Reference-Scientific Contributions

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The image of tropical geometry has emerged at the turn of millennia, combining the ideas from real algebraic geometry, mathematical physics and computer science, and since then has acquired numerous connections with various areas of mathematics and has shown a great potential for applications. In essence, its objects are piece-wise linear spaces endowed with integral affine structure. These spaces are not always familiar manifolds, such as tori – quotients of an affine space by a lattice, but may have junctions of several faces. For instance, metric graphs play the rôle of abstract tropical curves, where several edges may meet at a vertex. In many ways, tropical curves mimic the behavior of Riemann surfaces, also known as complex curves. A particular example is that the notion of Jacobian variety of a curve, controlling the values of integrals of 1-forms, makes perfect sense in tropical geometry, being a torus of real dimension equal to the genus of the tropical curve, where the genus here is the first Betti number of the graph. Observe again the general feature of a tropical counterpart – its real dimension is half of the dimension of the corresponding classical geometric object. A part of the explanation why tropical curves resemble so much complex curves is due to the fact that the former may be seen as degeneration of the latter when approaching the boundary of the moduli space and taking in to account the rate of vanishing of cycles. Such a procedure is sometimes referred to as an abstract tropicalization of curves. An analogue of this process is still far from being established in higher dimensions.

In contrast, the most conventional, i.e. embedded, tropicalization, inherently has no restriction on the dimension, dealing with arbitrary subvarieties of an algebraic torus and employs the notion of amoeba. An amoeba of a complex algebraic subvariety of $(\mathbb{C}\backslash\{0\})^n$ is the image in $\mathbb{R}^n$ under the phase-forgetful map, i.e. the coordinate-wise logarithm of the norm. Tropicalization procedure in this context is simultaneous sending of the base of the logarithm to infinity, taking a family of varieties, and going to the limit (perhaps on a subfamily) leads to a tropical subvariety of $\mathbb{R}^n$. A particular example is that of a constant family, in which case the result of tropicalization is a polyhedral fan with rational slope of each face. A more algebraic approach to tropicalization is via replacing the field of complex numbers by an algebraically closed non-Archimedean field

$\mathbb{K}$, i.e. endowed with a norm satisfying the ultra-metric inequality. Then, a tropicalization of a subvariety of $(\mathbb{K}\backslash\{0\})^n$ is the topological closure of the non-Archimedean amoeba, which is again the image under the logarithm of the norm of each coordinate. The geometric and algebraic approaches have their own virtues and are not fully equivalent. A somewhat middle ground is provided by choosing $\mathbb{K}$ to be some field of series in one variable with complex coefficients – there are a few choices for such fields depending on the nature of powers allowed and a convergence condition. In the case when the latter is assumed, indeed a variety over $\mathbb{K}$ may be thought of as a family of varieties over $\mathbb{C}$ by evaluating the parameter in every coefficient of defining system of polynomials.

To complete this extremely brief introduction (for a more detailed one I recommend "E. Brugalé, I. Itenberg, G. Mikhalkin, and K. Shaw. Brief introduction to tropical geometry. Proceedings of the 21st Gökova Geometry-Topology conference, 2015"), I need to mention an enhancement of tropicalization in the previous paragraph. Namely, the phase tropicalization, which is the same kind of procedure without forgetting the phase. Recall that in the geometric version of embedded tropicalization, we were changing the base of the logarithm of absolute value, which corresponds to a homothety on $\mathbb{R}^n$. Now, this homothety is lifted to a diffeomorphism of the original ambient space $(\mathbb{C}\backslash\{0\})^n$ by taking the powers of the radial part in polar decomposition of each coordinate. The resulting limit for a family of subvarieties under this diffeomorphisms is the phase tropical variety fibered over the usual tropicalization, with fibers being closures of coamoebas. An algebraic counterpart of phase tropicalization, in case of $\mathbb{K}$ being a field of series in t with complex coefficients, for a subvariety of $(\mathbb{K}\backslash\{0\})^n$, is an assignment of the closure of its image under a map $(\mathbb{K}\backslash\{0\})^n \to (\mathbb{C}\backslash\{0\})^n$ taking each coordinate given by $ct^\alpha + o(t^\alpha)$ to $\frac{c}{|c|}\exp(\alpha)$. Again, the two approaches are mostly equivalent, although, it seems, it is not proven anywhere in full generality. Initially, the phase tropical curves (under the name "complex tropical curves") have appeared in the context of proving what became known as Mikhalkin's correspondence theorem for counting curves on toric surfaces. More recently, it was shown that under a genericity assumption a phase tropical hypersurface restores the topology of a complex hypersurface with the same Newton polytope.

1 Non-abelian tropicalization

The idea suggested by Yakov Eliashberg, was to explore how the setup of amoebas and tropicalization could be modified. A particular way of such extensions, which was proposed by Grigory Mikhalkin, consisted of replacing an algebraic torus $(\mathbb{C}\backslash\{0\})^n$ by a possibly non-commutative complex group G, with the quotient map by a maximal compact subgroup taking the role of the phase-forgetful logarithmic projection. In the case of G being either $SL_2(\mathbb{C})$ or $PSL_2(\mathbb{C})$, maximal compact subgroups are $SU(2)$ and $SO(3)$ respectively, and the quotient is naturally the hyperbolic 3-space $\mathbb{H}^3$, thus the images of subvarieties of such G were baptized hyperbolic amoebas. A preliminary, yet systematic treatment:

"Mikhalkin, G. and Shkolnikov, M., 2022. Non-commutative amoebas. Bulletin of the London Mathematical Society, 54(2), pp.335-368."

apart from a few results already contained in my PhD thesis, among other things, contains a description of possible tropical limits of hyperbolic amoebas of curves in terms of $\mathbb{H}^3$-spherical complexes.

Until very recently, the case of tropical limits of hyperbolic amoebas of surfaces remained unresolved. The announcement of the corresponding result (to the proof of which a separate article in preparation is dedicated) appeared in the paper appeared last year:

"Shkolnikov, M. and Petrov, P., 2024, October. Introduction to PSL2 Phase Tropicalization. In Proceedings of the Bulgarian Academy of Sciences (Vol. 77, No. 10, pp. 1425-1432)."

claiming that such scaling limits are simply complements to geometric balls. This result implies that tropicalizations of hyperbolic amoebas of surfaces are insufficient to capture subtle geometric information of a classical surface, which motivates the enhancement of the process via restoring the phase. The main result of the above mentioned publication is finding the formula for the corresponding phase-valuation VAL: $PSL_2(K) \to PSL_2(\mathbb{C})$, as a couple of explicit examples of computations of images under it, showing that PSL_2-phase-tropical varieties exhibit a non-trivial behavior, as well as the discovery of a new direction for degeneration valid in arbitrary dimension. Exploring this new type of tropical geometry further, is the main topic of my current collaborative research with several mathematicians from IMI-BAS and elsewhere.

2 Tropical sandpile

Sandpile model is a cellular automaton providing a prototypical example of Self-organized criticality (SOC), a phenomenon that is believed to play a crucial role in explaning various natural systems. In the following paper we have reported on a discovery of the first continuous model demonstrating SOC:

"Kalinin, N., Guzmán-Sáenz, A., Prieto, Y., Shkolnikov, M., Kalinina, V. and Lupercio, E., 2018. Self-organized criticality and pattern emergence through the lens of tropical geometry. Proceedings of the National Academy of Sciences, 115(35), pp.E8135-E8142."

The definition of this new model is built on the results I obtained jointly with Nikita Kalinin, while studying perturbations of the maximal stable state on a large convex domain of the lattice, the global behavior of which is approximated by the dynamics on tropical series:

"Kalinin, N. and Shkolnikov, M., 2018. Introduction to tropical series and wave dynamic on them. Discrete and Continuous Dynamical Systems-Series A, 38(6).”

and the local structure is generically governed by sandpile solitons and their simple junctions, the existence of which was formally proven in

“Kalinin, N. and Shkolnikov, M., 2020. Sandpile Solitons via Smoothing of Superharmonic Functions. Communications in Mathematical Physics, 378(3), pp.1649-1675.”

The tropical sandpile model is not fully understood. For instance, the relaxation process is known to converge but it is still not clear if it terminates after a finite number of steps. Not so long ago, I went to the lowest dimension and, among other things, demonstrated the finiteness of relaxation:

“Shkolnikov, M., 2023. Relaxation in one-dimensional tropical sandpile. Communications in Mathematics, 31.”

We have also explored the situation on a hyperbolic lattice, where the perturbations of the maximal stable state admit an explicit solution consonant with hyperbolic amoebas:

“Kalinin, N. and Shkolnikov, M., 2016. Sandpiles on the heptagonal tiling. Journal of Knot Theory & Its Ramifications, 25(12).”

It is worth mentioning that the sandpile model, which may seem as a peculiar combinatorial gadget, is innately tropical due to the relation with a versionof Riemann-Roch theorem, originally proven for graphs by Baker and Norine, using the chip-firing game (essentially, a different branding for sandpile model), and a few months later extended to abstract tropical curves by Gathmann and Kerber, as well as Mikhalkin and Zharkov.

3 Tropical optics

The study of the interaction between tropical curves and large scale near-maximal sandpiles had the invention of tropical optics as its byproduct. The main objects of this new field under construction are tropical wave fronts and their caustics. A particular striking result is that tropical caustic, even as an abstract curve, of a convex contour knows everything about it. This implies the recipe for writing universal formulas, suggesting the relation of convex geometry to zeta-functions in complex analysis, in terms of vertices of caustics (organized by the free rank two commutative monoid $SL^+(2,\mathbb{Z})$) for such quantities as area or perimeter of the domain, some of which are published in:

“Kalinin, N. and Shkolnikov, M., 2019. Tropical formulae for summation over a part of $SL(2,\mathbb{Z})$. European Journal of Mathematics, 5, pp.909-928."

A fundamental treatment of planar tropical optics, accentuating wave fronts, their relation to toric surfaces with mild singularities, proving a version of Huygens principle, as well as clarifying the interplay of continued fractions and tropical trigonometry, was published in a non-indexed (thus, not included in the list of publications for the competition) but quite reputable proceedings:

"Mikhalkin, G., and Shkolnikov, M, 2024. Wave fronts and caustics in the tropical plane. Proceedings of 28th Gökova Geometry-Topology Conference, pp. 11-48."

Of course, the subject is still far from its ultimate form. Some directions, like dropping convexity, or extending the theory to higher dimensions, non-flat media and providing the bridge to classical geometric optics could serve as examples of excellent topics for my potential students.

4 Extended sandpile

The size of the sandpile group even on moderately large domains is enourmous (for example, on a 30-by-30 square it has approximately $3.3 \times 10^{4}62$ elements), moreover, its significant states, such as identity element, exhibit scale-transcendent visual patterns. These two observations suggest that the sandpile group might approximate a continous group. In addition, the work on tropical sandpile suggests that such a group should have tropical structure. All this, and more, was discovered to be indeed correct in:

"Lang, M. and Shkolnikov, M., 2019. Harmonic dynamics of the abelian sandpile. Proceedings of the National Academy of Sciences, 116(8), pp.2821-2830."

The main object put forward there was the so-called extended sandpile model. By simply allowing the recurrent states to have real values at vertices near the boundary of the domain on the lattice, results in the tropical Abelian variety whose set of integer points is the original sandpile group. In addition, quite remarkably at the first glance, the extended sandpile group behaves functorially with respect to the inclusions of the domains, which, in particular, allows to define its limit on the whole lattice, thus manifesting the renormalization.

By investigating the one-dimensional situation of the functoriality for the extended sandpile group and its possible failure for the usual sandpile group, we came to realize that there are monomorphisms (and via Pontryagin duality, epimorphisms) of the usual sandpile groups of the domains which are related by a certain type of tilings, which was published as

"Lang, M. and Shkolnikov, M., 2022. Sandpile monomorphisms and limits. Comptes Rendus. Mathématique, 360(G4), pp.333-341."

A particular application of this result is the divisibility for the numbers of spanning trees (via Kirchhoff theorem) of the suitably scaled domains.

5 Quantum topology

During my undergraduate studies, I was introduced to low-dimensional topology by Sergei Duzhin, who proposed to me the following problem: compute the HOMFLY-PT polynomial of all rational knots. The 8-letter abbreviation correspond to the last names of mathematicians co-invented the invariant generalizing both Alexander-Conway and Jones polynomials, shortly after the breakthrough discovery of the latter. All of them, as well as Kauffman polynomial and many others, are unified to a large family of quantum invariants inspired by the developments of Witten in quantum field theory and most rigorously treated as Reshetikhin-Turaev invariants defined using ribbon categories, the common example of which are categories of representations of quantum groups.

Rational knots form the first layer of complexity for testing computation techniques and conjectures, since they have Morse diagrams with at most two bridges. Counting the blocs of crossings on such a diagram, one extracts a finite sequence of integers which, after being interpreted as denominators of a continued fraction, produce a rational number. This correspondence of rational numbers of absolute value less than one and isotopy classes of rational knots (and links) is essentially one-to-one, with some rational numbers corresponding to the same isotopy class, which is naturally explained by the relation, via branched two-sheeted cover, to lens spaces and the problem of their classification up to a homeomorphism.

My solution to the Duzhin's problem, was published in a joint paper:

"Duzhin, S. and Shkolnikov, M., 2015. A formula for the HOMFLY polynomial of rational links. Arnold Mathematical Journal, 1(4), pp.345-359."

and relied on the following two observations. First, the HOMFLY-PT polynomial can be expressed as a state sum over resolutions of a matched diagram, where all crossings go in pairs. Second, is that every rational link can be represented by such a diagram, which follows from an arithmetic lemma telling that every reduced fraction $\frac{p}{q}$ with pq even has a continued fraction expansion with all denominators being even. Here, I have to mention that a different formula in the same context, as we learned after completing the manuscript, was earlier obtained by Shigekazu Nakabo.

The first observation in the previous paragraph motivated me to try extending the second observation to all knots, i.e. I spent a few months trying to prove that every knot has a matched diagram. At some point, Duzhin has informed me that the negative form of this problem is a part of a Przytycki's conjecture listed in the famous Kirby's list of open problems in low dimensional topology. Few more months later, we found a necessary condition for a knot to have a matched diagram in terms of its second Alexander ideal and a dozen concrete knots that do not satisfy it. The result was published as:

"Duzhin, S. and Shkolnikov, M., 2014. Bipartite knots. Fundamenta Mathematicae, 225(1), pp.95-102."

Some years later, the familiarity with quantum groups, acquired while working in knot theory, shown its usefulness in a successful collaboration in the field of condensed matter physics. A precise setup of this project is that of a quantum mechanical system consisting of a rotor (in reality it may be represented by a diatomic molecule) immersed into a bosonic bath (for instance, a superfluid helium-4 nanodroplet). In the absence of interaction of the bath with the rotor, the standard rotational symmetry group $SO(3)$ of the latter produces the Hamiltonian which is proportional to the Casimir operator, with discrete set of eigenvalues corresponding to rotational kinetic energy. In the presence of bath interacting with the rotor the rotational energies shift – our explanation for this is that $SO(3)$ is deformed to a quantum group $SO_q(3)$ with q being explicitly dependent on the coupling. In this approach, the full Hamiltonian of the many-body system is again proportional to the Casimir operator, but now of the deformed group. The proposed interpretation is in a good agreement both with experiment and the numerical simulation:

"Yakaboylu, E., Shkolnikov, M. and Lemeshko, M., 2018. Quantum Groups as Hidden Symmetries of Quantum Impurities. Physical review letters, 121(25)."