Gamma \in Z_1(U,G)$: $\psi(Z_0) \in \mathrm{Adm}_\mathcal{B}(\varphi_\#\Gamma) \overset{\text{def}}{\Leftrightarrow} \chi_\mathcal{B}(\psi(Z_0)) = \chi_\mathcal{A}(Z_0) \notin \chi_{\mathcal{B},\#}\varphi_\#\Gamma = (\chi_\mathcal{B} \circ \varphi)_\#\Gamma = \chi_{\mathcal{A},\#}\Gamma \overset{\text{def}}{\Leftrightarrow} Z_0 \in \mathrm{Adm}_\mathcal{A}(\Gamma)$. Thus, similarly, $\forall\, Z_0 \in \mathrm{Adm}_\mathcal{A}(\gamma)$:

$$\begin{aligned} \mathrm{Ind}_\mathcal{B}^\mathbb{Z}(\varphi_\#\gamma, \psi(Z_0)) &\overset{(2.2.9)}{=} \mathrm{ind}((\varphi_\#\gamma)^{\mathrm{sp}}, \psi(Z_0)^{\mathrm{sp}}) \overset{\text{def}}{=} \mathrm{ind}(\chi_{\mathcal{B},\#}\varphi_\#\gamma, \chi_\mathcal{B}(\psi(Z_0))) = \\ &= \mathrm{ind}((\chi_\mathcal{B} \circ \varphi)_\#\gamma, (\chi_\mathcal{B} \circ \psi)(Z_0)) = \mathrm{ind}(\chi_{\mathcal{A},\#}\gamma, \chi_\mathcal{A}(Z_0)) \overset{\text{def}}{=} \\ &\overset{\text{def}}{=} \mathrm{ind}(\gamma^{\mathrm{sp}}, z_0) \overset{(2.2.9)}{=} \mathrm{Ind}_\mathcal{A}^\mathbb{Z}(\Gamma, Z_0). \end{aligned}$$

The full claim now follows from $G \in \mathsf{Mod}(\mathbb{Z})$ by $\mathbb{Z}$-linearity. □

Definition 2.2.7 (Simple Loops in $\mathcal{A}$)**:** Let $\gamma \in \mathcal{L}\mathcal{A}$ and $Z_0 \in \mathcal{A}$. Then γ is called simple with respect to ρ and Z_0, really with respect to $[Z_0]_\rho \overset{\text{def}}{=} Z_0 + \mathrm{Sing}_\rho(\mathcal{A})$, if $Z_0 \in \mathrm{Adm}_\rho(\gamma)$ and $\mathrm{Ind}_\rho(\gamma, Z_0) = \mathrm{Ind}_\rho(\gamma, [Z_0]_\rho) = 1 \overset{\text{def}}{=} \mathrm{id}_M \in \mathrm{End}_\mathbb{C}(M)$.

Lemma 2.2.8 (Index is locally constant)**:** Let $\Gamma \in Z_1(U,G)$. Then the map

$$\begin{aligned} \mathrm{Ind}_\rho^G(\Gamma, -)\colon\ \mathrm{Adm}_\rho(\Gamma) &\to G \\ Z &\mapsto \int_\Gamma \frac{\varphi_\rho(\mathrm{d}W)}{\varphi_\rho(W - Z)} \end{aligned}$$

is locally constant. In particular, Γ divides $\mathrm{Adm}_\rho(\Gamma)$ in two types of components: points with zero and non-zero Ind_ρ^G.

Proof: We first prove the statement for $\Gamma := \gamma \in \mathcal{L}U$. This is a parameter integral with an integrand that is continuously φ_ρ-differentiable on the open subset $\mathrm{Adm}_\rho(\gamma) \subseteq \mathcal{A}$. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}Z}\,\mathrm{Ind}_{\varphi_\rho}^\mathbb{Z}(\gamma, Z) = \int_\gamma \frac{\mathrm{d}}{\mathrm{d}Z}\left(\frac{1}{\varphi_\rho(W - Z)}\right)\varphi_\rho(\mathrm{d}W) = \int_\gamma \frac{\varphi_\rho(\mathrm{d}W)}{\varphi_\rho(W - Z)^2} = 0$$

since $\varphi_\rho(Z-W)^{-1}$ is a global φ_ρ-primitive in the variable W of the last integrand. Thus

$$\mathrm{Ind}^{\mathbb{Z}}_\rho(\gamma,-) = \mathrm{Ind}^{\mathbb{Z}}_{\varphi_\rho}(\gamma,-)\colon\ \mathrm{Adm}_\rho(\Gamma)\to\mathbb{Z}$$

is a locally constant function. Finally, if $\Gamma := \sum_{\ell=1}^p g_\ell\gamma^\ell$ for some $g_\ell\in G$ and $\gamma^\ell\in\mathcal{L}U$, then $\mathrm{Ind}^G_\rho(\Gamma,-) \overset{\text{def}}{=} \sum_{\ell=1}^p g_\ell \,\mathrm{Ind}^{\mathbb{Z}}_\rho(\gamma^\ell,-)$ is a linear combination with constant coefficients of locally constant maps, hence a locally constant itself. □

Remark: The zero or non-zero component of $\mathrm{Ind}^G_\rho(\Gamma,-)$ is at least in principle allowed to be empty.

Definition 2.2.9 (Unital Component)**:** Let $\Gamma\in Z_1(U,\mathbb{Z})$. We put

$$\mathfrak{C}_\rho(\Gamma) := \{Z\in\mathrm{Adm}_\rho(\Gamma) : \mathrm{Ind}_\rho(\Gamma,Z)\in\mathrm{GL}(M)\}$$

or

$$\mathfrak{C}_\varphi(\Gamma) := \{Z\in\mathrm{Adm}_\varphi(\Gamma) : \mathrm{Ind}_\varphi(\Gamma,Z)\in\mathcal{B}^\times\},$$

respectively.

Corollary 2.2.10 (Components of Γ for $G=\mathbb{Z}$)**:** Let $\Gamma\in Z_1(U,\mathbb{Z})$ and $Z\in\mathrm{Adm}_\rho(\Gamma)$.

(1) Components of $\mathrm{Ind}_\rho(\Gamma,-)$: each component of Γ is of the form

$$\prod_{j=1}^n \begin{cases} C_j\times\mathfrak{m}_j, \text{ if } j\in I_\rho\\ \mathcal{A}_j, \text{ otherwise,}\end{cases}$$

where $C_j\subseteq\mathbb{C}$ is a component of the locally constant function $\mathrm{ind}(\Gamma^{\mathrm{sp}}_j,-)$, $j\in I_\rho$.

(2) Unital Component of $\mathrm{Ind}_\rho(\Gamma,-)$: we have $\mathrm{Ind}_\rho(\Gamma,Z)\in\mathrm{GL}(M)$ if and only if for all $j\in I_\rho$ the component of Γ^{sp}_j containing z_j is bounded in $\mathbb{C}$. In other words, we have

$$\mathfrak{C}_\rho(\Gamma) = \prod_{j=1}^n \begin{cases} C_j(\gamma)\times\mathfrak{m}_j, \text{ if } j\in I_\rho\\ \mathcal{A}_j, \text{ otherwise}\end{cases}$$

where $C_j(\gamma) := \{z_j\in\mathbb{C}\setminus\Gamma^{\mathrm{sp}}_j : \mathrm{ind}(\Gamma^{\mathrm{sp}}_j,z_j)\neq 0\}$, $1\le j\le n$.

(3) $\mathfrak{C}_\rho(\Gamma)$ is open, not necessarily connected, and spectrally closed, i.e. $\widehat{\mathfrak{C}_\rho(\Gamma)} = \mathfrak{C}_\rho(\Gamma)$.

Proof: To (1): If $G=\mathbb{Z}$, this is immediate from Eq. (2.2.12) and standard complex analysis.

To (2): Again by Eq. (2.2.12), we have that $\mathrm{Ind}_\rho(\Gamma,Z)\in\mathrm{GL}(M) \Leftrightarrow \forall\, j\in I_\rho\colon\ \mathrm{ind}(\Gamma^{\mathrm{sp}}_j,z_j)\neq 0 \Leftrightarrow$ the component of Γ^{sp}_j in $\mathbb{C}$ containing z_j is not the unbounded one (by standard complex analysis).

To (3): $\mathfrak{C}_\rho(\Gamma)$ is open, as a continuous preimage of $\mathrm{GL}(M)$, in $\mathrm{Adm}_\rho(\Gamma)$, which is itself open. Spectral closedness is obvious from the characterization in (2). □

Lemma 2.2.11: Let $Z\in\mathcal{A}$ and $\gamma,\delta\in\mathcal{L}(\mathcal{A}\setminus[Z]_\rho)$.

(1) $\gamma\simeq\delta$ in $\pi_1([Z]^\times_\rho) \Rightarrow \mathrm{Ind}_\rho(\gamma,Z) = \mathrm{Ind}_\rho(\delta,Z)$.

(2) $\gamma\simeq_{\mathrm{sp}}\delta$ in $\pi_1^{\mathrm{sp}}([Z]^\times_\rho) \Leftrightarrow \mathrm{Ind}_\rho(\gamma,Z) = \mathrm{Ind}_\rho(\delta,Z)$.

Proof: For $j\in I_\rho$ the character $\chi_j\colon\mathcal{A}\to\mathbb{C}$ induces a homomorphism $(\chi_j)_*\colon\pi_1(\mathcal{A}\setminus[Z]_\rho)\to\pi_1(\mathbb{C}\setminus\{z_j\})$, $[\gamma]\mapsto[\gamma^{\mathrm{sp}}_j]$. We thus have: $\gamma\simeq\delta$ in $\mathcal{A}\setminus[Z]_\rho \Rightarrow \forall\, j\in I_\rho\colon \gamma^{\mathrm{sp}}_j\simeq\delta^{\mathrm{sp}}_j$ in $\mathbb{C}\setminus\{z_j\} \Leftrightarrow \forall\, j\in I_\rho\colon\ \mathrm{ind}(\gamma^{\mathrm{sp}}_j,z_j) = \mathrm{ind}(\delta^{\mathrm{sp}}_j,z_j) \Leftrightarrow \mathrm{Ind}_\rho(\gamma,Z) = \mathrm{Ind}_\rho(\delta,Z)$ by Eq. (2.2.12). This proves both (1) and (2). □

Lemma 2.2.12 (Invariance of Ind_ρ under $\approx_{\mathrm{sp}}$)**:** Let $\Gamma,\Delta\in Z_1(U,\mathbb{Z})$. Then $\Gamma\approx_{\mathrm{sp}}\Delta$ if and only if $\forall\, Z\in\check{U}\colon\ \mathrm{Ind}_\rho(\Gamma,Z) = \mathrm{Ind}_\rho(\Delta,Z)$.

Proof: It suffices to show that $[\Gamma]=0$ in $H_1^{\mathrm{sp}}(U,\mathbb{Z}) \Leftrightarrow \forall\, Z\in\check{U}\colon\ \mathrm{Ind}_\rho(\Gamma,Z)=0$. Indeed, by Eq. (2.2.12) this is equivalent to $\forall\, j\in I_\rho\ \forall\, z_j\in(U^{\mathrm{sp}}_j)^c\colon\ \mathrm{ind}(\Gamma^{\mathrm{sp}}_j,z_j)=0 \Leftrightarrow \forall\, j\in I_\rho\colon \Gamma^{\mathrm{sp}}_j\in B_1(U^{\mathrm{sp}}_j,\mathbb{Z})$

(by standard complex analysis). □

Remark: Recall that if $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C})$, then $\check{U} = (\widehat{U})^c \subseteq \widehat{U^c}$, but if $\mathcal{A}$ is not local, then in general only $\check{U} \subseteq (\widehat{U})^c \subseteq \widehat{U^c}$ (cf. Lemma 0.1.43 (6)). Naively, one would have expected at least that $[\Gamma] = 0$ in $H_1^{\mathrm{sp}}(U, \mathbb{Z})$ would imply $\forall\, Z \in (\widehat{U})^c$: $\mathrm{Ind}_\rho(\Gamma, Z) = 0$, but alas this is not the case when $\mathcal{A}$ is not local. This goes to show once again that *the complement of the spectral closure is not the right notion* and should be replaced by that of the *spectral complement* as defined in Section 0.1.9.

2.3. The Generalized Cauchy Integral Formula with Index

We now move on to establish a few versions of Cauchy's Integral Formula for A-holomorphic functions in gradational order. The first issue to be mindful of is variation of the admissible loops (or 1-cycles) for a predefined point vs. variation of the admissible points for a predefined loop (or 1-cycle). To address this we will heavily rely on homotopy and d-closedness instead of other less conceptual, less powerful, and subsequently more cumbersome approaches.

Proposition 2.3.1 (Centered Local Cauchy Integral Formula over $\mathcal{A}$)**:** Let $Z_0 \in \mathcal{A}$ be a point, $r > 0$, $\Delta \coloneqq \Delta_r(Z_0)$, and $f \in \mathcal{O}_\rho(\Delta) \overset{\text{def}}{=} \mathcal{O}_{\mathcal{A}}(\Delta, {}_{\mathcal{A}}M)$. Then we have:

$$\forall\, \gamma \in \mathcal{L}(\Delta \setminus [Z_0]_\rho)\colon\ \operatorname{Ind}_\rho(\gamma, Z_0) f(Z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{\rho(\mathrm{d}Z)}{\rho(Z - Z_0)} f(Z). \tag{2.3.1}$$

Proof: It suffices to prove the claim for $Z_0 = 0$, otherwise simply consider $\tilde f(Z) \coloneqq f(Z + Z_0)$ and $\tilde\gamma(t) \coloneqq \gamma(t) - Z_0$, $t \in [0,1]$.[3] In particular $\Delta \setminus [Z_0]_\rho = \Delta^\times_\rho \coloneqq \Delta \cap \mathcal{A}^\times_\rho$. Moreover, note that in general f takes values in $M \in \mathsf{Mod}(\mathcal{A})$ and Ind_ρ takes values in $\operatorname{End}_{\mathbb{K}}(M)$.

We first consider the special case $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C})$. Putting $d \coloneqq \dim_{\mathbb{C}} \mathfrak{m}$, we have $\Delta^\times_\rho \overset{\text{def}}{=} \Delta^\times \overset{\text{def}}{=} \Delta \cap (\mathbb{C}^\times \times \mathbb{C}^d) = \mathbb{D}^*_r(0) \times \mathbb{D}_r(0)^d$. Therefore, $\Delta^\times$ has the same 1-homotopy structure as $\mathcal{A}^\times$, namely $\pi_0(\Delta^\times) = 1$ and $[\mathbb{S}^1, \Delta^\times] \cong [\mathbb{S}^1, \mathbb{D}^*_r(0)] \times [\mathbb{S}^1, \mathbb{D}_r(0)]^d \cong [\mathbb{S}^1, \mathbb{D}^*_r(0)] \cong \mathbb{Z}$, where explicitly $\gamma \simeq \gamma^{\text{sp}} \simeq \gamma^{\text{sp}}_\varepsilon$ with $\gamma^{\text{sp}}_\varepsilon\colon [0,1] \to \mathbb{D}^*_r(0) \times \{0_{\mathfrak{m}}\}$, $t \mapsto \varepsilon e^{2\pi i \mu t}$, for some arbitrary fixed $0 < \varepsilon < r$ and some winding number $\mu \in \mathbb{Z}$. We thus have

$$\frac{\rho(\dot\gamma^{\text{sp}}_\varepsilon(t))}{\rho(\gamma^{\text{sp}}_\varepsilon(t))} = 2\pi i \mu \overset{\text{def}}{=} 2\pi i \operatorname{ind}(\gamma^{\text{sp}}_\varepsilon, 0) \overset{(2.2.8)}{=} 2\pi i \operatorname{Ind}_\rho(\gamma, 0).$$

Since $f(Z) \in \mathcal{O}_\rho(\Delta)$, it follows from Proposition 1.1.19 that $g(Z) \coloneqq \rho(Z)^{-1}(f(Z) - f(0)) = \varphi_\rho(Z)^{-1}(f(Z) - f(0)) \in \mathcal{O}_\rho(\Delta^\times)$, hence

$$\omega \coloneqq \frac{\rho(\mathrm{d}Z)}{\rho(Z)}(f(Z) - f(0)) \overset{\text{def}}{=} \rho(\mathrm{d}Z) g(Z) \in \Omega^1_{\text{hol}}(\Delta^\times, M)$$

is d-closed by Proposition 1.4.8, and therefore $\int \omega$ is invariant under loop homotopy inside $\Delta^\times$. Thus,

$$\begin{aligned}
&\left\| \frac{1}{2\pi i} \oint_\gamma \frac{\rho(\mathrm{d}Z)}{\rho(Z)} f(Z) - \operatorname{Ind}_\rho(\gamma, 0) f(0) \right\|_M \overset{\text{def}}{=} \left\| \frac{1}{2\pi i} \oint_\gamma \omega \right\|_M = \left\| \frac{1}{2\pi i} \oint_{\gamma^{\text{sp}}_\varepsilon} \omega \right\|_M = \\
&= \left\| \frac{1}{2\pi i} \int_0^1 \frac{\rho(\dot\gamma^{\text{sp}}_\varepsilon(t))}{\rho(\gamma^{\text{sp}}_\varepsilon(t))} (f(\gamma^{\text{sp}}_\varepsilon(t)) - f(0)) \mathrm{d}t \right\|_M = \\
&= \left\| \operatorname{Ind}_\rho(\gamma, 0) \int_0^1 \big(f(\gamma^{\text{sp}}_\varepsilon(t)) - f(0)\big) \mathrm{d}t \right\|_M \le |\mu|\, \|f - f(0)\|_{M, \gamma^{\text{sp}}_\varepsilon} \xrightarrow{\varepsilon \to 0} 0 \text{ as desired.}
\end{aligned}$$

Next suppose that $\rho\colon \mathcal{A} \to \operatorname{End}_{\mathbb{C}}(M)$ is of the form

$$\rho = \times_{j=1}^n \rho_j \colon \mathcal{A} = \prod_{j=1}^n (\mathcal{A}_j, \mathfrak{m}_j, \mathbb{C}) \to \prod_{j=1}^n \operatorname{End}_{\mathbb{C}}(M_j),$$

where $M = \bigoplus_{j=1}^n M_j$. Since in that case $\Delta = \prod_{j=1}^n \Delta_j = \widehat{\Delta}$, we have by Proposition 1.2.9 that $f(Z) = f(Z_1, \dots, Z_n) = \oplus_{j=1}^n f_j(Z_j)$ for some $f_j \in \mathcal{O}_{\rho_j}(\Delta_j)$, $1 \le j \le n$. Using Eq. (2.2.6) and the first part of the proof, we now have

$$\begin{aligned}
\operatorname{Ind}_\rho(\gamma, 0) \cdot f(0) &= \times_{j=1}^n \operatorname{Ind}_{\rho_j}(\gamma_j, 0_{\mathcal{A}_j}) \cdot \oplus_{j=1}^n f_j(0_{\mathcal{A}_j}) = \oplus_{j=1}^n \operatorname{Ind}_{\rho_j}(\gamma_j, 0_{\mathcal{A}_j}) f_j(0_{\mathcal{A}_j}) = \\
&= \oplus_{j=1}^n \frac{1}{2\pi i} \oint_{\gamma_j} \frac{\rho_j(\mathrm{d}Z_j)}{\rho_j(Z_j)} f_j(Z_j) = \frac{1}{2\pi i} \oint_\gamma \frac{\rho(\mathrm{d}Z)}{\rho(Z)} f(Z),
\end{aligned}$$

[3] Of course, replacing $f(Z)$ by $\tilde f(Z) \coloneqq f(Z + Z_0) - f(Z_0)$ instead, one could without loss of generality further take f to be centered, i.e. $f(Z_0) = f(0) = 0$, but unfortunately that would inadvertently obscure parts of the proof.

where the last part of the equality follows in a way similar to the proof of Eq. (2.2.6).

Finally, if $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{C}}(M)$ is a general finite-dimensional representation with $\mathcal{A} = \prod_{j=1}^{n}(\mathcal{A}_j, \mathfrak{m}_j, \mathbb{C})$ and $\rho = \bar{\rho} \circ \Pi$ as in Corollary 0.1.83, then, following the notations of Proposition 1.2.12 and its proof, f can be written as

$$\begin{tikzcd} \Delta \arrow[r, "f"] \arrow[d, "\Pi"'] & M \\ \bar{\Delta} \arrow[ur, "\bar{f}"'] & \end{tikzcd} \tag{2.3.2}$$

where $\bar{\Delta} \stackrel{\text{def}}{=} \prod_{k=1}^{m} \Delta_{j(k)}$ and $\bar{f} \in \mathcal{O}_{\bar{\rho}}(\bar{\Delta})$ is of the form $\bar{f} = \times_{k=1}^{m} \bar{f}_k$ with $\bar{f}_k \in \mathcal{O}_{\bar{\rho}_k}(\Delta_{j(k)})$, $1 \leq k \leq m$, and $\bar{\rho} = \times_{k=1}^{m} \bar{\rho}_k$. Define $\bar{\gamma} \coloneqq \Pi_{\#}\gamma$ and note that $\mathrm{d}\bar{\gamma} \stackrel{\text{def}}{=} \mathrm{d}(\Pi \circ \gamma) = (\Pi \circ \gamma)'(t)\mathrm{d}t = (\Pi \circ \gamma')(t)\mathrm{d}t$ as Π is $\mathbb{R}$-linear. We now have

$$\begin{aligned}
\frac{1}{2\pi i} \oint_{\gamma} \frac{\rho(\mathrm{d}Z)}{\rho(Z)} f(Z) &= \frac{1}{2\pi i} \int_0^1 \frac{\rho(\gamma'(t))}{\rho(\gamma(t))} f(\gamma(t))\mathrm{d}t = \frac{1}{2\pi i} \int_0^1 \frac{(\bar{\rho} \circ \Pi)(\gamma'(t))}{(\bar{\rho} \circ \Pi)(\gamma(t))} (\bar{f} \circ \Pi)(\gamma(t))\mathrm{d}t = \\
&= \frac{1}{2\pi i} \int_0^1 \frac{\bar{\rho}(\bar{\gamma}'(t))}{\bar{\rho}(\bar{\gamma}(t))} \bar{f}(\bar{\gamma}(t))\mathrm{d}t = \frac{1}{2\pi i} \oint_{\bar{\gamma}} \frac{\bar{\rho}(\mathrm{d}\bar{Z})}{\bar{\rho}(\bar{Z})} \bar{f}(\bar{Z}) = \mathrm{Ind}_{\bar{\rho}}(\bar{\gamma}, 0)\bar{f}(0) = \\
&= \mathrm{Ind}_{\rho}(\gamma, 0) f(0)
\end{aligned}$$

by the previous case, where $\bar{f}(0) = f(0)$ by Eq. (2.3.2) and

$$\mathrm{Ind}_{\rho}(\gamma, 0) = \mathrm{Ind}_{\bar{\rho}\circ\Pi}(\gamma, 0) \stackrel{(2.2.1)}{=} \mathrm{Ind}_{\bar{\rho}}(\Pi_{\#}\gamma, \Pi(0)) = \mathrm{Ind}_{\bar{\rho}}(\bar{\gamma}, 0)$$

as desired. □

Remarks:

(1) A byproduct of the argument used in the proof of Proposition 2.3.1 is that every point $Z_0 \in \mathcal{A}$ admits an open neighbourhood containing an admissible loop of an arbitrary *integral* index $\mathrm{Ind}_{\rho}(\gamma, Z_0) \in \mathbb{Z}^m = \pi_1(\mathcal{B}^{\times})$.

(2) Putting $\mathcal{I}_{\gamma} \coloneqq \mathrm{Ind}_{\rho}(\gamma, -)$, it follows from Proposition 2.3.1 that $f|_{\gamma}$ determines f in all components of the set $\mathrm{Adm}_{\rho}(\gamma) \cap \mathcal{I}_{\gamma}^{-1}((\mathbb{Z} \setminus \{0\})^m)$, which actually contains stuff outside of Δ. In particular, this set is always unbounded as it contains a copy of $\mathrm{nil}(\mathcal{A})$. One thus obtains a ρ-holomorphic continuation of f to an unbounded subset of $\mathcal{A}$. Details follow.

The *local* Cauchy Integral Formula can be extended to very general domains in $\mathcal{A}$ under a simple topological assumption on the integration contour via the following simple topological construction:

Definition 2.3.2 (Subspace-Restricted Homotopy)**:** Let $(X, A) \in \mathsf{Top}_2$ and $\gamma_0, \gamma_1 \in \mathcal{L}X$. Then γ_0 and γ_1 are called homotopic through A, written $\gamma_0 \simeq_A \gamma_1$, if there exists a homotopy family $(\gamma_s)_{s\in[0,1]} \subset \mathcal{L}X$ such that $\forall\, s \in (0,1)\colon \gamma_s \in \mathcal{L}A$.

Remarks:

(1) We couldn't find any standard terminology for Definition 2.3.2, but apparently the concept occasionally appears in the literature, see [MSE03].

(2) A lot can be said about subspace-restricted homotopy, but we will focus here only on the minimum needed to establish the various generalized Cauchy Integral Formulas. □

Lemma 2.3.3: Let $(X, A) \in \mathsf{Top}_2$ and $\gamma_0 \in \mathcal{L}X$ with $\gamma_0 \simeq_A \gamma_1$. Then for every open neighbourhood $\mathcal{N} \supset \gamma_1$ there exists $s_{\mathcal{N}} \in (0,1)$ such that $\forall\, s_{\mathcal{N}} < s < 1\colon \gamma_s \in \mathcal{L}(\mathcal{N} \cap A)$.

Proof: Let $H\colon [0,1] \times [0,1] \to X$, $\gamma_s(t) \coloneqq H(s,t)$, be the corresponding homotopy. Since γ_s are loops, we actually have (by a small abuse of notation) $H\colon [0,1] \times \mathbb{S}^1 \to X$ continuous. Let $\mathcal{N}$ be an open neighbourhood of γ_1 and put $\mathcal{U} := H^{-1}(\mathcal{N})$, hence $\mathcal{U}$ is an open neighbourhood of $\{1\} \times \mathbb{S}^1$. In particular, if $p \in \mathbb{S}^1$ is an arbitrary point, then $\mathcal{U} \ni (1, p)$, hence $\mathcal{U}$ contains a basic open product neighbourhood $(s_p, 1] \times V_p$, where $0 \leq s_p < 1$ and $\mathbb{S}^1 \supseteq V_p \ni p$ is an open neighbourhood. By compactness of $\mathbb{S}^1$ there exist finitely many $p_1, \ldots, p_k \in \mathbb{S}^1$ such that $\{V_{p_1}, \ldots, V_{p_k}\}$ is an open cover of $\mathbb{S}^1$. Now define $s_{\mathcal{N}} := \max\{s_{p_1}, \ldots, s_{p_k}\}$. It follows that $(s_{\mathcal{N}}, 1] \times \mathbb{S}^1 \subseteq \mathcal{U}$ by construction. Finally observe that by hypothesis $H((0, 1) \times \mathbb{S}^1) \subseteq A$, hence in particular $H((s_{\mathcal{N}}, 1) \times \mathbb{S}^1) \subseteq A$, which completes the proof. □

We can now extend Proposition 2.3.1 to more general domains than Δ by a reduction to a local neighbourhood that is a direct product, via a subspace-restricted homotopy for a choice of a subspace that avoids the singularity.

Proposition 2.3.4 (Naive Null-Homotopical Cauchy Integral Formula over $\mathcal{A}$)**:**

(1) Pointed Naive Null-Homotopical Cauchy Integral Formula over $\mathcal{A}$ for General Domains: let $U \subseteq \mathcal{A}$ be open, $Z \in U$ a point, $f \in \mathcal{O}_\rho(U)$, and $\gamma \in \mathcal{L}(U \setminus [Z]_\rho)$. If $Z' \in \widetilde{\{Z\}}$ is arbitrary and $\gamma \simeq_{\widetilde{U} \cap [Z]_\rho^\times} Z'$, then

$$\operatorname{Ind}_\rho(\gamma, Z) f(Z) = \frac{1}{2\pi i} \oint_\gamma \frac{\rho(\mathrm{d}W)}{\rho(W - Z)} f(W). \tag{2.3.3}$$

(2) Local Cauchy Integral Formula over $\mathcal{A}$: let $\Delta \subseteq \mathcal{A}$ be a polydisc and $f \in \mathcal{O}_\rho(\Delta)$. Then:

$$\forall\, \gamma \in \mathcal{L}\Delta\ \forall\, Z \in \Delta \cap \operatorname{Adm}_\rho(\gamma)\colon\ \operatorname{Ind}_\rho(\gamma, Z) f(Z) = \frac{1}{2\pi i} \oint_\gamma \frac{\rho(\mathrm{d}W)}{\rho(W - Z)} f(W). \tag{2.3.4}$$

(3) (Nil)Radical Cauchy Integral Formula over $\mathcal{A}$: let

$$V := \prod_{j=1}^{n} \begin{cases} \mathbb{D}_{r_j}(0) \times \mathfrak{m}_j \text{ for some } r_j > 0, \text{ if } j \in I_\rho \\ V_j \subseteq \mathcal{A}_j \text{ arbitrary open, otherwise} \end{cases}$$

be a polycylindrical (with respect to ρ) neighbourhood of $\operatorname{nil}_\rho(\mathcal{A})$ and let $f \in \mathcal{O}_\rho(V)$. If $\gamma \in \mathcal{L}V$ is a simple loop with respect to 0 (hence with respect to $\operatorname{nil}_\rho(\mathcal{A})$), then

$$\forall\, X \in \operatorname{nil}_\rho(\mathcal{A})\colon\ f(X) = \frac{1}{2\pi i} \oint_\gamma \frac{\rho(\mathrm{d}Z)}{\rho(Z - X)} f(Z). \tag{2.3.5}$$

Proof: To (1): As already alluded, this will follow from Proposition 2.3.1 essentially for topological reasons. More precisely, note first that by definition $\widetilde{\{Z\}} \subseteq [Z]_\rho$ and

$$\widetilde{U} \cap [Z]_\rho^\times \overset{\text{def}}{=} \prod_{j=1}^{n} \begin{cases} U_j \cap [Z_j]^\times \overset{\text{def}}{=} U_j \cap (Z_j + \mathcal{A}_j^\times), \text{ if } j \in I_\rho \\ \mathcal{A}_j, \text{ otherwise.} \end{cases}$$

Following the notations in the proof of Proposition 2.3.1, we can assume as before without loss of generality that $Z = 0$ and accordingly put $\widetilde{U}_\rho^\times := \widetilde{U} \cap \mathcal{A}_\rho^\times$. Since γ_j is trivially null-homotopic in $\mathcal{A}_j$, $j \in [n] \setminus I_\rho$, the condition $\gamma \simeq_{\widetilde{U}_\rho^\times} Z'$ for some $Z' \in \widetilde{\{0\}}$ just means that $\forall\, j \in I_\rho\colon \gamma_j \simeq_{U_j^\times} 0_{\mathcal{A}_j}$, where $U_j^\times := U_j \cap \mathcal{A}_j^\times$. As in Proposition 1.2.12 write $\bar{\mathcal{A}} := \prod_{k=1}^m (\mathcal{A}_{j(k)}, \mathfrak{m}_{j(k)}, \mathbb{C})$ for the codomain of Π, $\bar{U} := \prod_{k=1}^m U_{j(k)} \subseteq \bar{\mathcal{A}}$, $\bar{U}^\times := \bar{U} \cap \bar{\mathcal{A}}^\times$, and $\bar{\gamma} := \Pi_* \gamma \in \mathcal{L}\bar{U}^\times$ (as $\gamma \in \mathcal{L}U_\rho^\times$ by assumption). We have that $\rho = \bar{\rho} \circ \Pi$ with $\bar{\rho}\colon \bar{\mathcal{A}} \to \prod_{k=1}^m \operatorname{End}_{\mathbb{C}}(M_k)$, $f = \bar{f} \circ \Pi$ with $\bar{f} \in \mathcal{O}_{\bar{\rho}}(\bar{U})$, and $\oint_\gamma \omega = \oint_{\bar{\gamma}} \bar{\omega}$, where

$$\bar{\omega} := \frac{\bar{\rho}(\mathrm{d}\bar{Z})}{\bar{\rho}(\bar{Z})} \bar{f}(\bar{Z}) \in \Omega^1_{\mathrm{hol}}(\bar{U}^\times, M)$$

is d-closed by Proposition 1.4.8. On the other hand, we now have $\bar{\gamma} \simeq_{\bar{U}^\times} 0_{\bar{\mathcal{A}}}$, hence by Lemma 2.3.3 there exists $\bar{\delta} \in \mathcal{L}\bar{\Delta}^\times$ such that $\bar{\gamma} \simeq \bar{\delta}$ in $\mathcal{L}\bar{U}^\times$, where as usual by now $\bar{\Delta} := \bar{\Delta}_r(0) \subseteq \bar{\mathcal{A}}$ denotes a polydisk in $\bar{\mathcal{A}}$ around $0_{\bar{\mathcal{A}}}$ for some sufficiently small $r > 0$ and $\bar{\Delta}^\times := \bar{\Delta} \cap \bar{\mathcal{A}}^\times$. In particular, $\bar{\delta}$ is admissible for $0_{\bar{\mathcal{A}}}$. Thus, $\oint_\gamma \omega = \oint_{\bar{\gamma}} \bar{\omega} = \oint_{\bar{\delta}} \bar{\omega}$ and $\operatorname{Ind}_\rho(\gamma, 0_{\mathcal{A}}) = \operatorname{Ind}_{\bar{\rho}}(\bar{\gamma}, 0_{\bar{\mathcal{A}}}) = \operatorname{Ind}_{\bar{\rho}}(\bar{\delta}, 0_{\bar{\mathcal{A}}})$, and the claim now follows from Proposition 2.3.1 applied to the triple $(\bar{\rho}, \bar{f}, \bar{\delta})$.

To (2): For notational simplicity we can again assume without loss of generality that $\Delta = \Delta_r(0)$ for some $r > 0$. Fix $\gamma \in \mathcal{L}\Delta$ and recall that $\Delta \cap \mathrm{Adm}_\rho(\gamma) \neq \varnothing$ (and open) since $\mathrm{Adm}_\rho(\gamma)$ is open dense in $\mathcal{A}$. Now fix $Z \in \Delta \cap \mathrm{Adm}_\rho(\gamma)$ arbitrary. Note first that by definition $\gamma \in \mathcal{L}(\Delta \cap [Z]_\rho^\times)$. We are going to show that $\gamma \simeq_{\Delta\cap[Z]_\rho^\times} Z$ and invoke (1). To this end write $\Delta_j := \mathrm{pr}_j(\Delta) \subseteq \mathcal{A}_j$, i.e. $\Delta_j \overset{\text{def}}{=} \mathbb{D}_r(0) \times \mathbb{D}_r(0)^{d_j}$, where $d_j := \dim_\mathbb{C} \mathfrak{m}_j$, $1 \leq j \leq n$, and note that $\Delta \overset{\text{def}}{=} \prod_{j=1}^n \Delta_j$. For $j \in I_\rho$ consider $\Delta_j \setminus [Z_j] = (\mathbb{D}_r(0) \setminus \{z_j\}) \times \mathbb{D}_r(0)^{d_j}$. We thus have $\pi_0(\Delta_j \setminus [Z_j]) = 1$ and $[S^1, \Delta_j \setminus [Z_j]] \cong [S^1, \mathbb{D}_r(0) \setminus \{z_j\}] \cong \mathbb{Z}$, where explicitly we have a *smooth* homotopy $\gamma_j \simeq (\iota_{\mathcal{A}_j})_\# \gamma_j^{\mathrm{sp}} \simeq (\iota_{\mathcal{A}_j})_\# \delta_j^{\mathrm{sp}}$ for $\delta_j^{\mathrm{sp}} \in \mathcal{L}(\mathbb{D}_r(0) \setminus \{z_j\})$ of the form $\delta_j^{\mathrm{sp}}(t) := z_j + \varepsilon_j e^{2\pi i \mu_j t}$, $t \in [0,1]$, for some (any) $0 < \varepsilon_j \leq r - |z_j|$ and some winding number $\mu_j \in \mathbb{Z}$. If $j \in [n] \setminus I_\rho$, we have some *smooth* null-homotopy $\gamma_j \simeq Z_j$ in Δ_j as $\pi_0(\Delta_j) = \pi_1(\Delta_j) = 1$. Now define $\delta(t) := (\delta_1(t), \ldots, \delta_n(t))$, $t \in [0,1]$, where

$$\delta_j := \begin{cases} (\iota_{\mathcal{A}_j})_\#(\delta_j^{\mathrm{sp}}), \text{ if } j \in I_\rho \\ Z_j, \text{ otherwise.} \end{cases}$$

By construction, $\gamma \simeq \delta$ in $\mathcal{L}(\Delta \cap [Z]_\rho^\times)$, and we clearly have $\delta \simeq_{\Delta\cap[Z]_\rho^\times} Z$, hence $\gamma \simeq_{\Delta\cap[Z]_\rho^\times} Z$. The claim now follows from (1).

To (3): We have the equivalences: $\gamma \in \mathcal{L}V$ is simple with respect to $0 \overset{\text{def}}{\Leftrightarrow} \mathrm{Ind}_\rho(\gamma, 0) = \mathrm{Ind}_\rho(\gamma, [0]_\rho) \overset{\text{def}}{=} \mathrm{Ind}_\rho(\gamma, \mathrm{nil}_\rho(\mathcal{A})) = 1 \in \mathrm{End}_\mathbb{C}(M) \Leftrightarrow \forall\, j \in I_\rho\colon \mathrm{ind}(\gamma_j^{\mathrm{sp}}, 0) = 1$. Explicitly, for $j \in I_\rho$ we have $\gamma_j \simeq (\iota_{\mathcal{A}_j})_\# \gamma_j^{\mathrm{sp}} \simeq X_j + \varepsilon e^{2\pi i t}$, $t \in [0,1]$, in $\mathcal{L}(\mathbb{D}_{r_j}^*(0) \times \mathfrak{m}_j)$ for some (any) $0 < \varepsilon < \min\{r_j : j \in I_\rho\}$. Thus, $\gamma \simeq_{\widetilde{V}\cap\mathcal{A}_\rho^\times} X$ similarly as before, and the claim follows from (1). □

Remarks:

(1) Observe in Proposition 2.3.4 (1) that even though only $f \in \mathcal{O}_\rho(U)$, it suffices that $\gamma \simeq_{\widetilde{U}\cap[Z]_\rho^\times} Z'$ for $Z' \in \widetilde{\{Z\}}$, which is a much weaker hypothesis than just $\gamma \simeq_{U\cap[Z]_\rho^\times} Z$. Of course, this traces back to the fact that $\mathcal{O}_\rho(U) \cong \mathcal{O}_\rho(\widetilde{U})$.

(2) In the same vein, the condition $\simeq_{\widetilde{U}}$ simply expresses that one only needs to check what happens for $j \in I_\rho$ since $j \in [n] \setminus I_\rho$ have no influence.

(3) The *point* of a formulation like Proposition 2.3.4 (1) is that it shifts the topological burden from the domain U to the integration contour γ and its relative "topological position" with respect to the point $Z \in U$. □

Theorem 2.3.5 (Automatic Analytic Continuation to Spectral Polycylinders)**:**

(1) Local Automatic Analytic Continuation: if $\Delta \subseteq \mathcal{A}$ is a polydisc and $f \in \mathcal{O}_\rho(\Delta)$, then f admits a unique ρ-holomorphic continuation to $\widehat{\Delta}$ given by:

$$\forall\, \gamma \in \mathcal{L}\Delta \;\forall\, Z \in \Delta \cap \mathfrak{C}_\rho(\gamma) \;\forall\, X \in \mathrm{nil}_\rho(\mathcal{A})\colon$$
$$\widehat{f}(Z + X) := \frac{1}{2\pi i \,\mathrm{Ind}_\rho(\gamma, Z)} \oint_\gamma \frac{\rho(\mathrm{d}W)}{\rho(W - Z - X)} f(W). \tag{2.3.6}$$

(2) Global Automatic Analytic Continuation: we have a natural isomorphism of $\mathcal{A}$-modules

$$\begin{aligned} \mathcal{O}_\rho(U) &\cong \mathcal{O}_\rho(\widehat{U}) \\ f &\mapsto \widehat{f}, \end{aligned} \tag{2.3.7}$$

whose inverse is given by restriction $g \mapsto g|_U$. For a morphism $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ in $\mathsf{FdCAlg}_\mathbb{C}$, the isomorphism $\mathcal{O}_\varphi(U) \cong \mathcal{O}_\varphi(\widehat{U})$ is one of $\mathcal{B}$-algebras instead.

Proof: To (1): To simplify notations we can again assume without loss of generality that $\Delta = \Delta_r(0)$ for some $r > 0$. Fix $\gamma \in \mathcal{L}\Delta$ and recall that by definition $\mathfrak{C}_\rho(\gamma) \subseteq \mathrm{Adm}_\rho(\gamma)$ and $\forall\, Z \in$

$\mathfrak{C}_\rho(\gamma)\colon\ \mathrm{Ind}_\rho(\gamma, Z) \in \mathrm{GL}(M)$. First, we need to show that $\Delta \cap \mathfrak{C}_\rho(\gamma) \neq \varnothing$ (it's clearly open). Indeed, since $\gamma \subset \Delta$, we have that $\gamma_j^{\mathrm{sp}} \subset \chi_j(\Delta) = \mathbb{D}_r(0)$ and $C_j(\gamma) := \{z_j \in \mathbb{C}\backslash\gamma_j^{\mathrm{sp}} : \mathrm{ind}(\gamma_j^{\mathrm{sp}}, z_j) \neq 0\} \subset \mathbb{D}_r(0)$ is the *bounded component*[4] of γ_j^{sp}, $j \in I_\rho$, hence explicitly

$$\Delta \cap \mathfrak{C}_\rho(\gamma) = \prod_{j=1}^{n} \begin{cases} C_j(\gamma) \times \mathbb{D}_r(0)^{d_j}, \text{ if } j \in I_\rho \\ \mathbb{D}_r(0)^{d_j+1}, \text{ otherwise,} \end{cases}$$

where as before $d_j := \dim_{\mathbb{C}} \mathfrak{m}_j \geq 0$, $1 \leq j \leq n$. Next notice that for $X = 0$ the RHS agrees with f on $\Delta \cap \mathfrak{C}_\rho(\gamma) \subseteq \Delta \cap \mathrm{Adm}_\rho(\gamma)$ by Eq. (2.3.4). Furthermore, since $X \in \mathrm{nil}_\rho(\mathcal{A})$ and $Z \in \mathrm{Adm}_\rho(\gamma)$, we have $Z + X \in \mathrm{Adm}_\rho(\gamma)$ and $\mathrm{Ind}_\rho(\gamma, Z + X) = \mathrm{Ind}_\rho(\gamma, Z) \neq 0$, so the RHS remains well-defined. More precisely, if $Z_1 + X_1 = Z_2 + X_2$ for some $Z_1, Z_2 \in \Delta$ and $X_1, X_2 \in \mathrm{nil}_\rho(\mathcal{A})$, i.e. $Z_2 = Z_1 + X$ for $X := X_1 - X_2 \in \mathrm{nil}_\rho(\mathcal{A})$, then $Z_1 \in \mathrm{Adm}_\rho(\gamma) \Leftrightarrow Z_2 \in \mathrm{Adm}_\rho(\gamma)$ and $\mathrm{Ind}_\rho(\gamma, Z_1 + X_1) = \mathrm{Ind}_\rho(\gamma, Z_1) = \mathrm{Ind}_\rho(\gamma, Z_2 - X) = \mathrm{Ind}_\rho(\gamma, Z_2) = \mathrm{Ind}_\rho(\gamma, Z_2 + X_2)$, hence

$$\forall\, \Xi \in \Delta \cap \mathfrak{C}_\rho(\gamma) + \mathrm{nil}_\rho(\mathcal{A})\colon\ \widehat{f}(\Xi) \overset{\text{def}}{=} \frac{1}{2\pi i\, \mathrm{Ind}_\rho(\gamma, \Xi)} \oint_\gamma \frac{\rho(\mathrm{d}W)}{\rho(W - \Xi)} f(W).$$

is itself is well-defined (also observe that $\Delta \cap \mathfrak{C}_\rho(\gamma) + \mathrm{nil}_\rho(\mathcal{A})$ is open). It follows from Proposition 1.1.19 that the so defined function is ρ-holomorphic since $\mathrm{Ind}_\rho(\gamma, Z)$ is locally constant by Lemma 2.2.8 and one can φ_ρ-differentiate under the integral. Since $\Delta \cap \mathfrak{C}_\rho(\gamma) \neq \varnothing$ and each of the connected components of $\mathfrak{C}_\rho(\gamma)$ clearly contains a corresponding connected (open) component of $\Delta \cap \mathfrak{C}_\rho(\gamma)$, it follows now that Eq. (2.3.6) defines a unique ρ-holomorphic continuation of f on each connected component of $\mathfrak{C}_\rho(\gamma)$.

Next we need to show that this procedure is independent of $\gamma \in \mathcal{L}\Delta$. Let $\delta \in \mathcal{L}\Delta$ be another loop and consider $\mathfrak{C}_\rho(\delta)$. If $\mathfrak{C}_\rho(\gamma) \cap \mathfrak{C}_\rho(\delta) = \varnothing$, there is nothing to show. Suppose now that $\mathfrak{C}_\rho(\gamma) \cap \mathfrak{C}_\rho(\delta) \neq \varnothing$. It follows that $\Delta \cap \mathfrak{C}_\rho(\gamma) \cap \mathfrak{C}_\rho(\delta) \neq \varnothing$ and again each connected component of $\mathfrak{C}_\rho(\gamma) \cap \mathfrak{C}_\rho(\delta)$ contains a connected (open) component of $\Delta \cap \mathfrak{C}_\rho(\gamma) \cap \mathfrak{C}_\rho(\delta)$. Thus, if f_γ and f_δ denote the corresponding analytic continuations of f to $\mathfrak{C}_\rho(\gamma)$ and $\mathfrak{C}_\rho(\delta)$, respectively, then by the identity principle f_γ and f_δ agree on each connected component of $\mathfrak{C}_\rho(\gamma) \cap \mathfrak{C}_\rho(\delta)$ as they do so inside Δ. Finally, each $Z \in \Delta$ admits a "local" $\gamma \in \mathcal{L}\Delta$ with $\gamma_j(t) = Z_j + \varepsilon e^{2\pi i t}$, $t \in [0,1]$, for $j \in I_\rho$ and some (any) $0 < \varepsilon < r - \max\{|z_j| : j \in I_\rho\}$, so Δ can be "exhausted" by such $\mathfrak{C}_\rho(\gamma)$, that is, $\bigcup_{\gamma \in \mathcal{L}\Delta} \mathfrak{C}_\rho(\gamma) = \widehat{\Delta}$.

To (2): More generally, the last arguments show that $\widehat{U} = \bigcup_{\gamma \in \mathcal{L}U} \mathfrak{C}_\rho(\gamma)$ and that $f \in \mathcal{O}_\rho(U)$ admits a unique ρ-holomorphic continuation $\widehat{f} \in \mathcal{O}_\rho(\widehat{U})$. Thus we get an isomorphism $\mathcal{O}_\rho(U) \cong \mathcal{O}_\rho(\widehat{U})$, $f \mapsto \widehat{f}$, of $\mathcal{A}$-modules (or $\mathcal{B}$-algebras) with inverse $g \mapsto g|_U$. $\square$

Remark: It follows from Lemma 0.1.46 that $(\widehat{U})^\rho = \widehat{\widehat{U}} = \widehat{U}$, hence applying Theorem 1.2.13 does not extend f further. It follows from Corollary 2.3.9 that $\widehat{U}$ are in fact *domains of ρ-holomorphy*. Likewise for a morphism φ. $\square$

Theorem 2.3.6 (Spectrally Null-Homological Generalized Cauchy Integral Formula)**:** Let $U \subseteq \mathcal{A}$ be open with $\pi_0(U) = 1$, $f \in \mathcal{O}_\rho(U)$, and $\Gamma \in Z_1(U, \mathbb{Z})$. If $[\Gamma] = 0$ in $H_1^{\mathrm{sp}}(U, \mathbb{Z})$, then

$$\forall\, Z \in U \cap \mathrm{Adm}_\rho(\Gamma)\colon\ \mathrm{Ind}_\rho(\Gamma, Z) f(Z) = \frac{1}{2\pi i} \int_\Gamma \frac{\rho(\mathrm{d}W)}{\rho(W - Z)} f(W). \tag{2.3.8}$$

In particular,

$$\int_\Gamma \rho(\mathrm{d}W) f(W) = 0. \tag{2.3.9}$$

Proof: As before, we can assume without loss of generality that $Z = 0$ and accordingly $\Gamma \in Z_1(U_\rho^\times, \mathbb{Z})$.

[4] itself possibly consisting of several connected components for each non-zero integer value of ind;

As in the proof of Proposition 2.3.1, we have by Proposition 1.2.12 that $\rho = \bar\rho \circ \Pi$ with $\bar\rho = \times_{k=1}^m \bar\rho_k$ and $f = \bar f \circ \Pi$ with $\bar f = \times_{k=1}^m \bar f_k$, where $\bar\rho_k\colon (\mathcal{A}_{j(k)}, \mathfrak{m}_{j(k)}, \mathbb{C}) \to \mathrm{End}_{\mathbb{C}}(M_k)$ and $\bar f_k \in \mathcal{O}_{\bar\rho_k}(U_{j(k)})$, $1 \le k \le m$. Write $\bar\Gamma \coloneqq \Pi_{\#}\Gamma \in Z_1(\bar U, \mathbb{Z})$, where $\bar U \coloneqq \prod_{k=1}^m U_{j(k)} \subseteq \bar{\mathcal{A}} \coloneqq \prod_{k=1}^m \mathcal{A}_{j(k)}$. Define

$$\bar\omega_k \coloneqq \bar\rho_k\Big(\frac{\mathrm{d}Z_k}{Z_k}\Big)\bar f_k(Z_k) \in \Omega^1_{\mathrm{hol}}(U^\times_{j(k)}, M_k) \text{ and } \omega_k^{\mathrm{sp}} \coloneqq \frac{\bar f_k(z_k)}{z_k}\mathrm{d}z_k \in \Omega^1_{\mathrm{hol}}(U^{\mathrm{sp}}_{j(k)} \setminus \{0\}, M_k)$$

where $U^\times_{j(k)} \overset{\mathrm{def}}{=} U_{j(k)} \cap (\mathcal{A}_{j(k)})^\times_{\bar\rho_k} \overset{\mathrm{def}}{=} U_{j(k)} \cap \mathcal{A}^\times_{j(k)}$ and $\pi_0(U^{\mathrm{sp}}_{j(k)}) = 1$, $1 \le k \le m$. By Theorem 2.3.5 we can replace $\bar f_k \in \mathcal{O}_{\bar\rho_k}(U_{j(k)})$ by $\bar f_k \in \mathcal{O}_{\bar\rho_k}(\widehat U_{j(k)})$ (slight abuse of notation), where recall $\widehat U_{j(k)} \overset{\mathrm{def}}{=} U^{\mathrm{sp}}_{j(k)} \times \mathfrak{m}_{j(k)}$ in $\mathcal{A}_{j(k)}$, hence actually $\bar\omega_k \in \Omega^1_{\mathrm{hol}}(\widehat U^\times_{j(k)}, M_k)$, $1 \le k \le m$. We now have

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma \omega &= \frac{1}{2\pi i}\int_{\bar\Gamma}\bar\omega = \frac{1}{2\pi i}\oplus_{k=1}^m \int_{\Gamma_{j(k)}}\bar\omega_k = \frac{1}{2\pi i}\oplus_{k=1}^m \int_{(\iota_{j(k)})_{\#}\Gamma^{\mathrm{sp}}_{j(k)}}\bar\omega_k = \oplus_{k=1}^m \frac{1}{2\pi i}\int_{\Gamma^{\mathrm{sp}}_{j(k)}}\omega_k^{\mathrm{sp}} = \\ &= \oplus_{k=1}^m \mathrm{ind}(\Gamma^{\mathrm{sp}}_{j(k)}, 0)\bar f_k(0) = (\times_{k=1}^m \mathrm{ind}(\Gamma^{\mathrm{sp}}_{j(k)}, 0)) \cdot \oplus_{k=1}^m \bar f_k(0) \overset{(2.2.12)}{=} \mathrm{Ind}_\rho(\Gamma, 0)\cdot \bar f(0) = \\ &= \mathrm{Ind}_\rho(\Gamma, 0)\cdot f(0)\end{aligned}$$

by applying the classical Homological Cauchy Integral Formula in $\mathbb{C}$ since $\forall\, 1 \le k \le m$:

- $\Gamma_{j(k)} \simeq (\iota_{j(k)})_{\#}\Gamma^{\mathrm{sp}}_{j(k)}$ in $\widehat U^\times_{j(k)} \overset{\mathrm{def}}{=} (U^{\mathrm{sp}}_{j(k)} \setminus \{0\}) \times \mathfrak{m}_{j(k)}$;
- $\Gamma^{\mathrm{sp}}_{j(k)} \in B_1(U^{\mathrm{sp}}_{j(k)}, \mathbb{Z})$ by hypothesis;
- $\bar\omega_k$ is d-closed by Proposition 1.4.8.

Finally, Eq. (2.3.9) follows by applying Eq. (2.3.8) to $\rho(W - Z)f(W)$. □

Remarks:

(1) We remark that Theorem 2.3.6 (and its corollaries) are the only generalized version of Cauchy's Integral Formula which we have proved in the present work by reduction to its counter-part in the complex plane. One reason is of course to give a clean demonstration of the role of the spectral POV and its importance in the *function theory* of $\mathcal{A}$-holomorphic functions. Another reason is that a completely "intrinsic" approach would have been unjustifiably long (for instance, recall Artin's theorem and its proof on the correspondence between homology and the analytic index in the complex plane). Having said that, we would like to point out that Dixon's proof[5] can be adapted to the $\mathcal{A}$-setting by introducing yet another generalized limit derivative

$$f'_{(W_0)}(Z_0) \coloneqq \lim_{\substack{H \to W_0 \\ H \in \mathcal{A}^\times_\rho \cap (U - Z_0)}} \rho(H)^{-1}(f(Z_0 + H) - f(Z_0)).$$

for $W_0 \in \mathrm{nil}_\rho(\mathcal{A})$ or even $W_0 \in \mathrm{Sing}_\rho(\mathcal{A})$. This is of course still assuming that one has established the relation between 1-homology and the index.

(2) A slight variation of the above proof would be to only demonstrate the local case $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C})$. Then the external product case is almost automatic, and one finishes off the proof by pulling back by Π. A self-contained exposition for the local case with ${}_{\mathcal{A}}M \coloneqq \mathcal{A}$ is given in our recent paper [Gen26].

(3) A similar result has been obtained by G.B. Rizza in [Riz52a], followed by [Riz52b] (see also the overview in [Riz54a]). Namely, in our language his main statement reads as follows: if $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C})$, $U \subseteq \mathcal{A}$ is a domain, $f \in \mathcal{O}_{\mathcal{A}}(U)$, and $\Gamma \in Z_1(U, \mathbb{Z})$ with $[\Gamma] = 0$ in $H_1(U, \mathbb{Z})$, then

$$\exists\, N \in \mathbb{N}\ \forall\, Z \in U \cap \mathrm{Adm}(\Gamma)\colon f(Z)2\pi i N = \int_\Gamma \frac{f(W)}{W - Z}\mathrm{d}W.$$

In comparison, we only require $[\Gamma^{\mathrm{sp}}] = 0$ in $H_1(U^{\mathrm{sp}}, \mathbb{Z})$. This is a substantially weaker topolog-

[5] see for instance [Rud87, p. 217–220];

ical requirement on Γ since generically null-homology of a cycle cannot be deduced merely from null-homology of (some or even all of) its coordinate projections, even though the converse implication is clearly true.

(4) Having Theorem 2.3.6 at our disposal it is immediate now how to derive a Laurent Series Expansion and Residue Theorems over $\mathcal{A}$ under our topologically weak "spectral" assumptions. In particular, this gives stronger forms of both theorems than the ones stated in [Ket28a] in the case ${}_{\mathcal{A}}M = \mathcal{A}$. The Laurent Series Expansion and Residue Theorems (and related results) under the topological hypothesis of [Riz52a; Riz52b] are stated for the case ${}_{\mathcal{A}}M = \mathcal{A}$ in [Riz54b]. Under our topological hypothesis, we will, however, state them elsewhere as they belong more organically to the $\mathcal{A}$-meromorphic theory, which is a very long topic on its own.

(5) Operator Calculus Interpretation: a byproduct of Theorem 2.3.6 and its proof is *a holomorphic $\mathcal{A}$-operator calculus* for *$\mathcal{A}$-valued* (or even more generally $\mathcal{A}$-module-valued) holomorphic functions:

$$f(Z)\,\mathrm{ind}(\Gamma^{\mathrm{sp}}, z) = \frac{1}{2\pi i}\int_{\Gamma^{\mathrm{sp}}} \frac{f(w)}{w - Z}\mathrm{d}w, \tag{2.3.10}$$

where $Z \in \mathcal{A}$ and $z \coloneqq \chi_{\mathcal{A}}(Z)$. This is a very natural idea in the infinite-dimensional setting as well, but as far as we know no (systematic) investigation of it exists.

Corollary 2.3.7 (Spectrally Null-Homotopical Cauchy Integral Formula)**:** Let $U \subseteq \mathcal{A}$ be open with $\pi_0(U) = 1$, $f \in \mathcal{O}_\rho(U)$, $\gamma \in \mathcal{L}U$ such that $[\gamma] = 1$ in $\pi_1^{\mathrm{sp}}(U)$. Then:

$$\forall\, Z \in U \cap \mathrm{Adm}_\rho(\gamma)\colon\ \mathrm{Ind}_\rho(\gamma, Z) f(Z) = \frac{1}{2\pi i}\oint_\gamma \frac{\rho(\mathrm{d}W)}{\rho(W - Z)} f(W). \tag{2.3.11}$$

Proof: This is a special case of Theorem 2.3.6: it follows from the definitions that if γ is spectrally null-homotopic in $U \subseteq \mathcal{A}$, then it is also spectrally null-homologous in U. $\square$

Proposition 2.3.8 (Spectrally Null-Homological Cauchy Differentiation Formula over $\mathcal{A}$)**:** Let $U \subseteq \mathcal{A}$ be open with $\pi_0(U) = 1$, $f \in \mathcal{O}_\rho(U)$, and $\Gamma \in Z_1(U, \mathbb{Z})$.

(1) We have $\mathcal{O}_\rho(U) = \mathscr{C}^\infty_\rho(U)$.

(2) If $[\Gamma] = 0$ in $H_1^{\mathrm{sp}}(U, \mathbb{Z})$, then:

$$\forall\, k \in \mathbb{N}_0\ \forall\, Z \in U \cap \mathrm{Adm}_\rho(\Gamma)\colon\ \mathrm{Ind}_\rho(\Gamma, Z) f^{(k)}(Z) = \frac{k!}{2\pi i}\int_\Gamma \frac{\rho(\mathrm{d}W)}{\rho(W - Z)^{k+1}} f(W). \tag{2.3.12}$$

Proof: To (1): This follows from Eq. (2.3.4). More precisely, let $Z \in U$ be an arbitrary point. Then Z admits a small local loop $\gamma \in \mathcal{L}U$ such that $\forall\, j \in I_\rho\colon \gamma_j(t) = Z_j + \varepsilon e^{2\pi i t}$, $t \in [0, 1]$, for some sufficiently small $\varepsilon > 0$. If $0 < r < \varepsilon$, then $\Delta_r(Z) \subseteq \mathrm{Adm}_\rho(\gamma)$ and $\forall\, W \in \Delta_r(Z)\colon \mathrm{Ind}_\rho(\gamma, W) = \mathrm{id}_M$. It follows that

$$\forall\, W \in \Delta_r(Z)\colon f(W) = \frac{1}{2\pi i}\oint_\gamma \frac{\rho(\mathrm{d}\Xi)}{\rho(\Xi - W)} f(\Xi),$$

hence $f \in \mathscr{C}^\infty_\rho(\Delta_r(Z))$ by φ_ρ-differentiability of the reproducing kernel.

To (2): Put $\mathcal{I}(Z) \coloneqq \mathrm{Ind}_\rho(\Gamma, Z)$ and recall that $U \cap \mathrm{Adm}_\rho(\Gamma) \neq \varnothing$ is open. By Lemma 2.2.8, the map $\mathcal{I}\colon U \cap \mathrm{Adm}_\rho(\Gamma) \to \mathrm{End}_{\mathbb{C}}(M)$ is locally constant, and by Proposition 2.3.8 we have

$$\forall\, Z \in U \cap \mathrm{Adm}_\rho(\Gamma)\colon \mathcal{I}(Z) f(Z) = \frac{1}{2\pi i}\int_\Gamma \frac{\rho(\mathrm{d}W)}{\rho(W - Z)} f(W).$$

Since now $f \in \mathscr{C}^\infty_\rho(U)$ by (1) and $\mathcal{I}$ is in particular φ_ρ-differentiable, φ_ρ-differentiating both sides and using the *mixed* product rule from Proposition 1.1.19 gives:

$$\mathcal{I}'(Z) f(Z) + \mathcal{I}(Z) f'(Z) = \mathcal{I}(Z) f'(Z) = \frac{1}{2\pi i}\int_\Gamma \frac{\rho(\mathrm{d}W)}{\rho(W - Z)^2} f(W).$$

The formula for the higher derivatives $f^{(k)}(Z)$, $k \in \mathbb{N}$, follows in the same way by induction. $\square$

Corollary 2.3.9 (Canonical Form)**:** Let $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C})$, $U \subseteq \mathcal{A}$ be open, and $f \in \mathcal{O}_\rho(U)$. Then:

(1) Nilpotent Expansion:

$$\forall\, Z \in \widehat{U}\ \forall\, X \in \mathfrak{m}\colon\ \widehat{f}(Z+X) = \sum_{k=0}^{\nu-1} \frac{\rho(X)^k}{k!} f^{(k)}(Z) \tag{2.3.13}$$

(2) Canonical Form (separation of spectral and nilpotent part):

$$\forall\, z \in U^{\mathrm{sp}}\ \forall\, X \in \mathfrak{m}\colon\ f(Z) \overset{\mathrm{def}}{=} f(z \oplus X) = \sum_{k=0}^{\nu-1} \frac{\rho(X)^k}{k!} f^{(k)}(z), \tag{2.3.14}$$

where observe that $f^{(k)} \circ \iota_{\mathcal{A}} \in \mathcal{O}_{\mathbb{C}}(U^{\mathrm{sp}}, M)$.

Proof: To (1): By Theorem 2.3.5, $f \in \mathcal{O}_\rho(U)$ admits a unique ρ-holomorphic continuation $\widehat{f}$ to $\widehat{U} \overset{\mathrm{def}}{=} U^{\mathrm{sp}} \times \mathfrak{m}$ and by Proposition 2.3.8 its higher ρ-derivatives exist. More precisely, if $\gamma \in \mathcal{L}\widehat{U}$ is a local simple loop for a sufficiently small polydisc $\Delta \subset \widehat{U} \overset{\mathrm{def}}{=} U^{\mathrm{sp}} \times \mathfrak{m}$, then using the geometric series we obtain $\forall\, Z \in \Delta\ \forall\, X \in \mathfrak{m}$:

$$\widehat{f}_\gamma(Z+X) \overset{(2.3.6)}{=} \frac{1}{2\pi i} \oint_\gamma \rho\left(\frac{\mathrm{d}W}{W-Z-X}\right) f(W) = \sum_{k=0}^{\nu-1} \rho(X)^k \frac{1}{2\pi i} \oint_\gamma \rho\left(\frac{\mathrm{d}W}{(W-Z)^{k+1}}\right) f(W) \overset{(2.3.12)}{=}$$

$$\overset{(2.3.12)}{=} \sum_{k=0}^{\nu-1} \rho(X)^k \frac{f^{(k)}(Z)}{k!},$$

where the RHS is clearly independent of γ and globally defined for $(Z, X) \in \widehat{U} \times \mathfrak{m}$.

To (2): This is a special case of (1) since by definition $U^{\mathrm{sp}} \subset \widehat{U}$ via $\iota_{\mathcal{A}}\colon \mathbb{C} \hookrightarrow \mathcal{A}$. $\square$

Corollary 2.3.10 (Cauchy's Inequality over $\mathcal{A}$)**:** Let $U \subseteq \mathcal{A}$ be open with $\pi_0(U) = 1$, $Z_0 \in U$, and $f \in \mathcal{O}_\rho(U)$.

(1) Global Cauchy Inequality over $\mathcal{A}$: if $\Gamma \in B_1^{\mathrm{sp}}(U, \mathbb{Z})$ such that $\mathfrak{C}_\rho(\Gamma) \ni Z_0$, then

$$\forall\, k \in \mathbb{N}_0\colon\ \left\| f^{(k)}(Z_0) \right\|_2 \leq \frac{k!}{2\pi} \left\| \frac{1}{\mathrm{Ind}_\rho(\Gamma, Z_0)} \right\|_{\mathrm{sp}} L_\rho(\Gamma) \left\| \frac{1}{\rho(Z-Z_0)^{k+1}} \right\|_{\mathrm{sp},\Gamma} \|f\|_{2,\Gamma}, \tag{2.3.15}$$

where we have equipped M with the Euclidean norm $\|\cdot\| \coloneqq \|\cdot\|_2$.

(2) Centered Cauchy Inequality over $\mathcal{A}$: if $\gamma_r(t) \coloneqq Z_0 + re^{2\pi i t}$, $t \in [0,1]$, for $r > 0$ sufficiently small, then:

$$\forall\, k \in \mathbb{N}_0\colon\ \left\| f^{(k)}(Z_0) \right\|_M \leq \frac{k!}{r^k} \|f\|_{M,\gamma_r}. \tag{2.3.16}$$

Proof: To (1): Write $\Gamma \coloneqq \sum_{\ell=1}^p m_\ell \gamma^\ell$ as before. Since $\mathfrak{C}_\rho(\Gamma) \ni Z_0$, using Eq. (2.3.12) and Section 0.3 we obtain:

$$\begin{aligned}
\left\| f^{(k)}(Z_0) \right\|_2 &\leq \frac{k!}{2\pi} \left\| \frac{1}{\mathrm{Ind}_\rho(\Gamma, Z_0)} \right\|_{\mathrm{sp}} \sum_{\ell=1}^p |m_\ell| \int_0^1 \left\| \frac{\rho(\dot{\gamma}^\ell(t))}{\rho(\gamma^\ell(t) - Z_0)^{k+1}} \right\|_{\mathrm{sp}} \left\| f(\gamma^\ell(t)) \right\|_2 \mathrm{d}t \leq \\
&\leq \frac{k!}{2\pi} \left\| \frac{1}{\mathrm{Ind}_\rho(\Gamma, Z_0)} \right\|_{\mathrm{sp}} \sum_{\ell=1}^p |m_\ell|\, L_\rho(\gamma^\ell) \left\| \frac{1}{\rho(Z-Z_0)^{k+1}} \right\|_{\mathrm{sp},\gamma^\ell} \|f\|_{M,\gamma^\ell} \leq \\
&\leq \frac{k!}{2\pi} \left\| \frac{1}{\mathrm{Ind}_\rho(\Gamma, Z_0)} \right\|_{\mathrm{sp}} \left(\sum_{\ell=1}^p |m_\ell|\, L_\rho(\gamma^\ell) \right) \max_{1 \leq \ell \leq p} \left\| \frac{1}{\rho(Z-Z_0)^{k+1}} \right\|_{\mathrm{sp},\gamma^\ell} \max_{1 \leq \ell \leq p} \|f\|_{2,\gamma^\ell} \overset{\mathrm{def}}{=} \\
&\overset{\mathrm{def}}{=} \frac{k!}{2\pi} \left\| \frac{1}{\mathrm{Ind}_\rho(\Gamma, Z_0)} \right\|_{\mathrm{sp}} L_\rho(\Gamma) \left\| \frac{1}{\rho(Z-Z_0)^{k+1}} \right\|_{\mathrm{sp},\Gamma} \|f\|_{2,\Gamma}.
\end{aligned}$$

To (2): This also follows from Eq. (2.3.12), by direct evaluation at the earliest stage:

$$\left\|f^{(k)}(Z_0)\right\|_M = \left\|\frac{k!}{2\pi i}\oint_{\gamma_r}\frac{\rho(\mathrm{d}Z)}{\rho(Z-Z_0)^{k+1}}f(Z)\right\|_M = \left\|\frac{k!}{2\pi i}\int_0^1\frac{\rho(2\pi i r e^{2\pi i t})}{\rho(re^{2\pi i t})^{k+1}}f(\gamma_r(t))\mathrm{d}t\right\|_M =$$
$$= \left\|\frac{k!}{r^k}\int_0^1\frac{f(\gamma_r(t))}{e^{2\pi i k t}}\mathrm{d}t\right\|_M \leq \frac{k!}{r^k}\left\|f\right\|_{M,\gamma_r}$$

as $\mathrm{Ind}_\rho(\gamma, Z_0) = \mathrm{id}_M$. □

Corollary 2.3.11 (Liouville over $\mathcal{A}$)**:** Let $U \subseteq \mathcal{A}$ be open with $\pi_0(U) = 1$ and let $f \in \mathcal{O}_\rho(U)$. Suppose that there exists a subset $V \subseteq U$ such that:

(i) $f|_V$ is bounded;

(ii) for every $Z_0 \in U$, V contains a collection of Z_0-admissible simple loops $\{\gamma_n\}_{n\in\mathbb{N}}$ such that

$$\left\{\sup_{t\in[0,1]}\left\|\frac{1}{\rho(\gamma_n(t))-\rho(Z_0)}\right\|_{\mathrm{sp}} : n\in\mathbb{N}\right\}$$

is unbounded with

$$L_\rho(\gamma_n)\sup_{t\in[0,1]}\left\|\frac{1}{\rho(\gamma_n(t))-\rho(Z_0)}\right\|_{\mathrm{sp}} \xrightarrow{n\to\infty} 0.$$

Then $f = \text{const}$. In particular, if $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C})$, $U \supseteq \mathbb{C}$, and $f\big|_{\mathbb{C}}$ is bounded, then $f = \text{const}$.

Proof: By Corollary 2.3.10 we have for $k = 1$

$$\left\|f'(Z_0)\right\|_2 \leq \frac{1}{2\pi}\left\|f\right\|_{2,\gamma_n} L_\rho(\gamma_n)\sup_{t\in[0,1]}\left\|\frac{1}{\rho(\gamma_n(t))-\rho(Z_0)}\right\|_{\mathrm{sp}} \to 0$$

as $n\to\infty$ since $\|f\|_{2,\gamma_n}$ is *bounded by assumption*. Thus f is locally constant and hence constant by path-connectedness of U. For the last statement it clearly suffices to take (scalar) loops in $\mathbb{C}$. □

Remark: The formulation of Liouville's theorem over $\mathcal{A}$ is possibly not the most elegant one, but at least it is very general.

Corollary 2.3.12 (Naive Scalar Maximum Modulus Principle over $\mathcal{A}$)**:** Let $U \subseteq \mathcal{A}$ be open, $f \in \mathcal{O}_\rho(U)$, $Z_0 \in U$, and let $\overline{\mathbb{D}}_r(Z_0) \subseteq U$ for some sufficiently small $r > 0$. Then $f|_{\overline{\mathbb{D}}_r(Z_0)}$ takes maximum norm on $\partial\overline{\mathbb{D}}_r(Z_0)$.

Proof: This follows directly from Eq. (2.3.16) with $k = 0$ since $\partial\overline{\mathbb{D}}_r(Z_0) = \gamma_r$. □

Corollary 2.3.13 (The Natural Bilinear Pairing)**:** Let $U \subseteq \mathcal{A}$ be open.

(1) Representation Setting: we get a well-defined $\mathbb{Z}$-$\mathcal{A}$-bilinear pairing

$$\begin{aligned}\beta\colon H_1^{\mathrm{sp}}(U,\mathbb{Z})\times\mathcal{O}_\rho(U) &\to {}_{\mathcal{A}}M\\ ([\Gamma], f) &\mapsto \int_\Gamma \rho(\mathrm{d}Z)f(Z).\end{aligned} \tag{2.3.17}$$

(2) Morphism Setting: We get a well-defined $\mathbb{Z}$-$\mathcal{B}$-bilinear pairing

$$\begin{aligned}\beta\colon H_1^{\mathrm{sp}}(U,\mathbb{Z})\times\mathcal{O}_\varphi(U) &\to \mathcal{B}\\ ([\Gamma], f) &\mapsto \int_\Gamma f(Z)\varphi(\mathrm{d}Z).\end{aligned} \tag{2.3.18}$$

□

Proposition 2.3.14 (Weierstraß Convergence Theorems over $\mathcal{A}$)**:** Let $U \subseteq \mathcal{A}$ be open with $\pi_0(U) = 1$ and $(f_n)_{n\in\mathbb{N}} \subset \mathcal{O}_\rho(U)$.

(1) If $f_n \to f$ *locally* uniformly[6], then $f \in \mathcal{O}_\rho(U)$ and $f_n' \to f'$ *locally* uniformly.
(2) If $\sum_{k=1}^n f_k \to f$ *locally* normally[7], then $f \in \mathcal{O}_\rho(U)$ and $\sum_{k=1}^n f_k' \to f'$ also *locally* normally.

Proof: To (1): follows completely analogously to the case of $\mathcal{A}$ being the complex plane itself.

To (2): Fix a "base point" $Z_0 \in U$ and $\gamma \in \mathscr{C}^1_{\mathrm{pw}}(\mathbb{S}^1, U)$ a Z_0-admissible loop with $\mathrm{Ind}_\rho(\gamma, Z_0) = \mathrm{id}_M$. By translation with $-Z_0$ we can assume without loss of generality that $Z_0 = 0$, which will allow us to pick γ to be a (sufficiently small) *scalar* curve. Note, however, that for an arbitrary Z_0 the "$\mathrm{Ind}_\rho(\gamma, Z_0) = \mathrm{id}_M$"-condition may prevent us in general from choosing γ to be a *scalar* curve as $\mathcal{A}$ need not be local. Thus take $\gamma(t) \coloneqq re^{2\pi i t}1_{\mathcal{A}}$, $t \in [0,1]$, for $r > 0$ sufficiently small, in particular $L_\rho(\gamma) = 2\pi r$, and fix $0 < \varepsilon < r$. As before, there exists a neighbourhood $V \ni Z_0 = 0$ of γ-admissible points such that $\forall\, Z \in V\colon\ \mathrm{Ind}_\rho(\gamma, Z) = \mathrm{id}_M$ and $\|\rho(Z)\|_{\mathrm{sp}} < \varepsilon$ by continuity of ρ. Using Lemma 0.3.18, we get

$$\forall\, Z \in V\colon\ \left\| \frac{1}{\rho(\gamma(t)) - \rho(Z)} \right\|_{\mathrm{sp}} \overset{\mathrm{def}}{=} \left\| \frac{1}{re^{2\pi i t}\big(\mathrm{id}_M - \frac{\rho(Z)}{re^{2\pi i t}}\big)} \right\|_{\mathrm{sp}} \le \frac{1}{r - \|\rho(Z)\|_{\mathrm{sp}}} < \frac{1}{r-\varepsilon}$$

Thus by Corollary 2.3.10 we obtain

$$\forall\, Z \in V\colon\ \big\| f_n'(Z) \big\|_2 \le \frac{L_\rho(\gamma)}{2\pi} \, \|f_n\|_{2,\gamma} \sup_{t\in[0,1]} \left\| \frac{1}{\rho(\gamma(t)) - \rho(Z)} \right\|_{\mathrm{sp}} \le \frac{r}{r-\varepsilon} \, \|f_n\|_{2,\gamma}\,,$$

which implies the claim. □

Definition 2.3.15 (Banach-space-valued measures)**:** Let (S, Σ) be a measurable space and $E \in \mathsf{Ban}_{\mathbb{K}}$. An E-valued measure on (S, Σ) is a map $\mu\colon \Sigma \to E$ such that
(i) $\mu(\varnothing) = 0$;
(ii) $\mu(\bigcup_{k\in\mathbb{N}} X_k) = \sum_{k\in\mathbb{N}} \mu(X_k)$ for any countable disjoint family $(X_k)_{k\in\mathbb{N}} \subseteq \Sigma$.

Remarks:
(1) The only difference to the usual notion of a measure is the lack of non-negativity, which per se does not make sense in an arbitrary Banach space E. Moreover, it is implied that the order of summation does not matter since the order of set-theoretic union does not.
(2) If $E = M \in \mathsf{FdMod}(\mathcal{A})$ and $\{v_1, \ldots, v_m\}$ is a $\mathbb{C}$-basis for M, then a measure $\mu\colon \Sigma \to M$ can be uniquely written as $\mu = \sum_{k=1}^m v_k \mu_k$, where $\mu_k\colon \Sigma \to \mathbb{C}$, $1 \le k \le m$, are $\mathbb{C}$-valued measures. □

We also have Cauchy's *producing kernel.* We only state here the version for $\mathsf{LocArt}_{\mathbb{C}}$.

Proposition 2.3.16 (Cauchy-Transform over $\mathcal{A}$)**:** Let $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C}) \in \mathsf{LocArt}_{\mathbb{C}}$, $M \in \mathsf{FdMod}(\mathcal{A})$ via $\rho\colon \mathcal{A} \to \mathrm{End}_{\mathbb{C}}(M)$, and write as usual $Z = z \oplus X$ for $Z \in \mathcal{A}$. Furthermore, let (S, Σ) be a measurable space, $\mu\colon S \to M$ a finite measure, $\gamma \coloneqq g \oplus G\colon S \to \mathcal{A}$ a measurable function with $g\colon S \to \mathbb{C}$ and $G\colon S \to \mathfrak{m}$, respectively, and $U \subseteq \mathcal{A}$ open such that

$$\forall\, Z \in U\colon\ \inf_{s\in S} \Big(|g(s) - z| - \big\|\rho\big(G(s) - X\big)\big\|_{\mathrm{End}} \Big) > 0 \tag{2.3.19}$$

Then $f\colon U \to M$ defined by

$$f(Z) \coloneqq \int_S (\gamma(s) - Z)^{-1} \cdot \mathrm{d}\mu(s) \overset{\mathrm{def}}{=} \int_S \rho\big(\gamma(s) - Z\big)^{-1} \mathrm{d}\mu(s)$$

[6] recall that, since $\mathcal{A}$ is locally compact, locally uniform convergence not only implies compact convergence, but is also equivalent to it;
[7] for the same reason, locally normal convergence not only implies compactly normal convergence, but is equivalent to it;

is ρ-analytic on U with derivatives

$$\frac{f^{(k)}(Z)}{k!} = \int_S (\gamma(s) - Z)^{-(k+1)} \cdot \mathrm{d}\mu(s),\ k \in \mathbb{N}_0.$$

Proof: First of all, observe that since $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C})$, we have $\mathcal{A}_\rho^\times = \mathcal{A}^\times$, which is why we can drop ρ from the integral kernel as we did above. Moreover, by hypothesis we have in particular that $\forall\, Z \in U\colon \inf_{s\in S} |g(s) - z| > 0$, hence indeed $\gamma(S) - U \subseteq \mathcal{A}^\times$ (as $\chi_{\mathcal{A}}(\gamma(S) - U) \not\ni 0$). Fix $Z_0 \in U$, write $Z_0 = z_0 \oplus X_0$ with $z_0 \in \mathbb{C}$ and $X_0 \in \mathfrak{m}$, and set

$$R := \inf_{s\in S} \big(|g(s) - z_0| - \big\|\rho\big(G(s) - X_0\big)\big\|_{\mathrm{End}} \big) > 0.$$

By (automatic) continuity of ρ choose $r > 0$ small enough such that $\mathbb{B}_{\mathcal{A}}(Z_0, r) \subseteq U$ and $\rho(\mathbb{B}_{\mathcal{A}}(Z_0, r)) \subseteq \mathbb{B}_{\mathrm{End}_{\mathbb{C}}(M)}(\rho(Z_0), R)$. By hypothesis we have

$$\forall\, s \in S\colon \left\| \frac{\rho\big(G(s) - X_0\big)}{g(s) - z_0} \right\|_{\mathrm{End}} = \frac{\big\|\rho\big(G(s) - X_0\big)\big\|_{\mathrm{End}}}{|g(s) - z_0|} < 1,$$

hence $\forall\, s \in S$:

$$\begin{aligned}
&\big\|\rho(\gamma(s) - Z_0)^{-1}\big\|_{\mathrm{End}} = \left\| \Big((g(s) - z_0) + \rho\big(G(s) - X_0\big) \Big)^{-1} \right\|_{\mathrm{End}} = \\
&= \frac{1}{|g(s) - z_0|} \left\| \left(1 - \frac{\rho\big(X_0 - G(s)\big)}{g(s) - z_0} \right)^{-1} \right\|_{\mathrm{End}} \leq \frac{1}{|g(s) - z_0|} \left(1 - \left\| \frac{\rho\big(G(s) - X_0\big)}{g(s) - z_0} \right\|_{\mathrm{End}} \right)^{-1} = \\
&= \big(|g(s) - z_0| - \big\|\rho\big(G(s) - X_0\big)\big\|_{\mathrm{End}} \big)^{-1} \leq \frac{1}{R},
\end{aligned}$$

where we have used Lemma 0.3.18 for the estimate in the middle row. Thus $\forall\, Z \in \mathbb{B}_{\mathcal{A}}(Z_0, r)\ \forall\, s \in S$:

$$q := \left\| \frac{\rho(Z - Z_0)}{\rho(\gamma(s) - Z_0)} \right\|_{\mathrm{End}} \leq \|\rho(Z - Z_0)\|_{\mathrm{End}} \left\| \frac{1}{\rho(\gamma(s) - Z_0)} \right\|_{\mathrm{End}} \leq \frac{\|\rho(Z - Z_0)\|_{\mathrm{End}}}{R} < 1.$$

Hence for all $Z \in \mathbb{B}_{\mathcal{A}}(Z_0, r)$ the geometric series

$$\frac{1}{\rho(\gamma(s) - Z)} = \frac{1}{\rho(\gamma(s) - Z_0)} \left(1 - \frac{\rho(Z - Z_0)}{\rho(\gamma(s) - Z_0)} \right)^{-1} = \frac{1}{\rho(\gamma(s) - Z_0)} \sum_{k=0}^{\infty} \left(\frac{\rho(Z - Z_0)}{\rho(\gamma(s) - Z_0)} \right)^k$$

converges uniformly in $s \in S$. Therefore, since μ is finite, we can interchange summation and integration to obtain

$$\begin{aligned}
f(Z) &\overset{\mathrm{def}}{=} \int_S \rho(\gamma(s) - Z)^{-1} \mathrm{d}\mu(s) = \int_S \left(\sum_{k=0}^{\infty} \frac{\rho(Z - Z_0)^k}{\rho(\gamma(s) - Z_0)^{k+1}} \right) \mathrm{d}\mu(s) = \\
&= \sum_{k=0}^{\infty} (Z - Z_0)^k \cdot \left(\int_S (\gamma(s) - Z_0)^{-(k+1)} \cdot \mathrm{d}\mu(s) \right)
\end{aligned}$$

for all $Z \in \mathbb{B}_{\mathcal{A}}(Z_0, r)$, which proves the claim. □

Remarks:

(1) To state and prove a more general version for any $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{C}}$ one could for example use Proposition 1.2.12 and Lemma-Definition 0.3.25

(2) Using Lemma 0.3.19 one can relax the condition in Eq. (2.3.19) somewhat to only require

$$\forall\, Z \in U\ \forall\, s \in S\colon\ |g(s) - z| \geq \big\|\rho\big(G(s) - X\big)\big\|_{\mathrm{End}} \ \text{ and } \ \inf_{s\in S} |g(s) - z| > 0.$$

Indeed, if we fix $Z_0 \in U$ and choose $R > 0$ small enough such that $|g(s) - z_0| \geq \nu R$, where $\nu \in \mathbb{N}$ is the smallest integer with the property that $\forall\, s \in S\colon \rho\big(G(s) - X_0\big)^{\nu} = 0$, then by

Lemma 0.3.19 we obtain

$$\left\|\rho\big(G(s)-X_0\big)^{-1}\right\|_{\mathrm{End}} \overset{\mathrm{def}}{=} \left\|\rho^{\mathrm{im}}\big(G(s)-X_0\big)^{-1}\right\|_{\mathrm{End}} \leq \frac{\nu}{|g(s)-z_0|} \leq \frac{1}{R},$$

where we have used that $\rho(\mathcal{A}) \in \mathsf{LocArt}_{\mathbb{C}}$ as well, and one proceeds from there analogously as in the proof above.

(3) Of course, we also have $\|\rho\big(G(s)-X\big)\|_{\mathrm{End}} \leq \|G(s)-X\|_{\mathcal{A}}$, but dropping ρ would have produced a less general statement despite $\|\rho\|_{\mathrm{op}} = 1$ since $|g(s)-z| - \|G(s)-X\|_{\mathcal{A}} > 0$ then implies $|g(s)-z| - \|\rho\big(G(s)-X\big)\|_{\mathrm{End}} > 0$, but not vice versa, while the latter one completely suffices.

(4) It is an interesting question whether one can improve the condition in Eq. (2.3.19) by using other estimates of the inverse (cf. Section 0.3). □

Proposition 2.3.17 (Analyticity)**:** Let $U \subseteq \mathcal{A}$ be open and $f \in \mathcal{O}_\rho(U)$. Then f is ρ-analytic in U.

Proof: Without loss of generality we can assume $Z_0 = 0$ and let $\Delta \subseteq U$ be a polydisc centered at Z_0. Following the argument in the the proof of Proposition 2.3.4 (2) we know constructively that there exists a sufficiently small smooth $\gamma \in \mathcal{L}(\Delta \setminus [Z_0]_\rho)$ such that $\mathrm{Ind}_\rho(\gamma, Z_0) = \mathrm{id}_M$. Denote by $C \ni Z_0$ the connected open component of Z_0, on which $\mathrm{Ind}_\rho(\gamma, -)$ stays $= \mathrm{id}_M$ (cf. Corollary 2.2.10). Then by Eq. (2.3.4) we have that

$$\forall\, Z \in \Delta \cap \mathrm{Adm}_\rho(\gamma) \cap C \colon\, f(Z) = \frac{1}{2\pi i} \oint_\gamma \frac{\rho(\mathrm{d}W)}{\rho(W-Z)} f(W).$$

One immediately checks that $S := I \overset{\mathrm{def}}{=} [0,1]$, $\gamma(s)$, and $\mathrm{d}\mu(s) := \rho(\gamma'(s))f(\gamma(s))\mathrm{d}s$ satisfy the hypothesis of Proposition 2.3.16. In particular, the condition in Eq. (2.3.19) presents no problem because S is compact and γ is continuous. □

Proposition 2.3.18 (Morera over $\mathcal{A}$)**:** Let $U \subseteq \mathcal{A}$ open with $\pi_0(U) = 1$ and let $f \in \mathscr{C}^0(U, M)$.

(1) If

$$\forall\, \gamma \in \mathcal{L}U \colon \oint_\gamma \rho(\mathrm{d}Z) f(Z) = 0,$$

then $f \in \mathcal{O}_\rho(U)$.

(2) If

$$\forall\, \triangle \subseteq U \colon \oint_{\partial\triangle} \rho(\mathrm{d}Z) f(Z) = 0,$$

then $f \in \mathcal{O}_\rho(U)$.

Proof: To (1): By Lemma 1.4.7 (1), f has a primitive F, which is analytic by Proposition 2.3.17, hence so is f.

To (2): Since $\mathcal{A}$ is automatically locally convex, U can be covered by (star-)convex open sets, over which f has local primitives by Lemma 1.4.7 (2). □

2.4. The Generalized Cauchy–Pompeiu Integral Formula

In the following we will assume $\mathcal{A} = (\mathcal{A}, \mathfrak{m}, \mathbb{C})$ and use the standard identifications $\mathcal{A} \cong \mathbb{C} \oplus \mathfrak{m} = \mathbb{C}^n \cong \mathbb{R}^{2n}$, $(z_1, \ldots, z_n)^{\mathrm{T}} = (x_1 + iy_1, \ldots, x_n + iy_n)^{\mathrm{T}} \mapsto (x_1, y_1, \ldots, x_n, y_n)^{\mathrm{T}}$, where also $z_1 \stackrel{\text{def}}{=} z =: x + iy$ is the distinguished first (=spectral) coordinate. In the same vein, for $W, Z \in \mathcal{A}$ we will write $Z = z \oplus X$, $W = w \oplus Y$. Note that by abuse of notation we will be using the latter not merely as coordinates, but also as *local parametrizations* (inverse coordinate functions).

Let $U \subseteq \mathcal{A}$ be open and $M \in \mathsf{FdMod}(\mathcal{A})$. The framework of $\mathcal{A}$-holomorphy allows to generalize the classical Cauchy–Pompeiu formula to *multi-variable mappings* $f \in \mathscr{C}^\nu(U, M)$ and compact connected orientable $\mathscr{C}^{\nu+1}$-surfaces $\Sigma \subset U$ with $\mathscr{C}^1_{\mathrm{pw}}$-boundary[8] $\partial\Sigma$ ($\partial\Sigma = \varnothing$ allowed) in place of bounded subdomains of $\mathbb{C}$, where recall $\nu \stackrel{\text{def}}{=} \nu_M(\mathfrak{m})$. Given $W = w \oplus Y \in \Sigma \setminus \partial\Sigma$, we will consider the $\mathscr{C}^\nu$-differential 1-forms

$$\omega := \frac{\mathrm{d}Z}{Z - W} \cdot f(Z) \stackrel{\text{def}}{=} \rho\left(\frac{\mathrm{d}Z}{Z - W}\right)(f(Z)) \text{ and } \omega_{\mathrm{sp}} := \frac{\mathrm{d}Z}{z - w} \cdot f(Z) \stackrel{\text{def}}{=} \rho(\mathrm{d}Z)\left(\frac{f(Z)}{z - w}\right).$$

Observe that both 1-forms share the same singularity locus given by the affine subspace $[W] \stackrel{\text{def}}{=} W + \mathfrak{m} = w + \mathfrak{m} = [w] \in \mathcal{A}/\mathfrak{m}$, and hence $\omega, \omega_{\mathrm{sp}} \in \Omega^1(U \setminus [W], M)$. In particular, $\omega|_\Sigma, \omega_{\mathrm{sp}}|_\Sigma \in \Omega^1(\Sigma \setminus \chi_\Sigma^{-1}(w), M)$, where $\chi_\Sigma \colon \Sigma \to \mathbb{C}$ denotes the obvious restriction of the character $\chi_{\mathcal{A}} \colon \mathcal{A} \twoheadrightarrow \mathbb{C}$. With a suitable "general position" assumption on Σ relative to $\mathfrak{m}$, the fiber $\chi_\Sigma^{-1}(w) \stackrel{\text{def}}{=} \Sigma \cap [w]$ becomes a finite set of hence isolated singularities of $\omega_{\mathrm{sp}}|_\Sigma$ and $\omega|_\Sigma$ with the corresponding integrals turning out to be only *weakly singular*.

Definition 2.4.1 (Transversality relative to the $\mathcal{A}$-structure)**:**
(1) A point $W_0 \in \Sigma$ will be called $\mathcal{A}$-*transverse* if $W_0 \notin \partial\Sigma$ and $[W_0] \cap \Sigma$ is transverse at W_0.
(2) A subset $S \subseteq \Sigma$ will be called $\mathcal{A}$-*transverse* if every $W_0 \in S$ is $\mathcal{A}$-transverse.

Observe that $\mathcal{A}$-transversality is an open condition. Indeed, if $j \colon \Sigma \hookrightarrow \mathcal{A}$ denotes the corresponding $\mathscr{C}^{\nu+1}$-embedding and $W_0 \in \Sigma \setminus \partial\Sigma$ is an arbitrary, consider the composition of tangent maps

$$T_{W_0}\Sigma \xhookrightarrow{j_{*,W_0}} T_{W_0}\mathcal{A} = \mathcal{A} = \mathbb{C} \oplus \mathfrak{m} \xrightarrow{\chi_{*,W_0}} T_{w_0}\mathbb{C},$$

where as usual $w_0 := \chi_{\mathcal{A}}(W_0)$. If W_0 is $\mathcal{A}$-transverse, then j_{*,W_0} is given by

$$T_{W_0}\Sigma \hookrightarrow T_{W_0}\Sigma \oplus T_{W_0}[W_0] = T_{W_0}\Sigma \oplus \mathfrak{m} = T_{W_0}\mathcal{A},$$

hence $\chi_{*,W_0} \circ j_{*,W_0} \colon T_{W_0}\Sigma \to T_{w_0}\mathbb{C}$ is an $\mathbb{R}$-linear isomorphism. Conversely, if $\chi_{*,W_0} \circ j_{*,W_0}$ is an isomorphism, then $\operatorname{Ker}(\chi_{*,W_0}) \cap \operatorname{Im}(j_{*,W_0}) = 0$, i.e. W_0 is an $\mathcal{A}$-transverse point. Furthermore, whenever $(\chi \circ j)_{*,W_0}$ is an isomorphism, the Inverse Function Theorem implies that $\chi_\Sigma = \chi \circ j$ is a local $\mathscr{C}^{\nu+1}$-diffeomorphism at W_0. In total, it follows that if $W_0 \in \Sigma$ is an $\mathcal{A}$-transverse point, then it admits a whole $\mathcal{A}$-transverse open neighbourhood $S \ni W_0$ in Σ together with a local parametrization ($\mathscr{C}^{\nu+1}$-diffeomorphic section of χ_Σ) of the form

$$\begin{aligned} W \colon S^{\mathrm{sp}} &\xrightarrow{\cong} S \stackrel{\text{def}}{=} W(S^{\mathrm{sp}}) \\ w &\mapsto w \oplus Y(w) \end{aligned} \tag{2.4.1}$$

with $W_0 = W(w_0)$, where $S^{\mathrm{sp}} \stackrel{\text{def}}{=} \chi_\Sigma(S) \subseteq \mathbb{C}$ and $Y \in \mathscr{C}^{\nu+1}(S^{\mathrm{sp}}, \mathfrak{m})$. Such W will be called a *spectral* (local) parametrization of the $\mathcal{A}$-transverse neighbourhood S, and we will write (S, W) for such a pair. In particular, we have

$$\forall\, w \in S^{\mathrm{sp}} \colon \frac{\partial W}{\partial w}(w) = 1 \oplus \frac{\partial Y}{\partial w}(w) \in \mathcal{A}^\times. \tag{2.4.2}$$

[8] regularity of f on $\partial\Sigma$ can likely be relaxed similarly to the various versions of the Stokes' package;

Notation: Given $\varepsilon > 0$ small enough, we set $\mathbb{D}_\Sigma(W_0; \varepsilon) := W\big(\mathbb{D}_\varepsilon(w_0)\big) \subseteq S$.

Lemma 2.4.2: Let $m \in \mathbb{N}$ and $g \in \mathscr{C}^{m+1}(\mathbb{D}_r(0), M)$. Then:

(1) We have:

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^m} \int_0^1 e^{-2\pi i m t} g(\varepsilon e^{2\pi i t}) \mathrm{d}t = \frac{1}{m!} \frac{\partial^m g}{\partial w^m}(0).$$

(2) We have:

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^m} \int_0^1 e^{-2\pi i (m+2) t} g(\varepsilon e^{2\pi i t}) \mathrm{d}t = 0.$$

Proof: Recall that for $m \in \mathbb{Z}$ we have the standard "orthogonality" integral

$$\int_0^1 e^{2\pi i m t} \mathrm{d}t = \begin{cases} 1, \text{ if } m = 0 \\ 0, \text{ otherwise} \end{cases}$$

Since $g \in \mathscr{C}^{m+1}(\mathbb{D}_r(0), M)$ and M is in particular a $\mathbb{C}$-vector space, we can Taylor-expand g in a small neighbourhood of 0 in complex coordinates

$$g(w) = \sum_{k+\ell \le m} \frac{1}{k!\ell!} \frac{\partial^{k+\ell} g}{\partial w^k \partial \bar{w}^\ell}(0) w^k \bar{w}^\ell + R(w),$$

where for the remainder term $R(w)$ we have the standard estimate $\|R(w)\|_M \le C\,|w|^{m+1}$ as $w \to 0$ for some constant $C > 0$ (depending only on m) due to the existence of $D^{m+1} g \in \mathscr{C}(\mathbb{D}_r(0), M)$. Hence for sufficiently small $\varepsilon > 0$ we have

$$g(\varepsilon e^{2\pi i t}) = \sum_{k+\ell \le m} \frac{1}{k!\ell!} \frac{\partial^{k+\ell} g}{\partial w^k \partial \bar{w}^\ell}(0) \varepsilon^{k+\ell} e^{2\pi i (k-\ell) t} + R(\varepsilon e^{2\pi i t}). \tag{2.4.3}$$

Thus

$$I_1 := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^m} \int_0^1 e^{-2\pi i m t} g(\varepsilon e^{2\pi i t}) \mathrm{d}t = \sum_{k+\ell \le m} \frac{1}{k!\ell!} \frac{\partial^{k+\ell} g}{\partial w^k \partial \bar{w}^\ell}(0) J_1^{k,\ell} + R_1,$$

$$I_2 := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^m} \int_0^1 e^{-2\pi i (m+2) t} g(\varepsilon e^{2\pi i t}) \mathrm{d}t = \sum_{k+\ell \le m} \frac{1}{k!\ell!} \frac{\partial^{k+\ell} g}{\partial w^k \partial \bar{w}^\ell}(0) J_2^{k,\ell} + R_2,$$

where

$$\begin{aligned} J_1^{k,\ell} &:= \lim_{\varepsilon\to 0} \varepsilon^{k+\ell-m} \int_0^1 e^{2\pi i (k-\ell-m) t} \mathrm{d}t, \\ R_1 &:= \lim_{\varepsilon\to 0} \int_0^1 e^{-2\pi i m t} \frac{R(\varepsilon e^{2\pi i t})}{\varepsilon^m} \mathrm{d}t, \\ J_2^{k,\ell} &:= \lim_{\varepsilon\to 0} \varepsilon^{k+\ell-m} \int_0^1 e^{2\pi i (k-\ell-m-2) t} \mathrm{d}t, \\ R_2 &:= \lim_{\varepsilon\to 0} \int_0^1 e^{-2\pi i (m+2) t} \frac{R(\varepsilon e^{2\pi i t})}{\varepsilon^m} \mathrm{d}t. \end{aligned}$$

Since $k, \ell \ge 0$ with $k + \ell \le m$, it follows that $k - \ell - m = 0$, i.e. $k - \ell = m$, is satisfied only for $k = m$ and $\ell = 0$, whereas $k - \ell = m + 2$ has no solutions in that range, hence

$$\begin{aligned} J_1^{k,\ell} &= \begin{cases} 1 \text{ for } k = m,\ \ell = 0 \\ 0, \text{ otherwise} \end{cases} \\ J_2^{k,\ell} &= 0. \end{aligned} \tag{2.4.4}$$

For the remainder integrands we have

$$\left\| e^{-2\pi imt}\frac{R(\varepsilon e^{2\pi it})}{\varepsilon^m}\right\|_M = \frac{\left\|R(\varepsilon e^{2\pi it})\right\|_M}{\varepsilon^m} \leq C\frac{\left|\varepsilon e^{2\pi it}\right|^{m+1}}{\varepsilon^m} = C\varepsilon,$$

hence the integrand is uniformly bounded as $\varepsilon \to 0$. It follows from the dominated convergence theorem that $R_1 = 0$. Analogously, $R_2 = 0$. Putting all this together implies both (1) and (2). $\square$

Lemma 2.4.3: Let $k \in \mathbb{N}_0$, $f \in \mathscr{C}^{k+1}(U, M)$, $W \in \mathscr{C}^{k+2}(S^{\mathrm{sp}}, \mathcal{A})$, $w_0 \in S^{\mathrm{sp}}$ arbitrary, and $W_0 := W(w_0)$. We have:

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\partial\mathbb{D}_\Sigma(W_0;\varepsilon)}\frac{\mathrm{d}Z}{(z-w_0)^{k+1}}f(Z) = \frac{1}{k!}\frac{\partial^k}{\partial w^k}\bigg|_{w_0}\left(\frac{\partial W}{\partial w}(w)f(W(w))\right). \tag{2.4.5}$$

Proof: For notational simplicity we can assume without loss of generality $w_0 = 0$. Write $\gamma_\varepsilon := \partial\mathbb{D}_\Sigma(W_0;\varepsilon)$ for the boundary loop, i.e. $\gamma_\varepsilon(t) \stackrel{\mathrm{def}}{=} W(\gamma_\varepsilon^{\mathrm{sp}}(t)) \stackrel{\mathrm{def}}{=} W(\varepsilon e^{2\pi it})$, $t \in [0,1]$, hence

$$\begin{aligned}\dot\gamma_\varepsilon(t) &= \frac{\partial W}{\partial w}(\varepsilon e^{2\pi it})2\pi i\varepsilon e^{2\pi it} + \frac{\partial W}{\partial\bar w}(\varepsilon e^{2\pi it})(-2\pi i\varepsilon)e^{-2\pi it} = \\ &= 2\pi i\varepsilon\big(W_w(\varepsilon e^{2\pi it})e^{2\pi it} - W_{\bar w}(\varepsilon e^{2\pi it})e^{-2\pi it}\big).\end{aligned}$$

We thus obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\partial\mathbb{D}_\Sigma(W_0;\varepsilon)}\frac{\mathrm{d}Z}{z^{k+1}}f(Z) \stackrel{\mathrm{def}}{=} \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\gamma_\varepsilon}\frac{\mathrm{d}Z}{z^{k+1}}f(Z) = \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_0^1\frac{\dot\gamma_\varepsilon(t)}{\gamma_\varepsilon^{\mathrm{sp}}(t)^{k+1}}f(\gamma_\varepsilon(t))\mathrm{d}t \stackrel{\mathrm{def}}{=} \\ &\stackrel{\mathrm{def}}{=} \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_0^1\frac{2\pi i\varepsilon\big(W_w(\varepsilon e^{2\pi it})e^{2\pi it} - W_{\bar w}(\varepsilon e^{2\pi it})e^{-2\pi it}\big)}{\varepsilon^{k+1}e^{2\pi i(k+1)t}}f(W(\varepsilon e^{2\pi it}))\mathrm{d}t = \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^k}\left(\int_0^1 e^{-2\pi ikt}W_w(\varepsilon e^{2\pi it})f(W(\varepsilon e^{2\pi it}))\mathrm{d}t - \int_0^1 e^{-2\pi i(k+2)t}W_{\bar w}(\varepsilon e^{2\pi it})f(W(\varepsilon e^{2\pi it}))\mathrm{d}t\right) = \\ &= \frac{1}{k!}\frac{\partial^k}{\partial w^k}\bigg|_0(W_w(w)f(W(w)))\end{aligned}$$

by Lemma 2.4.2 applied to $W_w(w)f(W(w))$ and $W_{\bar w}(w)f(W(w))$, respectively. $\square$

Remarks:

(1) For Eq. (2.4.5) to hold at a single fixed $\mathcal{A}$-transverse point $W_0 = W(w_0)$ in the case $k = 0$, it would have sufficed to construct a parametrization W that only satisfies $(DW)(w) = \left(\begin{smallmatrix} I_2 \\ * \end{smallmatrix}\right)$ at $w = w_0$. However, we want to have a spectral parametrization associated to a whole $\mathcal{A}$-transverse open neighbourhood rather than a whole myriad of different parametrizations "spectral" at each $\mathcal{A}$-transverse point *separately*.

(2) If $W_0 \in \Sigma$ is a non-$\mathcal{A}$-transverse point, then $\frac{\dot\gamma_\varepsilon(t)}{\gamma_\varepsilon^{\mathrm{sp}}(t)}$, $t \in [0,1]$, may have a non-resolvable zero in the denominator as $\varepsilon \to 0$. On the other hand, if $W_0 \in \Sigma$ is only an *isolated* non-$\mathcal{A}$-transverse point, then conceivably $\lim\limits_{\varepsilon\to 0}\int_0^1\frac{\dot\gamma_\varepsilon(t)}{\gamma_\varepsilon^{\mathrm{sp}}(t)}f(\gamma_\varepsilon(t))\mathrm{d}t$ may nevertheless exist, but we can offer no explicit calculation in that case.

Definition 2.4.4 (Fiber-wise $\mathcal{A}$-transversality)**:**

(1) A point $W \in \Sigma$ will be called *fiber-wise $\mathcal{A}$-transverse* if its fiber $[W] \cap \Sigma$ is an $\mathcal{A}$-transverse subset of Σ.

(2) A subset $S \subseteq \Sigma$ will be called *sheet-wise $\mathcal{A}$-transverse* if every $W \in S$ is fiber-wise $\mathcal{A}$-transverse.

Suppose now that $W \in \Sigma$ is a fiber-wise $\mathcal{A}$-transverse point with $\Sigma \pitchfork [W] =: \{p_1, \ldots, p_m\}$. Then, as before, each p_ℓ admits a local $\mathcal{A}$-transverse neighbourhood (S_ℓ, W_ℓ) with a *spectral* local $\mathscr{C}^{\nu+1}$-parametrization

$$W_\ell\colon S^{\mathrm{sp}} \xrightarrow{\cong} S_\ell := W_\ell(S^{\mathrm{sp}}),\ 1 \leq \ell \leq m,$$

where S^{sp} itself can be chosen independently of ℓ (possibly after shrinking S_ℓ) since $m < \infty$ (indeed, given some initial $W_\ell\colon S_\ell^{\mathrm{sp}} \xrightarrow{\cong} S_\ell$, $1 \leq \ell \leq m$, simply take $S^{\mathrm{sp}} := \bigcap_{\ell=1}^m S_\ell^{\mathrm{sp}}$ and restrict-corestrict each W_ℓ accordingly). In other words, if we take $S := \bigcup_{\ell=1}^m S_\ell \subseteq \Sigma$ (an open subset), then W_ℓ is a "sectional" parametrization of the $\mathcal{A}$-transverse sheet S_ℓ of the m-sheeted covering $\chi_\Sigma|S \twoheadrightarrow S^{\mathrm{sp}}$ of the open subset $S^{\mathrm{sp}} \subseteq \mathbb{C}$, $1 \leq \ell \leq m$, where $w \in S^{\mathrm{sp}}$ is a regular value of χ_Σ. Now put $\gamma_\varepsilon^{\mathrm{sp}}(t) := w + \varepsilon e^{2\pi i t}$, $t \in [0,1]$, with $\varepsilon > 0$ sufficiently small such that $\gamma_\varepsilon^{\mathrm{sp}} \subset S^{\mathrm{sp}}$, and define $\gamma_{\ell,\varepsilon} := W_\ell \circ \gamma_\varepsilon^{\mathrm{sp}}$ to be the corresponding parametrizations of $\partial\mathbb{D}_\Sigma(p_\ell;\varepsilon)$, $1 \leq \ell \leq m$. We will consider the surface

$$\Sigma_\varepsilon := \Sigma \setminus \dot{\bigcup_{\ell=1}^{m}} \mathbb{D}_\Sigma(p_\ell;\varepsilon),$$

i.e. Σ with deleted small open disks around the isolated singularities of $\omega_{\mathrm{sp}}|_\Sigma$. Putting

$$\epsilon_\ell := \deg(\chi_\Sigma|_{S_\ell}) = \operatorname{sgn}(\det((D\chi_\Sigma)(p_\ell))) \in \{-1,1\},\ 1 \leq \ell \leq m,$$

to signify whether χ_Σ is orientation-preserving or orientation-reversing on the corresponding sheet, we thus have

$$\partial\Sigma_\varepsilon = \partial\Sigma - \sum_{\ell=1}^{m} \epsilon_\ell \gamma_{\ell,\varepsilon}$$

as 1-cycles.

Proposition 2.4.5 (Global "Spectral" Cauchy–Pompeiu Formula)**:** With the above conventions we have $\forall\, w \in S^{\mathrm{sp}}$:

$$\begin{aligned}\sum_{\ell=1}^{m} \epsilon_\ell \frac{\partial W_\ell}{\partial w}(w) f(W_\ell(w)) &= \frac{1}{2\pi i}\int_{\partial\Sigma} \frac{\mathrm{d}Z}{z-w} f(Z) + \\ &+ \frac{1}{2\pi i}\int_\Sigma \left(\frac{\mathrm{d}z \wedge \mathrm{d}X}{(z-w)^2} f(Z) + \sum_{j=1}^{n} \left(\frac{\mathrm{d}Z \wedge \mathrm{d}z_j}{z-w} \frac{\partial f}{\partial z_j}(Z) + \frac{\mathrm{d}Z \wedge \mathrm{d}\bar{z}_j}{z-w} \frac{\partial f}{\partial \bar{z}_j}(Z) \right)\right)\end{aligned} \tag{2.4.6}$$

In particular, $\forall\, w \in S^{\mathrm{sp}}$:

$$\sum_{\ell=1}^{m} \epsilon_\ell \frac{\partial W_\ell}{\partial w}(w) = \frac{1}{2\pi i}\int_{\partial\Sigma} \frac{\mathrm{d}Z}{z-w} + \frac{1}{2\pi i}\int_\Sigma \frac{\mathrm{d}z \wedge \mathrm{d}X}{(z-w)^2} \tag{2.4.7}$$

Proof: By Stokes' theorem we have

$$\sum_{\ell=1}^{m} \epsilon_\ell \int_{\gamma_{\ell,\varepsilon}} \omega_{\mathrm{sp}} = \int_{\partial\Sigma} \omega_{\mathrm{sp}} - \int_{\Sigma_\varepsilon} \mathrm{d}\omega_{\mathrm{sp}},$$

where one calculates

$$\mathrm{d}\omega_{\mathrm{sp}} = -\frac{\mathrm{d}z \wedge \mathrm{d}X}{z^2} f(Z) - \sum_{j=1}^{n} \frac{\mathrm{d}Z \wedge \mathrm{d}z_j}{z} \frac{\partial f}{\partial z_j}(Z) - \sum_{j=1}^{n} \frac{\mathrm{d}Z \wedge \mathrm{d}\bar{z}_j}{z} \frac{\partial f}{\partial \bar{z}_j}(Z).$$

On the other hand, by Lemma 2.4.3 we have:

$$\forall\, 1 \leq \ell \leq m\colon \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\gamma_{\ell,\varepsilon}} \omega_{\mathrm{sp}} = \frac{\partial W_\ell}{\partial w}(w) f(W_\ell(w)).$$

Letting $\varepsilon \to 0$ and combining the three statements implies the first claim. The second claim follows as usual by taking $M := {}_{\mathcal{A}}\mathcal{A}$ and $f := 1_{\mathcal{A}}$. □

Remarks:

(1) Observe that the "spectral" 1-form

$$\frac{\mathrm{d}Z}{z-w} \in \Omega^1_{\mathrm{hol}}(\mathcal{A} \setminus [w], \mathcal{A}) \stackrel{\mathrm{def}}{=} \Omega^1_{\mathrm{hol}}(\mathcal{A} \setminus [W], \mathcal{A})$$

is **not** ∂-closed, hence **not** d-closed, unlike its "full" $\mathcal{A}$-holomorphic counterpart

$$\frac{\mathrm{d}Z}{Z-W} \in \Omega^1_{\mathrm{hol}}(\mathcal{A}\setminus[w],\mathcal{A}) \overset{\mathrm{def}}{=} \Omega^1_{\mathrm{hol}}(\mathcal{A}\setminus[W],\mathcal{A}),$$

which has the same domain of definition.

(2) In the degenerate case $\nu = 1$, i.e. $\mathfrak{m} = 0$ and $\mathcal{A} = \mathbb{C}$, hence $n = 1$ and $m = 1$, we recover the classical Cauchy–Pompeiu formula since $\mathrm{d}X = 0$, $\mathrm{d}Z = \mathrm{d}z$, and $W = W_1 = w$ in that case.

Lemma 2.4.6: Let $k \in \mathbb{N}_0$, $V \subseteq \mathbb{C}$ open, $g \in \mathscr{C}^k(V,\mathcal{A})$, and $h \in \mathscr{C}^k(V,M)$. Then:

$$\forall\, w \in V\colon \frac{1}{k!}\frac{\partial^k}{\partial u^k}\bigg|_w \Big(\big(g(w)-g(u)\big)^k h(u)\Big) = \Big(-\frac{\partial g}{\partial w}(w)\Big)^k h(w).$$

Proof: We have $\forall\, 0 \le j \le k$:

$$\frac{\partial^j}{\partial u^j}\bigg|_w \big(g(w)-g(u)\big)^k = \sum_{\substack{j_1,\dots,j_k\ge 0\\ j_1+\dots+j_k=j}} \binom{j}{j_1,\dots,j_k} \prod_{\ell=1}^k \frac{\partial^{j_\ell}}{\partial u^{j_\ell}}\bigg|_w \big(g(w)-g(u)\big) = \begin{cases} 0, \text{ if } j<k \\ k!(-\frac{\partial g}{\partial w}(w))^k, \text{ if } j=k \end{cases}$$

since if $\{j_1,\dots,j_k\} \ni 0$, then the corresponding product contains a factor $(g(w)-g(w)) = 0$, and therefore the only non-zero product in the sum corresponds to the maximal partition $(j_1,\dots,j_k) = (1,\dots,1)$. It now follows that

$$\frac{\partial^k}{\partial u^k}\bigg|_w \Big(\big(g(w)-g(u)\big)^k h(u)\Big) = \sum_{j=0}^k \binom{k}{j} \frac{\partial^j}{\partial u^j}\bigg|_w \big(g(w)-g(u)\big)^k \frac{\partial^{k-j}}{\partial u^{k-j}}\bigg|_w h(u) = k!\Big(-\frac{\partial g}{\partial w}(w)\Big)^k h(w)$$

as desired. □

Assuming Problem 2 has a positive solution, then we obtain an action $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \curvearrowright \mathcal{A}$, that is, the complex conjugation $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ extends to a $\sigma \in \mathrm{Aut}_{\mathbb{R}}(\mathcal{A})$, $a \mapsto \bar{a} := \sigma(a)$. Now fix some $\mathbb{C}$-basis $\{a_1 = 1_{\mathcal{A}}, a_2,\dots,a_n\}$ for $\mathcal{A}$ and recall from Section 1.3 the "complementary set" of $\mathcal{A}$-variables for $Z_1 := Z \overset{\mathrm{def}}{=} z + z_2a_2 + \dots + z_na_n$, given simply by

$$Z_j := z_j,\ 2 \le j \le n.$$

Hence

$$\bar{Z}_1 \overset{\mathrm{def}}{=} \bar{Z} = \bar{z} + \bar{z}_2\bar{a}_2 + \dots + \bar{z}_n\bar{a}_n,\ \bar{Z}_j \overset{\mathrm{def}}{=} \bar{z}_j,\ 2 \le j \le n,$$

and

$$\begin{aligned} \mathrm{d}Z_1 &\overset{\mathrm{def}}{=} \mathrm{d}Z = \mathrm{d}z + a_2\mathrm{d}z_2 + \dots + a_n\mathrm{d}z_n,\\ \mathrm{d}\bar{Z}_1 &\overset{\mathrm{def}}{=} \mathrm{d}\bar{Z} = \mathrm{d}\bar{z} + \bar{a}_2\mathrm{d}\bar{z}_2 + \dots + \bar{a}_n\mathrm{d}\bar{z}_n,\\ \mathrm{d}Z_j &\overset{\mathrm{def}}{=} \mathrm{d}z_j,\ \mathrm{d}\bar{Z}_j \overset{\mathrm{def}}{=} \mathrm{d}\bar{z}_j,\ 2 \le j \le n. \end{aligned}$$

Accordingly, we have

$$\begin{aligned} &\frac{\partial}{\partial Z_1} \overset{\mathrm{def}}{=} \frac{\partial}{\partial Z} = \frac{\partial}{\partial z} \text{ and } \frac{\partial}{\partial \bar{Z}_1} \overset{\mathrm{def}}{=} \frac{\partial}{\partial \bar{Z}} = \frac{\partial}{\partial \bar{z}},\\ &\frac{\partial}{\partial Z_j} = \frac{\partial}{\partial z_j} - a_j\frac{\partial}{\partial z} \text{ and } \frac{\partial}{\partial \bar{Z}_j} = \frac{\partial}{\partial \bar{z}_j} - \bar{a}_j\frac{\partial}{\partial \bar{z}},\ 2 \le j \le n. \end{aligned} \tag{2.4.8}$$

In fact, it is not difficult to see that one can choose $\{a_1 = 1_{\mathcal{A}}, a_2,\dots,a_n\}$ to be σ-invariant, i.e. $\forall\, 2 \le j \le n\colon \bar{a}_j = a_j$, which somewhat simplifies the above $\mathcal{A}$-valued (co)frames, so from this point on we will assume this to be the case.

However, if one prefers an unconditional statement, one can restrict attention only to the class of $\mathcal{A} \in \mathsf{LocArt}_{\mathbb{C}}$ that admit real forms, which in particular includes all $\mathcal{A}$ up to $\dim_{\mathbb{C}}\mathcal{A} = 6$ (cf. isomorphism types table in [Poo08b]). Alternatively, one can introduce *basis-dependent conjugation* by simply declaring:

$$\bar{Z}_1 \stackrel{\text{def}}{=} \bar{Z} := \bar{z} + \bar{z}_2 a_2 + \cdots + \bar{z}_n a_n \text{ and } \bar{Z}_j := \bar{z}_j,\ 2 \le j \le n.$$

This is of course no longer an automorphism of $\mathcal{A}$ and hence not intrinsic.

Theorem 2.4.7 (Generalized Cauchy–Pompeiu Formula over $\mathcal{A}$)**:** Let $W \in \{W_1, \dots, W_m\}$ be any of the sheet parametrizations as defined previously. We have $\forall\, w \in S^{\mathrm{sp}}$:

$$\begin{aligned}\sum_{\ell=1}^{m} \epsilon_\ell f(W_\ell(w)) = \frac{1}{2\pi i} \int_{\partial\Sigma} \frac{\mathrm{d}Z}{Z-W} f(Z) + \\ + \frac{1}{2\pi i} \sum_{j=2}^{n} \int_{\Sigma} \frac{\mathrm{d}Z \wedge \mathrm{d}Z_j}{Z-W} \frac{\partial f}{\partial Z_j}(Z) + \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\Sigma} \frac{\mathrm{d}Z \wedge \mathrm{d}\bar{Z}_j}{Z-W} \frac{\partial f}{\partial \bar{Z}_j}(Z).\end{aligned} \tag{2.4.9}$$

Proof: We finally get to consider

$$\omega \stackrel{\text{def}}{=} \frac{\mathrm{d}Z}{Z-W} f(Z) \in \Omega^1(\Sigma_\varepsilon, M)$$

(of differentiability class $\mathscr{C}^\nu$). We calculate

$$\begin{aligned}\mathrm{d}\omega &= \sum_{j=2}^{n} \mathrm{d}Z_j \wedge \mathrm{d}Z \frac{\partial}{\partial Z_j}\Big(\frac{f(Z)}{Z-W}\Big) + \sum_{j=1}^{n} \mathrm{d}\bar{Z}_j \wedge \mathrm{d}Z \frac{\partial}{\partial \bar{Z}_j}\Big(\frac{f(Z)}{Z-W}\Big) = \\ &= \sum_{j=2}^{n} \frac{\mathrm{d}Z_j \wedge \mathrm{d}Z}{Z-W} \frac{\partial f}{\partial Z_j}(Z) + \sum_{j=1}^{n} \frac{\mathrm{d}\bar{Z}_j \wedge \mathrm{d}Z}{Z-W} \frac{\partial f}{\partial \bar{Z}_j}(Z)\end{aligned}$$

since $\frac{1}{Z-W}$ is not only holomorphic, but also $\mathcal{A}$-holomorphic in Z. On the other hand, by Stokes' theorem:

$$\sum_{\ell=1}^{m} \epsilon_\ell \int_{\gamma_{\ell,\varepsilon}} \omega = \int_{\partial\Sigma} \omega - \int_{\Sigma_\varepsilon} \mathrm{d}\omega.$$

Furthermore,

$$\frac{\mathrm{d}Z}{Z-W} \stackrel{\text{def}}{=} \frac{\mathrm{d}Z}{(z-w) \oplus (Z - Y(w))} = \sum_{k=0}^{\nu-1} (Y(w) - X)^k \frac{\mathrm{d}Z}{(z-w)^{k+1}}.$$

Putting $f_k(Z) := (Y(w) - X)^k f(Z) \stackrel{\text{def}}{=} (Y(w) - X)^k f(z, X)$, $0 \le k \le \nu - 1$, a combination of Lemma 2.4.3 and Lemma 2.4.6 now yields

$$\begin{aligned}\lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int_{\gamma_{\ell,\varepsilon}} \omega &= \sum_{k=0}^{\nu-1} \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\gamma_{\ell,\varepsilon}} \frac{\mathrm{d}Z}{(z-w)^{k+1}} f_k(Z) = \sum_{k=0}^{\nu-1} \frac{1}{k!} \frac{\partial^k}{\partial u^k}\Big|_w \Big(\frac{\partial W_\ell}{\partial u}(u) f_k(W_\ell(u))\Big) \stackrel{\text{def}}{=} \\ &\stackrel{\text{def}}{=} \sum_{k=0}^{\nu-1} \frac{1}{k!} \frac{\partial^k}{\partial u^k}\Big|_w \Big(\frac{\partial W_\ell}{\partial u}(u) (Y(w) - Y_\ell(u))^k f(W_\ell(u))\Big) = \\ &= \sum_{k=0}^{\nu-1} \frac{1}{k!} \frac{\partial^k}{\partial u^k}\Big|_w \Big((Y(w) - Y_\ell(u))^k \frac{\partial W_\ell}{\partial u}(u) f(W_\ell(u))\Big) = \\ &= \sum_{k=0}^{\nu-1} \Big(-\frac{\partial Y_\ell}{\partial w}(w)\Big)^k \frac{\partial W_\ell}{\partial w}(w) f(W_\ell(w)) = \frac{1}{1 + \frac{\partial Y_\ell}{\partial w}(w)} \frac{\partial W_\ell}{\partial w}(w) f(W_\ell(w)) = \\ &= \frac{1}{1 + \frac{\partial Y_\ell}{\partial w}(w)} \Big(1 + \frac{\partial Y_\ell}{\partial w}(w)\Big) f(W_\ell(w)) = f(W_\ell(w)),\end{aligned}$$

where recall Eq. (2.4.2). Hence

$$\lim_{\varepsilon\to 0}\epsilon_\ell \sum_{\ell=1}^{m}\int_{\gamma_{\ell,\varepsilon}}\omega = \sum_{\ell=1}^{m}\epsilon_\ell \lim_{\varepsilon\to 0}\int_{\gamma_{\ell,\varepsilon}}\omega = \sum_{\ell=1}^{m}\epsilon_\ell f(W_\ell(w))$$

as desired. Finally, the second claim follows as before by taking $M := {}_{\mathcal{A}}\mathcal{A}$ and $f := 1_{\mathcal{A}}$. □

Remarks:

(1) In particular, we recover from Theorem 2.4.7 the rather well-known fact that

$$\deg(\chi_\Sigma|_S) \overset{\text{def}}{=} \sum_{\ell=1}^{m}\epsilon_\ell = \frac{1}{2\pi i}\int_{\partial\Sigma}\frac{\mathrm{d}Z}{Z-W} \overset{\text{def}}{=} \mathrm{Ind}_{\mathcal{A}}(\partial\Sigma, W) = \mathrm{ind}(\partial\Sigma^{\mathrm{sp}}, w) \in \mathbb{Z} \tag{2.4.10}$$

for all $w \in S^{\mathrm{sp}}$.

(2) Notice that unlike Eq. (2.4.6) the $\mathcal{A}$-holomorphic framework allows for Eq. (2.4.9) to lose the derivatives of the local parametrizations. Similarly, Eq. (2.4.10) has not only no derivatives of the local parametrizations, but also no surface integral terms, unlike Eq. (2.4.7).

(3) The astute will observe that Equation (2.4.10) is just a special case of differential-topological degree theory for the map χ_Σ, restricted to $S \subseteq \Sigma$.

(4) The last calculation in the above proof shows that we need $f\circ W$ to be at least of differentiability class $\mathscr{C}^\nu$ and W at least $\mathscr{C}^{\nu+1}$ in order to be able to invoke Lemma 2.4.3. This in turn traces back to estimating the Taylor series remainder in the proof of Lemma 2.4.3.

(5) Comparison with the Bochner–Martinelli (BM) formula:

  (i) In (BM) the domain of integration D has $\dim_{\mathbb{R}} D = 2n$ (hence $\dim_{\mathbb{R}}\partial D = 2n-1$), while here we have $\dim_{\mathbb{R}}\Sigma = 2$ (hence $\dim_{\mathbb{R}}\partial\Sigma = 1$).

  (ii) When f is smooth, both formulas involve both a boundary integral and a domain integral. However, when f is holomorphic, (BM) reduces to only a boundary integral involving only f, whereas in Eq. (2.4.9) there is a remaining surface integral term that also involves the first partial derivatives of f.

  (iii) The integral kernels in Eq. (2.4.9) are holomorphic for both integrals, whereas the (BM)-kernels are not holomorphic.

  (iv) When f is holomorphic, (BM) reproduces all the values of f over D, whereas Eq. (2.4.9) "reproduces" the values only over certain subsets of Σ (essentially the parts, where χ_Σ is a local diffeomorphism).

  (v) When the $\mathcal{A}$-transversality "covering" is "multi-sheeted", i.e. $p > 1$, then the LHS of Eq. (2.4.9) involves the (signed) values of f on all the "sheets".

(6) A very surprising Cauchy–Pompeiu formula for all $\mathcal{A} \in \mathsf{FdCAlg}_{\mathbb{R}}$ appears in [Sny90, Eq. 5.9 on p. 167], in particular without necessarily assuming that $\mathcal{A}$ carries an underlying complex structure. However, both the boundary kernel and the surface kernel involve powers of $\|\cdot\|_{\mathcal{A}}$ and hence are not themselves $\mathcal{A}$-differentiable or even holomorphic in the special case when $\mathcal{A}$ is defined over $\mathbb{C}$. Indeed, removing $\|\cdot\|_{\mathcal{A}}$ from the denominator *without passing to distributions and currents* in the absence of an underlying complex structure is likely impossible since $\|\cdot\|_{\mathcal{A}}$ is the most natural way to get denominator $\neq 0$ from non-zero *singular* elements and moreover $\mathrm{Sing}(\mathcal{A})^c \overset{\text{def}}{=} \mathcal{A}^\times$ is not path-connected, so there is literally no way around it. □

Corollary 2.4.8: Suppose that $m = 1$, i.e. we have only a single $\mathcal{A}$-transverse sheet (S, W), and without loss of generality that χ_Σ is orientation-preserving on S. Then:

(1) We have $\forall\, w \in S^{\mathrm{sp}}$:

$$f(W) = \frac{1}{2\pi i}\int_{\partial\Sigma}\frac{\mathrm{d}Z}{Z-W}f(Z) + \frac{1}{2\pi i}\sum_{j=2}^{n}\int_{\Sigma}\frac{\mathrm{d}Z\wedge\mathrm{d}Z_j}{Z-W}\frac{\partial f}{\partial Z_j}(Z) + \frac{1}{2\pi i}\sum_{j=1}^{n}\int_{\Sigma}\frac{\mathrm{d}Z\wedge\mathrm{d}\bar{Z}_j}{Z-W}\frac{\partial f}{\partial \bar{Z}_j}(Z).$$

(2) If additionally $f \in \mathcal{O}_{\mathbb{C}^n}(U, M)$, then $\forall\, w \in S^{\mathrm{sp}}$:

$$f(W) = \frac{1}{2\pi i}\int_{\partial\Sigma} \frac{\mathrm{d}Z}{Z-W} f(Z) + \frac{1}{2\pi i}\sum_{j=2}^{n}\int_{\Sigma} \frac{\mathrm{d}Z \wedge \mathrm{d}Z_j}{Z-W}\frac{\partial f}{\partial Z_j}(Z) =$$
$$= \frac{1}{2\pi i}\int_{\partial\Sigma} \frac{\mathrm{d}Z}{Z-W} f(Z) + \frac{1}{2\pi i}\sum_{j=2}^{n}\int_{\Sigma} \frac{\mathrm{d}Z \wedge \mathrm{d}z_j}{Z-W}\Big(\frac{\partial f}{\partial z_j}(Z) - a_j\frac{\partial f}{\partial z}(Z)\Big).$$

(3) If additionally $f \in \mathcal{O}_{\mathbb{C}^n}(U, M)$ and Σ is closed, then $\forall\, w \in S^{\mathrm{sp}}$:

$$f(W) = \frac{1}{2\pi i}\sum_{j=2}^{n}\int_{\Sigma} \frac{\mathrm{d}Z \wedge \mathrm{d}z_j}{Z-W}\Big(\frac{\partial f}{\partial z_j}(Z) - a_j\frac{\partial f}{\partial z}(Z)\Big).$$

Proof: To (1): immediate from Theorem 2.4.7.

To (2): special case of (1) and an immediate application of Eq. (2.4.8).

To (3): special case of (2). □

Example (Complex Dual Numbers)**:** Let $\mathcal{A} := \mathbb{C}[t]/(t^2) = \mathbb{C}[\epsilon]$, $\epsilon \neq 0$, $\epsilon^2 = 0$, hence $\nu = 2$, and write $Z = z_1 \oplus \epsilon z_2, W = w_1 \oplus \epsilon w_2 \in \mathcal{A}$. Let Σ be a compact orientable $\mathscr{C}^3$-surface in $\mathcal{A}$, $m = 1$, and let $(S \subseteq \Sigma, W = W(w_1) = w_1 \oplus w_2(w_1))$ be the corresponding $\mathcal{A}$-transverse neighbourhood with χ_Σ assumed orientation-preserving on S. Finally, let $f = f(Z) = f_1(z_1, z_2) + \epsilon f_2(z_1, z_2) \in \mathcal{O}_{\mathbb{C}^2}(U, \mathcal{A})$ for some open $U \supseteq \Sigma$. We are thus in the setting of Corollary 2.4.8 (2). We calculate:

$$\mathrm{RHS}_{\partial\Sigma} := \frac{1}{2\pi i}\int_{\partial\Sigma}\frac{\mathrm{d}Z}{Z-W} f(Z) =$$
$$= \frac{1}{2\pi i}\int_{\partial\Sigma}(f_1(Z) + \epsilon f_2(Z))\Big(\frac{1}{z_1-w_1} - \epsilon\frac{z_2-w_2}{(z_1-w_1)^2}\Big)(\mathrm{d}z_1 + \epsilon\mathrm{d}z_2) =$$
$$= \frac{1}{2\pi i}\int_{\partial\Sigma} f_1(Z)\frac{\mathrm{d}z_1}{z_1-w_1} + \epsilon\frac{1}{2\pi i}\int_{\partial\Sigma}\Big(f_2(Z)\frac{\mathrm{d}z_1}{z_1-w_1} + f_1(Z)\Big(\frac{\mathrm{d}z_2}{z_1-w_1} - \frac{z_2-w_2}{z_1-w_1}\mathrm{d}z_1\Big)\Big)$$

and

$$\mathrm{RHS}_{\Sigma} := \frac{1}{2\pi i}\int_{\Sigma}\frac{\mathrm{d}Z \wedge \mathrm{d}z_2}{Z-W}\Big(\frac{\partial f}{\partial z_2}(Z) - \epsilon\frac{\partial f}{\partial z_1}(Z)\Big) =$$
$$= \frac{1}{2\pi i}\int_{\Sigma}\frac{(\mathrm{d}z_1 + \epsilon\mathrm{d}z_2)\wedge\mathrm{d}z_2}{(z_1-w_1)+\epsilon(z_2-w_2)}\Big(\frac{\partial f_1}{\partial z_2}(Z) + \epsilon\frac{\partial f_2}{\partial z_2}(Z) - \epsilon\Big(\frac{\partial f_1}{\partial z_1}(Z) + \epsilon\frac{\partial f_2}{\partial z_1}(Z)\Big)\Big) =$$
$$= \frac{1}{2\pi i}\int_{\Sigma}\Big(\frac{1}{z_1-w_1} - \epsilon\frac{z_2-w_2}{(z_1-w_1)^2}\Big)\Big(\frac{\partial f_1}{\partial z_2}(Z) + \epsilon\Big(\frac{\partial f_2}{\partial z_2}(Z) - \frac{\partial f_1}{\partial z_1}(Z)\Big)\Big)\mathrm{d}z_1\wedge\mathrm{d}z_2 =$$
$$= \frac{1}{2\pi i}\int_{\Sigma}\frac{\partial f_1}{\partial z_2}(Z)\frac{\mathrm{d}z_1\wedge\mathrm{d}z_2}{z_1-w_1} + \epsilon\frac{1}{2\pi i}\int_{\Sigma}\Big(\frac{\partial f_2}{\partial z_2}(Z) - \frac{\partial f_1}{\partial z_1}(Z) - \frac{z_2-w_2}{z_1-w_1}\frac{\partial f_1}{\partial z_2}(Z)\Big)\frac{\mathrm{d}z_1\wedge\mathrm{d}z_2}{z_1-w_1}.$$

Hence $\forall\, W \in S$:

$$f_1(W) \overset{\text{def}}{=} f_1(w_1, w_2(w_1)) = \frac{1}{2\pi i}\int_{\partial\Sigma} f_1(Z)\frac{\mathrm{d}z_1}{z_1-w_1} + \frac{1}{2\pi i}\int_{\Sigma}\frac{\partial f_1}{\partial z_2}(Z)\frac{\mathrm{d}z_1\wedge\mathrm{d}z_2}{z_1-w_1} \tag{2.4.11}$$

and

$$\begin{aligned} f_2(W) \overset{\text{def}}{=} f_2(w_1, w_2(w_1)) &= \frac{1}{2\pi i}\int_{\partial\Sigma}\Big(f_2(Z)\frac{\mathrm{d}z_1}{z_1-w_1} + f_1(Z)\Big(\frac{\mathrm{d}z_2}{z_1-w_1} - \frac{z_2-w_2}{z_1-w_1}\mathrm{d}z_1\Big)\Big) + \\ &+ \frac{1}{2\pi i}\int_{\Sigma}\Big(\frac{\partial f_2}{\partial z_2}(Z) - \frac{\partial f_1}{\partial z_1}(Z) - \frac{z_2-w_2}{z_1-w_1}\frac{\partial f_1}{\partial z_2}(Z)\Big)\frac{\mathrm{d}z_1\wedge\mathrm{d}z_2}{z_1-w_1}. \end{aligned} \tag{2.4.12}$$

In particular, we get two very different formulas for "reproducing" a $\mathbb{C}$-valued holomorphic function of two complex variables. The reader should feel free to verify that of course $f_2(W)$ does not actually depend on f_1 in the last equation (this is perhaps not quite obvious). This in turn yields uncountably infinitely many holomorphic "reproducing kernels", one for each f_1, at least in principle. □

A. Structure Tensors

A.1. The Structure Tensor of a Finite-Dimensional Algebra

The purpose of this and the next two sections is the careful bookkeeping, in a few easy lemmas, of various relations involving the structure constants of finite-dimensional algebras. As with any bookkeeping, it will be thorough and boring. We will omit the most trivial proofs. We start with some general definitions.

Definition A.1.1: A finite-dimensional based vector space is a finite-dimensional vector space V together with a choice of an ordered basis $e_1, \ldots, e_n$.

Let K be an arbitrary field, $\mathcal{A} := (\mathcal{A}, a_1, \ldots, a_n)$ a finite-dimensional based K-algebra, and $V := (V, e_1, \ldots, e_d) \cong K^d$ a d-imensional based vector K-space together with a K-bilinear map $\beta\colon \mathcal{A} \times V \to V$. Write $av := \beta(a, v)$ and for $a \in \mathcal{A}$ define $\lambda_a\colon V \to V$, $v \mapsto av$. Thus β induces a morphism of vector K-spaces $\lambda\colon \mathcal{A} \to \mathrm{End}_K(V) \cong \mathrm{M}_d(K)$, $a \mapsto \lambda_a$, which is a monomorphism $\mathcal{A} \hookrightarrow \mathrm{End}_K(V) \cong \mathrm{M}_d(K)$ iff β is non-degenerate. We define the matrices $A_i := \lambda(a_i) \in \mathrm{M}_d(K)$, $1 \leq i \leq n$.

Definition A.1.2: The structure tensor[1] (structure constants) $(\alpha^i_{jk})_{1 \leq i,j,k \leq n}$ of $\mathcal{A}$ with respect to the basis $a_1, \ldots, a_n$ is given by

$$a_j a_k = \sum_{r=1}^{n} \alpha^r_{jk} a_r \tag{A.1.1}$$

for all $1 \leq j, k \leq n$, or more succinctly,

$$a_j \cdot (a_1, \ldots, a_n) = (a_1, \ldots, a_n) \cdot (\alpha^i_{jk})_{1 \leq i,k \leq n} \tag{A.1.2}$$

for all $1 \leq j \leq n$.

Hence the structure tensor represents the multiplication in $\mathcal{A}$ from left with respect to the given basis. Notice that we also have

$$(a_1, \ldots, a_n) \cdot a_j = (a_1, \ldots, a_n) \cdot (\alpha^i_{kj})_{1 \leq i,k \leq n}, \tag{A.1.3}$$

so there is no need to distinguish between left and right structure tensor, it already contains all the information about multiplication in $\mathcal{A}$. Obviously, $\mathcal{A}$ is uniquely determined by its structure constants. Furthermore, if $\mathcal{A}$ is unital with unit $1_{\mathcal{A}}$, we set:

$$1_{\mathcal{A}} =: \sum_{r=1}^{n} \epsilon_r a_r \tag{A.1.4}$$

Lemma A.1.3: The following are equivalent:

(i) $\mathcal{A}$ is unital with unit $1_{\mathcal{A}}$;

(ii) $\forall\, 1 \leq i \leq n\colon \lambda(1_{\mathcal{A}} a_i) = \lambda(a_i 1_{\mathcal{A}}) = \lambda(a_i)$, provided β is non-degenerate;

(iii) $\exists\, (\epsilon_r)_{1 \leq r \leq n} \subseteq K\ \forall\, 1 \leq i, k \leq n\colon$

$$\sum_{r=1}^{n} \epsilon_r \alpha^i_{rk} = \sum_{r=1}^{n} \epsilon_r \alpha^i_{kr} = \delta^i_k \tag{A.1.5}$$

In the special case $a_1 = 1_{\mathcal{A}}$, we have $\forall\, 1 \leq i, k \leq n\colon \alpha^i_{1k} = \alpha^i_{k1} = \delta^i_k$.

[1] later on, we may let them vary and we will give them a differential-geometric interpretation;

Proof: (iii) is restatement of $\forall\, 1 \le k \le n\colon 1_{\mathcal{A}} a_k = a_k 1_{\mathcal{A}} = a_k$. □

Lemma A.1.4: The following are equivalent:

(i) $\mathcal{A}$ is commutative;

(ii) $\forall\, 1 \le i, j \le n\colon \lambda(a_i a_j) = \lambda(a_j a_i)$, provided β is non-degenerate;

(iii) $\forall\, 1 \le i, j, k \le n\colon \alpha^i_{jk} = \alpha^i_{kj}$.

Proof: $\mathcal{A}$ commutative $\Leftrightarrow$ $\mathcal{A}$ commutative at the basis. □

Lemma A.1.5: The following are equivalent:

(i) $\mathcal{A}$ is associative;

(ii) $\forall\, 1 \le i, j, k \le n\colon \lambda(a_i(a_j a_k)) = \lambda((a_i a_j)a_k)$, provided β is non-degenerate;

(iii) $\forall\, 1 \le i, j, k, \ell \le n$:

$$\sum_{r=1}^{n} \alpha^r_{jk}\alpha^i_{r\ell} = \sum_{r=1}^{n} \alpha^r_{k\ell}\alpha^i_{jr} \tag{A.1.6}$$

Proof: Likewise, $\mathcal{A}$ associative $\Leftrightarrow$ $\mathcal{A}$ associative at the basis. □

But one can do better. Namely, let $V := \mathcal{A}$ and let $\beta := \mu\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $(a, b) \mapsto ab$, be the multiplication in $\mathcal{A}$. In this case, $\lambda\colon \mathcal{A} \to \operatorname{End}_{\mathsf{Vec}}(\mathcal{A}) \cong \mathrm{M}_n(K)$ is called the left regular representation of $\mathcal{A}$. Additionally, μ also induces K-linear maps $\rho_b\colon \mathcal{A} \to \mathcal{A}$, $a \mapsto ab$. Then $\rho\colon \mathcal{A} \to \operatorname{End}_{\mathsf{Vec}}(\mathcal{A})$, $b \mapsto \rho_b$, is called the right regular representation of $\mathcal{A}$. Notice that by definition neither λ nor ρ are assumed to have any additional properties beyond K-linearity.

Lemma A.1.6: In the setup $(V, \beta) = (\mathcal{A}, \mu)$ the following are equivalent:

(i) $\mathcal{A}$ is associative;

(ii) The K-linear map $\lambda\colon \mathcal{A} \to \operatorname{End}_{\mathsf{Vec}}(\mathcal{A}) \cong \mathrm{M}_n(K)$ is multiplicative. In other words, the left regular representation λ is a morphism of (non-unital) associative K-algebras;

(iii) $\forall\, 1 \le i, j \le n\colon \lambda(a_i a_j) = A_i A_j$;

(iv) λ induces an epimorphism $\bar{\lambda}\colon \mathcal{A} \twoheadrightarrow K[A_1, \ldots, A_n] \subseteq \mathrm{M}_n(K)$ of (non-unital) associative K-algebras. □

Lemma A.1.7: In the setup $(V, \beta) = (\mathcal{A}, \mu)$ the following are equivalent:

(i) $\mathcal{A}$ is associative;

(ii) The K-linear map $\rho\colon \mathcal{A} \to \operatorname{End}_{\mathsf{Vec}}(\mathcal{A}) \cong \mathrm{M}_n(K)$ is anti-multiplicative. In other words, the right regular representation ρ is an antihomomorphism of (non-unital) associative K-algebras;

(iii) $\forall\, 1 \le i, j \le n\colon \rho(a_i a_j) = \rho(a_j)\rho(a_i)$;

(iv) ρ induces an anti-epimorphism $\bar{\rho}\colon \mathcal{A} \twoheadrightarrow K[A_1, \ldots, A_n] \subseteq \mathrm{M}_n(K)$ of (non-unital) associative K-algebras. □

Definition A.1.8: A morphism $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ of unital K-algebras, also called a **unital** morphism, is a morphism of K-algebras such that $\varphi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$.

Corollary A.1.9: If $\mathcal{A}$ is associative and unital, then:

(1) $\lambda\colon \mathcal{A} \hookrightarrow \operatorname{End}_{\mathsf{Vec}}(\mathcal{A}) \cong \mathrm{M}_n(K)$ is a monomorphism of **unital** associative K-algebras, inducing $\bar{\lambda}\colon \mathcal{A} \cong K[A_1, \ldots, A_n] \subseteq \mathrm{M}_n(K)$;

(2) $\rho\colon \mathcal{A} \hookrightarrow \operatorname{End}_{\mathsf{Vec}}(\mathcal{A}) \cong \mathrm{M}_n(K)$ is an anti-monomorphism of **unital** associative K-algebras, inducing an anti-isomorphism $\bar{\rho} : \mathcal{A} \xrightarrow{\sim} K[A_1, \ldots, A_n] \subseteq \mathrm{M}_n(K)$.

Lemma A.1.10: In the setup $(V, \beta) = (\mathcal{A}, \mu)$ suppose $\mathcal{A}$ is commutative. Equivalent:

(i) $\mathcal{A}$ is unital;

(ii) There exist $\epsilon_1, \dots, \epsilon_n \in K$ such that:

$$\sum_{r=1}^{n} \epsilon_r A_r = I_n \tag{A.1.7}$$

Proof: This is just Eq. (A.1.5) in the case $\mathcal{A}$ is commutative. □

Remark: Notice that it is not required for $\lambda\colon \mathcal{A} \to \mathrm{M}_n(K)$ to be a morphism of associative K-algebras.

With all this in mind, let us now have another look at commutativity.

Lemma A.1.11: In the setup $(V, \beta) = (\mathcal{A}, \mu)$ suppose $\mathcal{A}$ is associative. If $\mathcal{A}$ is also commutative, then

$$A_j A_k = A_k A_j \tag{A.1.8}$$

in $\mathrm{M}_n(K)$ for all $1 \leq j, k \leq n$, which in turn is equivalent to

$$\sum_{r=1}^{n} \alpha^i_{jr} \alpha^r_{k\ell} = \sum_{r=1}^{n} \alpha^i_{kr} \alpha^r_{j\ell} \tag{A.1.9}$$

for all $1 \leq i, j, k, \ell \leq n$. □

Now, a comparison between Eq. (A.1.9) and Eq. (A.1.6) gives an amusing partial converse:

Lemma A.1.12: In the setup $(V, \beta) = (\mathcal{A}, \mu)$ let $\mathcal{A}$ be a commutative (but not necessarily associative) K-algebra such that $A_j A_k = A_k A_j$ in $\mathrm{M}_n(K)$ for all $1 \leq j, k \leq n$. Then $\mathcal{A}$ is in fact associative.

Proof: Since $A_j A_\ell = A_\ell A_j$, we have Eq. (A.1.9) for the (i, k)-th element in the form

$$\sum_{r=1}^{n} \alpha^i_{jr} \alpha^r_{\ell k} = \sum_{r=1}^{n} \alpha^i_{\ell r} \alpha^r_{jk}$$

for all $1 \leq i, k \leq n$. By commutativity of $\mathcal{A}$ itself, $\alpha^r_{\ell k} = \alpha^r_{k\ell}$ and $\alpha^i_{\ell r} = \alpha^i_{r\ell}$, hence:

$$\sum_{r=1}^{n} \alpha^i_{jr} \alpha^r_{k\ell} = \sum_{r=1}^{n} \alpha^i_{r\ell} \alpha^r_{jk},$$

which is precisely Eq. (A.1.6). □

Remark: By commutativity of $\mathcal{A}$, we necessarily have $\lambda(a_j a_k) = \lambda(a_k a_j)$ on the one hand, and by assumption, we have $\lambda(a_j)\lambda(a_k) = \lambda(a_k)\lambda(a_j)$ on the other hand. Lemma A.1.12 tells us that then we get mulitplicativity of λ for free.

A.2. The Structure Constants of a Morphism

Let $\mathcal{B} := (\mathcal{B}, b_1, \dots, b_m)$ be another finite-dimensional based K-algebra with corresponding structure constants $(\beta^i_{jk})_{1\le i,j,k\le m}$ and matrices $B_j := (\beta^i_{jk})_{1\le i,k\le m}$ representing the left multiplication by b_j, $1 \le j \le m$. Let $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ a *non-unital* K-algebra morphism with representing matrix $\Phi := (\phi^j_k)_{1\le j\le m, 1\le k\le n}$, that is

$$\forall\ 1 \le k \le n \colon \varphi(a_k) \stackrel{\text{def}}{=} \sum_{j=1}^{m} \phi^j_k b_j, \tag{A.2.1}$$

or more succintly,

$$\varphi(a_1, \dots, a_n) \stackrel{\text{def}}{=} (\varphi(a_1), \dots, \varphi(a_n)) = (b_1, \dots, b_m)\Phi$$

For $V := \mathcal{B}$, φ induces in general two different K-bilinear maps $\beta \colon \mathcal{A} \times \mathcal{B} \to \mathcal{B}$, $ab := \varphi(a)b$, and $\tilde{\beta} \colon \mathcal{A} \times \mathcal{B} \to \mathcal{B}$, $\tilde{\beta}(a, b) := ba := b\varphi(a)$. These in turn give rise to K-linear maps

$$\widetilde{\lambda}_a \colon \mathcal{B} \to \mathcal{B},\ b \mapsto ab$$
$$\widetilde{\rho}_a \colon \mathcal{B} \to \mathcal{B},\ b \mapsto ba$$

and

$$\widetilde{\lambda} \colon \mathcal{A} \to \mathrm{End}_{\mathsf{Vec}}(\mathcal{B}) \cong \mathrm{M}_m(K),\ a \mapsto \widetilde{\lambda}_a$$
$$\widetilde{\rho} \colon \mathcal{A} \to \mathrm{End}_{\mathsf{Vec}}(\mathcal{B}) \cong \mathrm{M}_m(K),\ a \mapsto \widetilde{\rho}_a,$$

respectively. We reserve the notations λ and ρ for the regular representations of $\mathcal{B}$. We briefly mention that there is also an induced pair of K-linear maps $\mathcal{A} \to \mathcal{B}$, namely $a \mapsto ab$ and $a \mapsto ba$. However, these are nothing else but the compositions $\widetilde{\rho}_b \circ \varphi$ and $\widetilde{\lambda}_b \circ \varphi$, and we get K-linear maps $\mathcal{B} \to \mathrm{Hom}_{\mathsf{Vec}}(\mathcal{A}, \mathcal{B}) \cong \mathrm{Mat}_{m\times n}(K)$, $b \mapsto \widetilde{\rho}_b \circ \varphi$ and $b \mapsto \widetilde{\lambda}_b \circ \varphi$, respectively. Moreover, we have commutative diagrams of K-linear maps:

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\widetilde{\lambda}"] \arrow[dr, "\varphi"'] & & \mathrm{End}_{\mathsf{Vec}}(\mathcal{B}) \\ & \mathcal{B} \arrow[ur, "\lambda"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{A} \arrow[rr, "\widetilde{\rho}"] \arrow[dr, "\varphi"'] & & \mathrm{End}_{\mathsf{Vec}}(\mathcal{B}) \\ & \mathcal{B} \arrow[ur, "\rho"'] & \end{tikzcd} \tag{A.2.2}$$

Define

$$\begin{aligned} \Gamma_j &:= (\gamma^i_{jk})_{1\le i,k\le m} := \widetilde{\lambda}(a_j) \\ \widehat{\Gamma}_j &:= (\widehat{\gamma}^i_{jk})_{1\le i,k\le m} := \widetilde{\rho}(a_j) \end{aligned} \tag{A.2.3}$$

to be the representing matrices of the left and right multiplication by the basis element a_j, $1 \le j \le n$. That is

$$\begin{aligned} a_j(b_1, \dots, b_m) &= (b_1, \dots, b_m)\Gamma_j \\ (b_1, \dots, b_m)a_j &= (b_1, \dots, b_m)\tilde{\Gamma}_j \end{aligned} \tag{A.2.4}$$

for all $1 \le j \le n$.

Definition A.2.1: $(\gamma^i_{jk})_{1\le i,k\le m, 1\le j\le n}$ and $(\widehat{\gamma}^i_{jk})_{1\le i,k\le m, 1\le j\le n}$ are called the left and right structure constants (left and right structure tensor) of the morphism $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$.

Remark: Notice that, unlike before, the definition of γ^i_{jk} is not symmetric in j and k, hence the need to introduce $\widehat{\gamma}^i_{jk}$. However, in the special case $\mathcal{A} \xrightarrow{\mathrm{id}} \mathcal{A}$, we do have $(\gamma^i_{jk}) = (\alpha^i_{jk})$ and $(\widehat{\gamma}^i_{jk}) = (\alpha^i_{kj})$.

Lemma A.2.2: The left and right structure constants of φ are computed from its representation matrix and the structure tensor of $\mathcal{B}$ as follows:

$$\begin{aligned}\gamma^i_{jk} &= \sum_{r=1}^m \beta^i_{rk}\phi^r_j \\ \widehat{\gamma}^i_{jk} &= \sum_{r=1}^m \beta^i_{kr}\phi^r_j\end{aligned} \tag{A.2.5}$$

for all $1 \le j \le n$, $1 \le i, k \le m$.

Proof: $a_j b_k \stackrel{\text{def}}{=} \varphi(a_j) b_k$ and $b_k a_j \stackrel{\text{def}}{=} b_k \varphi(a_j)$. □

Lemma A.2.3: We have $\forall\, 1 \le j, k \le n\ \forall\, 1 \le i \le m$:

$$\sum_{r=1}^m \gamma^i_{jr}\phi^r_k = \sum_{r=1}^m \tilde{\gamma}^i_{kr}\phi^r_j \tag{A.2.6}$$

Proof: $a_j \varphi(a_k) \stackrel{\text{def}}{=} \varphi(a_j)\varphi(a_k) \stackrel{\text{def}}{=} \varphi(a_j) a_k$ □

Lemma A.2.4: The following are equivalent:

(i) φ is multiplicative;

(ii) $\forall\, 1 \le j, k \le n\ \forall\, 1 \le i \le m$:

$$\sum_{r=1}^n \phi^i_r \alpha^r_{jk} = \sum_{s=1}^m \gamma^i_{js}\phi^s_k; \tag{A.2.7}$$

(iii) $\forall\, 1 \le j \le n$: $\Phi A_j = \Gamma_j \Phi$.

Proof: φ multiplicative $\Leftrightarrow \forall\, 1 \le j, k \le n : \varphi(a_j a_k) = \varphi(a_j)\varphi(a_k)$. □

If $\mathcal{B}$ is unital, set

$$1_{\mathcal{B}} =: \sum_{s=1}^m \eta_s b_s, \tag{A.2.8}$$

that is $\forall\, 1 \le i, k \le m$:

$$\sum_{s=1}^m \eta_s \beta^i_{sk} = \sum_{s=1}^m \eta_s \beta^i_{ks} = \delta^i_k \tag{A.2.9}$$

Lemma A.2.5: Let $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ be unital. Then we have the relations:

(1) $\forall\, 1 \le i, k \le m$:

$$\sum_{r=1}^n \epsilon_r \gamma^i_{rk} = \sum_{r=1}^n \epsilon_r \widehat{\gamma}^i_{rk} = \delta^i_k \tag{A.2.10}$$

(2) $\forall\, 1 \le i \le m\ \forall\, 1 \le j \le n$:

$$\sum_{s=1}^m \eta_s \gamma^i_{js} = \sum_{s=1}^m \eta_s \widehat{\gamma}^i_{js} = \phi^i_j \tag{A.2.11}$$

(3) $\forall\, 1 \le i \le m$:

$$\eta_i = \sum_{r=1}^n \epsilon_r \phi^i_r \tag{A.2.12}$$

Proof: To (1): This is a restatement of $1_{\mathcal{A}} b_k = b_k 1_{\mathcal{A}} = b_k$.

To (2): This is a restatement of $a_j 1_{\mathcal{B}} = 1_{\mathcal{B}} a_j = \varphi(a_j)$.

To (3): This is just $\varphi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. □

Lemma A.2.6: The relation

$$\forall\, 1 \leq i, j, k, \ell \leq m\colon \sum_{r=1}^{m} \beta_{jk}^{r} \beta_{r\ell}^{i} = \sum_{r=1}^{m} \beta_{k\ell}^{r} \beta_{jr}^{i} \tag{A.2.13}$$

(1) implies the relation

$$\forall\, 1 \leq j \leq n\ \forall\, 1 \leq i, k, \ell \leq m\colon \sum_{r=1}^{m} \beta_{r\ell}^{i} \gamma_{jk}^{r} = \sum_{r=1}^{m} \gamma_{jr}^{i} \beta_{k\ell}^{r} \tag{A.2.14}$$

(2) implies the relation

$$\forall\, 1 \leq j \leq n\ \forall\, 1 \leq k \leq m\colon B_k \widehat{\Gamma}_j = \widehat{\Gamma}_j B_k \tag{A.2.15}$$

in $\mathrm{M}_m(K)$, or more explicitly,

$$\forall\, 1 \leq j \leq n\ \forall\, 1 \leq i, k, \ell \leq m\colon \sum_{r=1}^{m} \beta_{kr}^{i} \widehat{\gamma}_{j\ell}^{r} = \sum_{r=1}^{m} \widehat{\gamma}_{jr}^{i} \beta_{k\ell}^{r} \tag{A.2.16}$$

(3) implies the relation

$$\forall\, 1 \leq j \leq n\ \forall\, 1 \leq i, k, \ell \leq m\colon \sum_{r=1}^{m} \beta_{r\ell}^{i} \widehat{\gamma}_{jk}^{r} = \sum_{r=1}^{m} \beta_{kr}^{i} \gamma_{j\ell}^{r} \tag{A.2.17}$$

(4) implies the relation

$$\forall\, 1 \leq j \leq n\ \forall\, 1 \leq k \leq m\colon \Gamma_k \widehat{\Gamma}_j = \widehat{\Gamma}_j \Gamma_k \tag{A.2.18}$$

in $\mathrm{M}_m(K)$, or more explicitly,

$$\forall\, 1 \leq j, k \leq n\ \forall\, 1 \leq i, \ell \leq m\colon \sum_{r=1}^{m} \widehat{\gamma}_{kr}^{i} \gamma_{jl}^{r} = \sum_{r=1}^{m} \gamma_{jr}^{i} \widehat{\gamma}_{kl}^{r} \tag{A.2.19}$$

(5) together with multiplicativity of φ (A.2.7) implies the relation

$$\forall\, 1 \leq j, k \leq n\ \forall\, 1 \leq i, \ell \leq m\colon \sum_{r=1}^{n} \gamma_{r\ell}^{i} \alpha_{jk}^{r} = \sum_{s=1}^{m} \gamma_{js}^{i} \gamma_{k\ell}^{s} \tag{A.2.20}$$

(6) together with multiplicativity of φ (A.2.7) implies the relation

$$\forall\, 1 \leq j, k \leq n\ \forall\, 1 \leq i, \ell \leq m\colon \sum_{r=1}^{n} \widehat{\gamma}_{r\ell}^{i} \alpha_{jk}^{r} = \sum_{s=1}^{m} \widehat{\gamma}_{ks}^{i} \widehat{\gamma}_{j\ell}^{s} \tag{A.2.21}$$

Conversely, if φ is an epimorphism, then any one of these implies Eq. (A.2.13), and therefore in that case all are equivalent.

Proof: Equation (A.2.13) is associativity of $\mathcal{B}$.

To (1): $(a_j b_k) b_\ell = a_j (b_k b_\ell), 1 \leq j \leq n, 1 \leq k, \ell \leq m$.

To (2): $b_k (b_\ell a_j) = (b_k b_\ell) a_j, 1 \leq j \leq n, 1 \leq k, \ell \leq m$.

To (3): $(b_k a_j) b_\ell = b_k (a_j b_\ell), 1 \leq j \leq n, 1 \leq k, \ell \leq m$.

To (4): $(a_j b_\ell) a_k = a_j (b_\ell a_k), 1 \leq j, k \leq n, 1 \leq \ell \leq m$.

To (5): $(a_j a_k) b_\ell = a_j (a_k b_\ell), 1 \leq j, k \leq n, 1 \leq \ell \leq m$.

To (6): $b_\ell (a_j a_k) = (b_\ell a_j) a_k, 1 \leq j, k \leq n, 1 \leq \ell \leq m$. □

Corollary A.2.7: If $\mathcal{B}$ is associative and commutative, then $\forall\, 1 \leq \ell \leq m\ \forall\, 1 \leq j \leq n$:

$$B_\ell \Gamma_j = \Gamma_j B_\ell \tag{A.2.22}$$

in $\mathrm{M}_m(K)$.

Proof: By commutativity of $\mathcal{B}$, $\beta^i_{r\ell} = \beta^i_{\ell r}$ and $\beta^r_{k\ell} = \beta^r_{\ell k}$. Inserting these in Eq. (A.2.14), we get:

$$\sum_{r=1}^{m} \beta^i_{\ell r}\gamma^r_{jk} = \sum_{r=1}^{m} \gamma^i_{jr}\beta^r_{\ell k},$$

which is precisely the (i,k)-th element of the desired matrix identity. □

Remarks:

(1) Notice that in comparison $B_k\widehat{\Gamma}_j = \widehat{\Gamma}_j B_k$ is satisfied regardless of commutativity of $\mathcal{B}$.

(2) Similarly, we have $\Gamma_k\widehat{\Gamma}_j = \widehat{\Gamma}_j\Gamma_k$, even though neither $\mathcal{A}$ nor $\mathcal{B}$ is assumed commutative.

Lemma A.2.8: Let $\mathcal{B}$ associative. Then multiplicativity of φ (A.2.7) implies:

(1) $\widetilde{\lambda}\colon \mathcal{A} \to \mathrm{End}_{\mathsf{Vec}}(\mathcal{B})$ is multiplicative, i.e. $\widetilde{\lambda}$ is a morphism of non-unital K-algebras;

(2) $\widetilde{\rho}\colon \mathcal{A} \to \mathrm{End}_{\mathsf{Vec}}(\mathcal{B})$ is anti-multiplicative, i.e. $\widetilde{\rho}$ is antihomomorphism of non-unital K-algebras. □

Proof: If $\mathcal{B}$ is associative, then the left and right regular representations are (anti-)multiplicative. Claim then follows by the commutative diagrams Eq. (A.2.2). □

Lemma A.2.9: Let φ be epic. If $\widetilde{\lambda}\colon \mathcal{A} \to \mathrm{End}_{\mathsf{Vec}}(\mathcal{B}) \cong \mathrm{M}_m(K)$ is multiplicative, then $\mathcal{B}$ is associative.

Proof: By the commutative diagram Eq. (A.2.29), it follows that the left regular representation λ is multiplicative, hence $\mathcal{B}$ is associative. □

Lemma A.2.10 (Partial converse of Corollary A.2.7 and Lemma A.2.8)**:** Let φ be epic and $\mathcal{B}$ commutative $(\beta^i_{jk}) = (\beta^i_{kj})$ such that $\forall\, 1 \le \ell \le m\ \forall\, 1 \le j \le n:$

$$B_\ell\Gamma_j = \Gamma_j B_\ell$$

Then multiplicativity of φ (A.2.7) implies that $\mathcal{B}$ is associative, i.e. the relation $\forall\, 1 \le i,j,k,\ell \le m$:

$$\sum_{r=1}^{m} \beta^r_{jk}\beta^i_{r\ell} = \sum_{r=1}^{m} \beta^r_{k\ell}\beta^i_{jr}$$

Proof: We use a coordinate-free approach. We have $\forall\, 1 \le j,k \le n\ \forall\, 1 \le \ell \le m$:

$$\begin{aligned}\widetilde{\lambda}(a_j a_k)(b_\ell) &= \lambda(b_\ell)\big(\varphi(a_j a_k)\big) = \lambda(b_\ell)\big(\varphi(a_j)\varphi(a_k)\big) = \lambda(b_\ell)\Big(\widetilde{\lambda}(a_j)\big(\varphi(a_k)\big)\Big) = \\ &= \widetilde{\lambda}(a_j)\Big(\lambda(b_\ell)\big(\varphi(a_k)\big)\Big) = \widetilde{\lambda}(a_j)\big(\widetilde{\lambda}(a_k)(b_\ell)\big) = \big(\widetilde{\lambda}(a_j)\widetilde{\lambda}(a_k)\big)(b_\ell),\end{aligned}$$

i.e. $\widetilde{\lambda}(a_j a_k) = \widetilde{\lambda}(a_j)\widetilde{\lambda}(a_k)$ in $\mathrm{End}_{\mathsf{Vec}}(\mathcal{B})$. Since φ is epic, multiplicativity of $\widetilde{\lambda}$ implies associativity of $\mathcal{B}$. □

Remark: In particular, notice that there is no requirement for $\mathcal{A}$ to be commutative or associative nor that $B_j B_\ell = B_\ell B_j, 1 \le j,\ell \le m$ (cfg. Lemma A.1.12).

Unless $\ker\varphi$ is trivial, β and $\tilde{\beta}$ are never non-degenerate (in particular, $\widetilde{\lambda}$ and $\widetilde{\rho}$ are not injective), which somewhat limits our options of testing the basic properties of $\mathcal{A}$ against $\mathcal{B}$ or $\mathrm{End}_{\mathsf{Vec}}(\mathcal{B})$. Still:

Lemma A.2.11: The relation

$$\forall\, 1 \le i,j,k \le m\colon \beta^i_{jk} = \beta^i_{kj}$$

implies the relations:

(1) $\forall\, 1 \leq i \leq m\ \forall\, 1 \leq j,k \leq n$:

$$\sum_{r=1}^{n} \phi_r^i \alpha_{jk}^r = \sum_{r=1}^{n} \phi_r^i \alpha_{kj}^r \tag{A.2.23}$$

(2) $\forall\, 1 \leq j \leq n\ \forall\, 1 \leq i,k \leq m$: $\gamma_{jk}^i = \widehat{\gamma}_{jk}^i$.

Proof: The first relation is just commutativity of $\mathcal{B}$.

To (1): $\forall\, 1 \leq j,k \leq n$: $\varphi(a_j a_k) = \varphi(a_k a_j)$ and see LHS of Eq. (A.2.7).

To (2): $a_j b_k = b_k a_j$. □

Remark: Notice that $\mathcal{A}$ is not assumed commutative.

In a similar spirit:

Lemma A.2.12: The relation

$$\forall\, 1 \leq i,j,k \leq n\colon \alpha_{jk}^i = \alpha_{kj}^i$$

together with multiplicativity of φ (A.2.7) implies the relation $\forall\, 1 \leq i \leq m\ \forall\, 1 \leq j,k \leq n$:

$$\sum_{r=1}^{m} \gamma_{jr}^i \phi_k^r = \sum_{r=1}^{m} \gamma_{kr}^i \phi_j^r \tag{A.2.24}$$

Conversely, if φ is a monomorphism, then multiplicativity of φ (A.2.7) implies equivalence between both relations.

Proof: The first relation is just commutativity of $\mathcal{A}$. Then $\forall\, 1 \leq j,k \leq n$: $\varphi(a_j)\varphi(a_k) = \varphi(a_k)\varphi(a_j)$ and see RHS of Eq. (A.2.7). □

Remark: Notice once again that $\mathcal{B}$ is not assumed commutative.

Lemma A.2.13: Let φ be monic. Then the relation

$$\forall\, 1 \leq j \leq n\ \forall\, 1 \leq i,k \leq m\colon \gamma_{jk}^i = \widehat{\gamma}_{jk}^i$$

implies the relation

$$\forall\, 1 \leq i,j,k \leq n\colon \alpha_{jk}^i = \alpha_{kj}^i$$

Proof: The first condition means $\forall\, 1 \leq j \leq n\ \forall\, 1 \leq k \leq m$: $a_j b_k = b_k a_j$. Indeed, since φ is injective, for $\mathcal{A}$ to be commutative, it suffices that the image elements of φ commute with the basis elements of $\mathcal{B}$. □

Lemma A.2.14: Let $\mathcal{A}$ and $\mathcal{B}$ unital with $a_1 = 1_{\mathcal{A}}$ and $b_1 = 1_{\mathcal{B}}$. Then $\varphi(1_{\mathcal{A}}) = 1_{\mathcal{B}} \Rightarrow \forall\, 1 \leq j \leq m$: $\phi_1^j = \delta_1^j \Rightarrow \forall\, 1 \leq i,k \leq m$: $\gamma_{1k}^i = \widehat{\gamma}_{1k}^i = \delta_k^i$. □

We have a partial converse:

Lemma A.2.15: Let φ be monic. Suppose that $a_1 \in \mathcal{A}$ such that $\Gamma_1 = \widehat{\Gamma}_1 = I_m \in \mathrm{M}_m(K)$, i.e. we have the relation

$$\forall\, 1 \leq i,k \leq m\colon \gamma_{1k}^i = \widehat{\gamma}_{1k}^i = \delta_k^i$$

Then both $\mathcal{A}$ and $\mathcal{B}$ are unital with $a_1 = 1_{\mathcal{A}}$, and φ is automatically a unital morphism, i.e. $\varphi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$.

Proof: By hypothesis, $\forall\, 1 \leq j \leq n$: $\varphi(a_1)b_j = b_j\varphi(a_1) = b_j$. Hence $\varphi(a_1) = 1_{\mathcal{B}}$ and $\forall\, 1 \leq i \leq n$: $\varphi(a_1 a_i) = \varphi(a_i a_1) = \varphi(a_i)$, therefore $a_1 = 1_{\mathcal{A}}$ by injectivity of φ. □

Lemma A.2.16 (φ-induced $\mathcal{A}$-structures on $\mathcal{B}$)**:**

(1) $\mathcal{A}$ unital, $\mathcal{B}$ unital associative $\Rightarrow$ $\mathcal{B}$ left $\mathcal{A}$-module;

(2) $\mathcal{A}$ unital commutative, $\mathcal{B}$ unital associative $\Rightarrow$ $\mathcal{B}$ $\mathcal{A}$-bimodule;

(3) $\mathcal{A}$ unital commutative, $\mathcal{B}$ unital associative commutative $\Rightarrow$ $\mathcal{B}$ $\mathcal{A}$-algebra. □

Remark: Notice that associativity of $\mathcal{A}$ is not required, although usually R-modules are defined for an associative R and R-algebras are defined for commutative associative R.

Now we would like to examine in more detail two important special cases of φ, namely being monic or epic, as well as their interplay with the structure tensors of $\mathcal{A}$ and $\mathcal{B}$. Moreover, we remark that any morphism $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ (of associative algebras) factors as a composition $p \circ i$ of a monomorphism i and an epimorphism p:

$$\mathcal{A} \overset{p}{\twoheadrightarrow} \mathcal{A}/\ker(\varphi) \overset{i}{\hookrightarrow} \mathcal{B} \tag{A.2.25}$$

More on monicness of φ. We have $\mathrm{Ker}(\varphi) = 0$, necessarily $n \leq m$, and our commutative diagrams become:

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\widetilde{\lambda}"] \arrow[dr, "\varphi"'] & & \mathrm{End}_{\mathsf{Vec}}(\mathcal{B}) \\ & \mathcal{B} \arrow[ur, "\lambda"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{A} \arrow[rr, "\widetilde{\rho}"] \arrow[dr, "\varphi"'] & & \mathrm{End}_{\mathsf{Vec}}(\mathcal{B}) \\ & \mathcal{B} \arrow[ur, "\rho"'] & \end{tikzcd} \tag{A.2.26}$$

Lemma A.2.17: Let φ monic. Then multiplicativity of φ (A.2.7) implies equivalence of the following two relations:

$$\begin{aligned} \forall\, 1 \leq i \leq m \ \forall\, 1 \leq k \leq n\colon \ \sum_{r=1}^{n} \phi^i_r \alpha^r_{1k} &= \sum_{r=1}^{n} \phi^i_r \alpha^r_{k1} = \phi^i_k \\ \forall\, 1 \leq i \leq m \ \forall\, 1 \leq k \leq n\colon \ \sum_{s=1}^{m} \gamma^i_{1s} \phi^s_k &= \sum_{s=1}^{m} \gamma^i_{ks} \phi^s_j = \phi^i_k \end{aligned} \tag{A.2.27}$$

Proof: The first relation is a restatement of $\mathcal{A}$ being unital with $a_1 := 1_{\mathcal{A}}$. The second one is a restatement of $\varphi(a_1)\varphi(a_k) = \varphi(a_k)\varphi(a_1) = \varphi(a_k)$, $1 \leq k \leq n$. □

Lemma A.2.18: Let φ be monic. Then:

(1) The relation

$$\forall\, 1 \leq i, j, k \leq m\colon \beta^i_{jk} = \beta^i_{kj}$$

implies the relation

$$\forall\, 1 \leq i, j, k \leq n\colon \alpha^i_{jk} = \alpha^i_{kj}$$

(2) The relation

$$\forall\, 1 \leq i, j, k, \ell \leq m\colon \ \sum_{r=1}^{n} \beta^r_{jk} \beta^i_{r\ell} = \sum_{r=1}^{n} \beta^r_{k\ell} \beta^i_{jr}$$

implies the relation

$$\forall\, 1 \leq i, j, k, \ell \leq n\colon \ \sum_{r=1}^{n} \alpha^r_{jk} \alpha^i_{r\ell} = \sum_{r=1}^{n} \alpha^r_{k\ell} \alpha^i_{jr}$$

Proof: $\mathcal{A} \cong \varphi(\mathcal{A}) \subseteq \mathcal{B}$.

To (1): $\mathcal{B}$ commutative $\Rightarrow$ $\mathcal{A}$ commutative.

To (2): Likewise, $\mathcal{B}$ associative $\Rightarrow$ $\mathcal{A}$ associative. □

Lemma A.2.19: Let φ be monic. Multiplicativity of φ (A.2.7) implies equivalence of the following three relations:

$$\begin{aligned} \forall\, 1 \le i,j,k,\ell \le n\colon \sum_{r=1}^{n} \alpha^r_{jk}\alpha^i_{r\ell} &= \sum_{r=1}^{n} \alpha^r_{k\ell}\alpha^i_{jr} \\ \forall\, 1 \le j,k,\ell \le n\ \forall\, 1 \le i \le m\colon \sum_{s,t=1}^{m} \widehat{\gamma}^i_{\ell t}\gamma^t_{js}\phi^s_k &= \sum_{s,t=1}^{m} \gamma^i_{jt}\gamma^t_{ks}\phi^s_\ell \\ \forall\, 1 \le j,k,\ell \le n\ \forall\, 1 \le i \le m\colon \sum_{r=1}^{n}\sum_{t=1}^{m} \widehat{\gamma}^i_{\ell t}\phi^t_r\alpha^r_{jk} &= \sum_{r=1}^{n}\sum_{t=1}^{m} \gamma^i_{jt}\phi^t_r\alpha^r_{kl} \end{aligned} \tag{A.2.28}$$

Proof: As φ is a monomorphism of K-algebras, associativity of $\mathcal{A} \Leftrightarrow \big(\varphi(a_j)\varphi(a_k)\big)\varphi(a_\ell) = \varphi(a_j)\big(\varphi(a_k)\varphi(a_\ell)\big)$, $1 \le j,k,\ell \le n$. The second and third relation are always equivalent by multiplicativity of φ (A.2.7) regardless of its injectivity. □

Lemma A.2.20: Let φ be monic and $\mathcal{B}$ unital. Then:

(1) $\mathcal{A}$ is unital with unit $a_1 = 1_{\mathcal{A}}$ iff $\forall\, 1 \le j \le n\colon \widetilde{\lambda}(a_1a_j) = \widetilde{\lambda}(a_ja_1) = \Gamma_j$;
(2) $\mathcal{A}$ is commutative iff $\forall\, 1 \le i,j \le n\colon \widetilde{\lambda}(a_ia_j) = \widetilde{\lambda}(a_ja_i)$;
(3) $\mathcal{A}$ is associative iff $\forall\, 1 \le i,j,k \le n\colon \widetilde{\lambda}((a_ia_j)a_k) = \widetilde{\lambda}(a_i(a_ja_k))$.

Proof: Since $\mathcal{B}$ is unital, λ is injective, hence $\widetilde{\lambda}$ is injective as φ injective. □

Lemma A.2.21: Let φ be monic. If $\widetilde{\lambda}$ is multiplicative, i.e. $\forall\, 1 \le j,k \le n\ \forall\, 1 \le i,\ell \le m$:

$$\sum_{r=1}^{n} \gamma^i_{r\ell}\alpha^r_{jk} = \sum_{s=1}^{m} \gamma^i_{js}\gamma^s_{k\ell},$$

then multiplicativity of φ (A.2.7) implies that $\mathcal{A}$ is associative, i.e. $\forall\, 1 \le i,j,k,\ell \le n$:

$$\sum_{r=1}^{n} \alpha^r_{jk}\alpha^i_{r\ell} = \sum_{r=1}^{n} \alpha^r_{k\ell}\alpha^i_{jr}$$

Proof: $\widetilde{\lambda}$ multiplicative $\Leftrightarrow \forall\, 1 \le j,k \le n\ \forall\, 1 \le \ell \le m\colon (a_ja_k)b_\ell = a_j(a_kb_\ell)$ (first relation) $\overset{\text{def}}{\Leftrightarrow}$ $\forall\, 1 \le j,k \le n\colon \widetilde{\lambda}(a_ja_k) = \widetilde{\lambda}(a_j)\widetilde{\lambda}(a_k) \Leftrightarrow \forall\, 1 \le j,k \le n\ \forall\, 1 \le \ell \le m\colon (\varphi(a_j)\varphi(a_k))b_\ell = \varphi(a_j)(\varphi(a_k)b_\ell)$ (by multiplicativity of φ (A.2.7)) $\Rightarrow$ $\mathcal{A}$ associative by injectivity and multiplicativity of φ (A.2.7). □

More on epicness of φ. We have $\mathrm{Im}(\varphi) = \mathcal{B}$, necessarily $n \ge m$, and our commutative diagrams become:

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\widetilde{\lambda}"] \arrow[dr, "\varphi"'] & & \mathrm{End}_{\mathsf{Vec}}(\mathcal{B}) \\ & \mathcal{B} \arrow[ur, "\lambda"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{A} \arrow[rr, "\widetilde{\rho}"] \arrow[dr, "\varphi"'] & & \mathrm{End}_{\mathsf{Vec}}(\mathcal{B}) \\ & \mathcal{B} \arrow[ur, "\rho"'] & \end{tikzcd} \tag{A.2.29}$$

Lemma A.2.22: Let φ be epic. We have:

(1) If $\mathcal{A}$ is unital, then $\mathcal{B}$ is unital, and φ is automatically a morphism of unital K-algebras;
(2) The relation

$$\forall\, 1 \le i,j,k \le n\colon \alpha^i_{jk} = \alpha^i_{kj}$$

implies the relation

$$\forall\, 1 \leq i,j,k \leq m\colon \beta^i_{jk} = \beta^i_{kj}$$

(3) The relation

$$\forall\, 1 \leq i,j,k,\ell \leq n\colon \sum_{r=1}^{n} \alpha^r_{jk}\alpha^i_{r\ell} = \sum_{r=1}^{n} \alpha^r_{k\ell}\alpha^i_{jr}$$

implies the relation

$$\forall\, 1 \leq i,j,k,\ell \leq m\colon \sum_{r=1}^{n} \beta^r_{jk}\beta^i_{r\ell} = \sum_{r=1}^{n} \beta^r_{k\ell}\beta^i_{jr}$$

Proof: To (1): $\forall\, a \in \mathcal{A}\colon \varphi(1_{\mathcal{A}})\varphi(a) = \varphi(a)\varphi(1_{\mathcal{A}}) = \varphi(a)$. By surjectivity of φ, it follows that $\varphi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$.

To (2): $\mathcal{A}$ commutative $\Rightarrow$ $\mathcal{B}$ commutative.

To (3): $\mathcal{A}$ associative $\Rightarrow$ $\mathcal{B}$ associative. $\square$

Recall that if $\mathcal{A}$ is associative, then the left regular representation $\varphi \coloneqq \lambda\colon \mathcal{A} \to \mathcal{B} \coloneqq \mathrm{End}_{\mathsf{Vec}}(\mathcal{A}) \cong \mathrm{M}_n(K)$ is a morphism of K-algebras. We get commutative diagrams of associative K-algebras:

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\widetilde{\lambda}"] \arrow[dr, "\varphi"'] & & \mathrm{M}_{n^2}(K) \\ & \mathrm{M}_n(K) \arrow[ur, "\lambda"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{A} \arrow[rr, "\widetilde{\rho}"] \arrow[dr, "\varphi"'] & & \mathrm{M}_{n^2}(K) \\ & \mathrm{M}_n(K) \arrow[ur, "\rho"'] & \end{tikzcd} \tag{A.2.30}$$

Let us now examine how the structure constants behave under basis change. Let $a'_1, \dots, a'_n$ and $b'_1, \dots, b'_m$ be another set of ordered bases for $\mathcal{A}$ and $\mathcal{B}$, respectively, and let $\Lambda \coloneqq (\lambda^i_j)_{1 \leq i,j \leq n} \in \mathrm{GL}_n(K)$ and $\Theta \coloneqq (\theta^k_\ell)_{1 \leq k,\ell \leq m} \in \mathrm{GL}_m(K)$ be the corresponding transition matrices with $\Lambda^{-1} \eqqcolon (\tilde{\lambda}^i_j)_{1 \leq i,j \leq n}$ and $\Theta^{-1} \eqqcolon (\tilde{\theta}^k_\ell)_{1 \leq k,\ell \leq m}$. That is:

$$a_j = \sum_{r=1}^{n} \lambda^r_j a'_r,\ a'_j = \sum_{r=1}^{n} \tilde{\lambda}^r_j a_r,\ b_k = \sum_{s=1}^{m} \theta^s_k b'_s,\ b'_k = \sum_{s=1}^{m} \tilde{\theta}^s_k b_s$$

Denote by $(\alpha'^i_{jk})_{1 \leq i,j,k \leq n}$, $(\gamma'^i_{jk})_{1 \leq i,k \leq m, 1 \leq j \leq n}$, and Φ' the respective representing matrices after basis change.

Lemma A.2.23 (Basis Change)**:** We have:

(1) $\Phi' = \Theta\Phi\Lambda^{-1}$, or explicitly, $\forall\, 1 \leq i \leq m\ \forall\, 1 \leq j \leq m$:

$$\phi'^i_j = \sum_{r=1}^{m}\sum_{s=1}^{n} \theta^i_r \phi^r_s \tilde{\lambda}^s_j \tag{A.2.31}$$

(2) $\forall\, 1 \leq i,k \leq m\ \forall\, 1 \leq j \leq n$:

$$\gamma'^i_{jk} = \sum_{s=1}^{n}\sum_{r,t=1}^{m} \theta^i_r \gamma^r_{st} \tilde{\lambda}^s_j \tilde{\theta}^t_k \tag{A.2.32}$$

(3) $\forall\, 1 \leq i,j,k \leq n$:

$$\alpha'^i_{jk} = \sum_{r,s,t=1}^{n} \lambda^i_r \alpha^r_{st} \tilde{\lambda}^s_j \tilde{\lambda}^t_k \tag{A.2.33}$$

$\square$

Exercise: Verify that, as expected, φ remains multiplicative after basis change, i.e. $\Phi' A'_j = \Gamma'_j \Phi', 1 \leq j \leq n$.

Let us now check when two different sets of structure constants produce isomorphic K-algebra structures. Suppose $\mathcal{A}$ and $\mathcal{B}$ are same as *based* vector spaces and $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ is a linear transformation having a representing matrix $\Phi \coloneqq (\phi^j_k)_{1 \leq j,k \leq n}$.

Lemma A.2.24: φ is an isomorphism of K-algebras iff $\Phi \in \mathrm{GL}_n(K)$ and $\forall\, 1 \leq i, j, k \leq n$:

$$\sum_{r=1}^{n} \phi^i_r \alpha^r_{jk} = \sum_{s,t=1}^{n} \beta^i_{st} \phi^s_j \phi^t_k \tag{A.2.34}$$

In other words, the structure tensors (α^i_{jk}) and (β^i_{jk}) produce isomorphic algebras iff there is $\Phi \in \mathrm{GL}_n(K)$ with $\Phi^{-1} =: (\tilde{\phi}^i_j)_{1 \leq i,j \leq n}$ such that $\forall\, 1 \leq i, j, k \leq n$:

$$\alpha^i_{jk} = \sum_{r,s,t=1}^{n} \tilde{\phi}^i_r \beta^r_{st} \phi^s_j \phi^t_k \tag{A.2.35}$$

Proof: Successive application of Eq. (A.2.7) and Eq. (A.2.5). □

Remark: Notice that we don't need to assume φ to be unital: since φ is in particular a linear epimorphism and multiplicative, it automatically preserves 1.

A.3. The Structure Constants of a Representation

Let $\mathcal{A} \in \mathsf{FdAlg}_{\mathbb{K}}$ and $M \in \mathsf{FdLMod}(\mathcal{A})$ via a representation $\lambda\colon \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(M)$. If $\{a_1, \ldots, a_n\}$ is a $\mathbb{K}$-basis of $\mathcal{A}$ and $\{v_1, \ldots, v_m\}$ is a $\mathbb{K}$-basis of M, then the *structure tensor* $(\lambda^k_{j\ell})_{1 \le k,\ell \le m,\, 1 \le j \le n}$ of the action $\mathcal{A} \curvearrowright M$ is defined by

$$a_j \cdot v_\ell =: \sum_{k=1}^{m} \gamma^k_{j\ell} v_k. \tag{A.3.1}$$

As we now have $\mathrm{End}_{\mathbb{K}}(M) \cong \mathrm{M}_m(\mathbb{K})$, we get

$$\forall\, 1 \le j \le n\colon A_j \coloneqq \lambda(a_j) = (\gamma^k_\ell)_{1 \le k,\ell \le m} \in \mathrm{M}_m(\mathbb{K}). \tag{A.3.2}$$

for the representing matrices of the action. Indeed

$$\forall\, 1 \le j \le n\colon a_j \cdot (v_1, \ldots, v_m) \stackrel{\text{def}}{=} (v_1, \ldots, v_m) A_j.$$

If $a_1 = 1_{\mathcal{A}}$, then additionally we have

$$\forall\, 1 \le k,\ell \le m\colon \gamma^k_{1\ell} = \delta^k_\ell. \tag{A.3.3}$$

Lemma A.3.1: We have:

$$\forall\, 1 \le i,j \le n\ \forall\, 1 \le k,\ell \le m\colon\ \sum_{r=1}^{n} \alpha^r_{ij} \gamma^k_{r\ell} = \sum_{s=1}^{m} \gamma^k_{is} \gamma^s_{j\ell} \tag{A.3.4}$$

Proof: This follows from the "associativity" of the action:

$$\forall\, 1 \le i,j \le n\ \forall\, 1 \le \ell \le m\colon (a_i a_j) \cdot v_\ell = a_i \cdot (a_j \cdot v_\ell).$$

□

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