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Symmetry and metric geometry in Banach spaces

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Chapter 1

Introduction

During the last thirty years the theory of Banach spaces branched in many different directions. Results and techniques from this area were applied in nonlinear analysis, metric geometry, computer science, theory of operators, logic, and others. At the same time, there was also an extensive study of the structure of Banach spaces themselves, especially of separable spaces and, in particular, those with a Schauder basis. For recent developments in Banach spaces theory, see the two volumes of the Handbook on the geometry of Banach spaces, edited by W. B. Johnson and J. Lindenstrauss [23, 24].

A main line of investigation in the structure theory of Banach spaces is whether any infinite-dimensional space contains an infinite-dimensional subspace which is isomorphic to a space from a list of spaces with "nice" properties. The most natural first question was if any X contains an isomorphic copy of c_0 or ℓ_p, $1 \leq p < \infty$. All the known classical examples of Banach spaces do contain an isomorphic copy of c_0 or ℓ_p, $1 \leq p < \infty$. Additionally, there exist deep results for embedding copies of ℓ_2^n or ℓ_p^n for an arbitrary dimension n, due to Dvoretzky and Krivine, respectively. However, the case of an infinite dimensional subspace turned out to be different. In 1974 Boris Tsirelson [47], inspired by ideas in logic, constructed a reflexive space with an unconditional Schauder basis which has no subspace isomorphic to c_0 or ℓ_p, $1 \leq p < \infty$.

Tsirelson inductively defined the unit ball B of his space as follows:

Let $K^0 = \{\pm e_n : \ n \in \mathbb{N}\}$. Given K^m, $m \geq 0$,

$$K^{m+1} = K^m \cup \left\{ \frac{1}{2}\sum_{i=1}^{d} f_i \ : \ \begin{array}{l} f_i \in K^m, i = 1, 2, \ldots, d, \quad d \in \mathbb{N}, \text{ and} \\ d \leq \min \mathrm{supp} f_1 \leq \max \mathrm{supp} f_1 < \min \mathrm{supp} f_2 \leq \\ \leq \max \mathrm{supp} f_2 < \cdots < \min \mathrm{supp} f_d \end{array} \right\}$$

Finally, let $K = \bigcup_{m=0}^{\infty} K^m$, and the unit ball B is the closed convex hull of K.

In actuality, what is now referred to as the Tsirelson space T is the construction given by Figiel and Johnson [13], while the original space defined by Tsirelson is its dual T^*. The definition of the norm in T is given either as the limit of a sequence of norms, or, equivalently, as the implicit solution of an equation.

Consider the linear space of finitely supported real-valued sequences c_{00}. For any $x = \sum_n a_n t_n \in c_{00}$, and for any nonempty finite set E of natural numbers, we define

$$Ex = \sum_{n \in E} a_n t_n \ .$$

Here (t_n) is the canonical basis of c_{00} and (a_n) is an arbitrary sequence of real numbers. We define inductively a sequence of norms $(\| \cdot \|_j)_{j=0}^{\infty}$ on c_{00} as follows:

$$\begin{aligned}&\text{for any } x = \textstyle\sum_n a_n t_n \in c_{00}, \ \text{let } \|x\|_0 = \max_n \ |a_n|, \ \text{and for } m \geq 0,\\ &\|x\|_{m+1} = \max\left\{\|x\|_m, \ \tfrac{1}{2}\max\left[\textstyle\sum_{i=1}^{d} \|E_i x\|_m\right], \ \ d \in \mathbb{N}\right\},\end{aligned}$$

where the inner maximum is taken over all choices of d and all choices of finite subsets $(E_i)_{i=1}^{d}$ of $\mathbb{N}$ as d varies such that

$$d \leq \min E_1 \leq \max E_1 < \min E_2 \leq \max E_2 < \cdots < \min E_d \ .$$

Then $\|x\|$ is defined as $\lim_{m\to\infty} \|x\|_m$, and what is now called the Tsirelson space T is the completion of $(c_{00}, \| \cdot \|)$.

Figiel and Johnson proved that the norm of T satisfies the following implicit equation.

$$\forall x = \sum_{n=1}^{\infty} a_n t_n \in T,$$

$$\|x\| = \max\left\{\max_n |a_n|, \ \frac{1}{2}\sup\sum_{i=1}^{d} \|E_i x\|\right\},$$

where the inner supremum is taken over all choices of d and all choices of finite subsets $(E_i)_{i=1}^{d}$ of $\mathbb{N}$ as d varies such that

$$d \leq \min E_1 \leq \max E_1 < \min E_2 \leq \max E_2 < \cdots < \min E_d \ .$$

Tsirelson space had an enormous impact on the theory of Banach spaces. Many constructions, similar in spirit, were defined to solve other important problems. Among those, we mention a construction by Tzafriri [48]. Since their introduction, the notions of type and cotype (cf. e.g. [33]) played a major role in the theory of normed spaces and their many applications.

Tzafriri answered a question of Pisier by defining a space which has an equal-norm type p, but has no type p, for $1 < p < 2$. The book of Casazza and Shura [8] gathered most of the results related to Tsirelson space and its variations.

The next natural big question was whether every space contained an unconditional basic sequence, that is, a subspace with an unconditional basis. A partial result for weakly null sequences was due to Maurey and Rosenthal, cf. e.g. [32].

In 1990 E. Odell showed, in an unpublished note, that T is distortable, by constructing in every subspace sequences with two different asymptotic behaviour; this was the first example of a distortable space.

Definition 1. *Let $\lambda > 1$. A Banach space $(X, \|\cdot\|)$ is λ-distortable if there exists an equivalent norm $|\cdot|$ on X such that for every infinite dimensional subspace Y of X,*

$$\sup\left\{\frac{|y|}{|x|} : y, x \in Y, \|y\| = \|x\| = 1\right\} \geq \lambda$$

X is arbitrarily distortable if it is λ-distortable for every $\lambda > 1$.

James proved that c_0 and ℓ_1 are not distortable. Currently, there is still no example of a space which is distortable but not arbitrarily distortable, although the Tsirelson space T is a candidate.

Using a similar idea to that of Odell, Th. Schlumprecht [44] constructed in 1991 the first arbitrarily distortable space S that has an unconditional basis. Schlumprecht's space was a launching point for a renewed interest in Tsirelson type spaces and led to remarkable developments in Banach space theory.

In 1993, W. T. Gowers and B. Maurey [19], using Schlumprecht's construction, solved the famous unconditional basic sequence problem by constructing a space without such a sequence. Their space has a stronger property, namely that it is hereditarily indecomposable.

Definition 2. *A Banach space is heriditarily indecomposable (H.I.) if no subspace can be written as a topological direct sum of two infinite dimensional closed subspaces.*

In 1994, Odell and Schlumprecht [39] solved the distortion problem. In particular, by transferring sets from S, they showed that the Hilbert space ℓ_2 is arbitrarily distortable. This illustrates the impact of the Tsirelson-type spaces on the understanding of the classical Banach spaces.

Briefly jumping to the present time in order to note another example, Baudier, Lancien and Schlumprecht [49] recently used the original Tsirelson space T^* for solving a problem in metric geometry.

Gowers used the notion of hereditary incomposability to solve several other long-standing problems. He showed [15] that the H.I. property is a consequence of the absence of unconditionality, in the sense that every Banach space which does not contain any unconditional basic sequences has an H.I. subspace. Among others, he solved Banach's hyperplane problem [17] by constructing a space which is not isomorphic to any of its hyperplanes. He also [16] proved dichotomies for spaces with a basis and used them for a partial classification of Banach spaces. In 1998, Gowers was awarded a Fields medal for his outstanding work (see [18]).

Further results in this area followed, especially by Argyros and his students. Argyros and Deliyanni [4] answered a question of Gowers by constructing an asymptotic ℓ_1 herediatry indecomposable space. For that purpose, they first defined a new class of asymptotic ℓ_1 spaces with unconditional basis, namely the mixed Tsirelson spaces.

Following the properties of T, Milman and Tomczak-Jaegermann [35] defined asymptotic ℓ_p spaces with respect to a basis to be those for which, given any n, all collections of n successive normalized block-vectors supported, say, after position n of the basis are uniformly equivalent to the unit vector basis of ℓ_p^n. In the beginning, it was hoped that asymptotic ℓ_p spaces might have nicer local structure, that is to say, nicer finite-dimensional subspaces. That turned out to be incorrect as it was proved in [54] that some asymptotic ℓ_1 spaces contain uniform copies of ℓ_∞^n's. The universality of ℓ_∞ implies that these spaces contain arbitrary finite-dimensional subspaces.

One of the main problems in the isomorphic theory of Banach spaces is the classification of the basic sequences of a certain type. This question is formulated in a proper way using the notion of equivalence of basic sequences. Recall that a sequence in a Banach space is a basic sequence if it is a (Schauder) basis of its closed linear span. The most important category of sequences in which this classification is studied is that of symmetric sequences.

Definition 3. *A basic sequence is called symmetric if it is equivalent to all of its permutations.*

This class of sequences includes the canonical unit vector basis of the ℓ_p and c_0 spaces. Closely related to symmetry is the notion of subsymmetry.

Definition 4. *A basic sequence is said to be subsymmetric if it is unconditional,and is equivalent to all of its subsequences.*

The question whether a symmetric basic sequence exists in every Banach space was a driving force in the development of the theory for many decades. This question was solved in the negative by the Tsirelson space.

The class of subsymmetric basic sequences is more general. In practice, the only feature that one needs about symmetric basic sequences in many situations is their subsymmetry, to the extent that when symmetric bases were introduced these two concepts were believed to be equivalent until Garling [14] provided a counterexample that disproved it.

However, subsymmetric bases, far from being just a capricious generalization of symmetric bases, played a relevant role by themselves within the general theory. Indeed, the study of Banach spaces with a subsymmetric but not symmetric basis led to the solution of major problems in the field. In example, the space S constructed by Schlumprecht has one such basis. It follows from the minimality of the Schlumprecht space and the "yardstick" construction of Kutzarova-Lin [28], which for every natural number n provides uniform copies of disjointly supported ℓ_∞^n's, that S does not contain symmetric basic sequences.

If a Banach space has a given special type of structure, it is always an important question whether that structure is unique. Vice versa, if we know that a given structure is unique in that Banach space, that leads to the interesting question what can we say about both the structure and the space itself.

Albiac, Ansorena and Wallis [3] used Garling-type spaces to provide the first example of a Banach space with a unique subsymmetric basis which is not symmetric. However, as shown in [2], that space contains a continuum of non-equivalent subsymmetric basic sequences.

Altshuler [1] (see also [32]) constructed a nontrivial space in which all symmetric basic sequences are equivalent to its symmetric basis. It was remarked in [27] that a careful look at the paper of Altshuler shows that his proof works similarly for the more general case of all subsymmetric basic sequences. In the case of the Altshuler's space the uniqueness of the subsymmetric structure implies that it is actually symmetric. This observation led to the following question, asked first in [27] and restated in [2] :

Question 5. *Does there exist a Banach space in which all subsymmetric basic sequences are equivalent to one basis, and that basis is not symmetric?*

Recently, the first example of a Banach space with a subsymmetric basis with a unique (up to equivalence) subsymmetric basic sequence which is not symmetric was given in [7]. The space under consideration was $Su(T^*)$ [8], the subsymmetric version of T^*, the original space defined by Tsirelson.

In Chapter 2, we give more examples of spaces with a unique, up to equivalence, subsymmetric basic sequence.

In the spirit of the Tsirelson space T, Tzafriri [48] constructed a space (with a symmetric basis) which is of equal-norm type p, but is not of type p, for $1 < p < 2$.

For convenience, we will define the following:

$$\text{Average}_{\theta_j=\pm 1}\left\|\sum_{j=1}^{n}\theta_j x_j\right\| = 2^{-n}\sum_{\theta_j=\pm 1}\left\|\sum_{j=1}^{n}\theta_j x_j\right\|$$

The important notions of type and cotype were defined by J. Hoffmann-Jørgensen in [50].

Definition 6. *A Banach space X is said to be of type p for some $1 < p \leq 2$, and respectively of cotype q for some $q \geq 2$, if there exists a constant $M < \infty$ so that for every finite set of vectors $(x_j)_{j=1}^{n}$ in X we have*

$$\text{Average}_{\theta_j=\pm 1}\left\|\sum_{j=1}^{n}\theta_j x_j\right\| \leq M\left(\sum_{j=1}^{n}\|x_j\|^p\right)^{\frac{1}{p}}$$

and respectively

$$\text{Average}_{\theta_j=\pm 1}\left\|\sum_{j=1}^{n}\theta_j x_j\right\| \geq M^{-1}\left(\sum_{j=1}^{n}\|x_j\|^q\right)^{\frac{1}{q}}$$

Any constant M satisfying the first or the second inequality is called a type p, respectively cotype q, constant of X.

Equal-norm type p, and equal-norm cotype q are defined if the above inequalities are required only for vectors $(x_j)_{j=1}^{n}$ in X of equal norm.

The Tirilman space $Ti(p,\gamma)$, where $1 < p < \infty$ and $0 < \gamma < 1$, was introduced and studied by Casazza and Shura [8]. It is a version of the original space of Tzafriri whose Romanian surname was Tirilman.

We prove that for $1 < p < \infty$ and sufficiently small $0 < \gamma < 1$, the dual space $Ti^*(p,\gamma)$, whose canonical basis is subsymmetric and not symmetric, has a unique, up to equivalence, subsymmetric basis sequence. While the normalized block bases (x_j) of the canonical basis of $Su(T^*)$, whose ℓ_∞ norm $\|x_j\|_\infty$ tends to 0 as $j \to \infty$, are asymptotic c_0, the similar block bases in $Ti^*(p,\gamma)$ are asymptotic ℓ_q basis sequences, where

$$\frac{1}{p} + \frac{1}{q} = 1$$

Theorem 7. *Let $1 < p < \infty$ and $\gamma > 0$ be sufficiently small. Every subsymmetric basic sequence of the dual space $Ti^*(p,\gamma)$ is equivalent to the subsymmetric canonical basis $(e_j^*)_{j=1}^{\infty}$ which is not symmetric.*

Towards the proof of this main result we also obtain the following statements that are interesting on their own:

Lemma. *25*

Let (e_i) be a 1-unconditional basis of a reflexive Banach space X which is K-dominated by its normalized block bases, where $K \geq 1$. Then (e_i^*) K-dominates all normalized block bases of (e_i^*) in the dual space X^*.

Lemma. *27*

$Ti^*(p,\gamma)$ does not contain an isomorphic copy of ℓ_q (where $\frac{1}{p} + \frac{1}{q} = 1$).

As a corollary to a theorem from [7] and the uniqueness, up to equivalence, of a subsymmetric basic sequence in $Ti^*(p,\gamma)$, we remark that the canonical basis of $Ti^*(p,\gamma)$ has a continuum of non-equivalent subsymmetric block bases.

An important hereditary property of Banach spaces, first defined by Rosenthal, is the notion of minimality.

Definition 8. *An infinite dimensional Banach space X is called minimal if every infinite dimensional subspace $Y \subset X$ contains a further subspace Z such that Z is isomorphic to X.*

Recall that

Definition 9. *Two normed spaces X and Z are called isomorphic if there exists a bounded linear operator $T : X \to Z$ that is a bijection and its inverse $T^{-1} : Z \to X$ is also a bounded linear operator.*

There are only a few examples of minimal spaces. Clearly, c_0 and ℓ_p, for $1 \leq p < \infty$, are minimal. In [51] it was proved that T^* is a nontrivial minimal space. Later, Schlumprecht [52] proved that his space S is minimal. S is also reflexive but not super-reflexive. However, based on the Schlumprecht space S, [53] give examples of super-reflexive minimal spaces. They also show that S^* is also minimal. The basis of T^* is asymptotic ℓ_∞, thus, highly nonsymmetric. The bases of S and S^* are subsymmetric, but not symmetric.

An important folklore question is whether there exists a nontrivial (not a subspace of c_0 or ℓ_p) example of a minimal space with a symmetric basis.

As explained later, the symmetrizations of T^* and S are not minimal. The quest for such an example is one of the motivations for the consideration of the symmetrization of S^*. The theorem in Chapter 3 leaves the question of symmetric minimal space open.

The natural symmetrization $S(T)$ of the Tsirelson space T contains a subspace isomorphic to ℓ_1, while the symmetrization $S(T^*)$ of the original Tsirelson space is reflexive, so it does not have the same property [8]. In an unpublished note Schlumprecht showed that the symmetric version of his space S does contain a subspace isomorphic to ℓ_1 (a similar result can be found in [34]).

In Chapter 3 we use the "yardstick" construction in Schlumprecht space S to prove the following theorem.

Theorem 10. *In contrast to the case of $S(T^*)$, the symmetrization $S(S^*)$ of the dual of the Schlumprecht space does contain a subspace isomorphic to the space ℓ_1.*

One of the most natural way to grasp the geometry of a metric space is to understand in which metric spaces, in particular which Banach spaces, it does, or it does not, bi-Lipschitzly embed.

Definition 11. *Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f\colon X \to Y$ is called a bi-Lipschitz embedding if there exist $s > 0$ and $D \geq 1$ such that for all $x, y \in X$,*

$$s \cdot d(x,y) \leq d_Y(f(x), f(y)) \leq D \cdot s \cdot d(x,y). \tag{1.1}$$

As usual,

$$c_Y(X) := \inf\{D \geq 1 \mid \text{equation (1.1) holds for some embedding } f\}$$

denotes the Y-distortion of X. If there is no bi-Lipschitz embedding from X into Y then we set $c_Y(X) = \infty$. A sequence $(X_k)_{k\in\mathbb{N}}$ of metric spaces is said to equi-bi-Lipschitzly embed into a metric space Y if $\sup_{k\in\mathbb{N}} c_Y(X_k) < \infty$.

If two normed spaces are uniformly homeomorphic then all finite-dimensional subspaces of one of the spaces are uniformly embeddable in the other by means of a linear mapping.

Ribe's rigidity theorem [41], namely that if two normed spaces are uniformly homeomorphic then all finite-dimensional subspaces of one of the spaces are uniformly embeddable in the other by means of a linear mapping, suggests that it is reasonable to believe that local geometric properties of Banach spaces could be characterized in purely metric terms. This result spawned what later came to be known as "the Ribe programme".

James [21] introduced the important property of super-reflexivity: a Banach space X is super-reflexive if every Banach space Y which is finitely representable in X is reflexive. Enflo [12] showed that super-reflexivity of X is equivalent to X having an equivalent uniformly convex norm.

The first successful step in the Ribe programme was obtained by Bourgain [6] when he showed that the sequence $(B_k)_{k\in\mathbb{N}}$ of binary trees of height k is a uniformly characterizing sequence for super-reflexivity, i.e. that spaces which are not super-reflexive as those for which the binary trees B_n of depth n embed with uniformly bounded distortion.

We refer to [5] and [36] for a thorough description of the Ribe programme and its successful achievements.

Subsequently, Johnson and Schechtman [25] found two new uniformly characterizing sequences for super-reflexivity, namely the sequence $(D_k^2)_{k\in\mathbb{N}}$ of (2-branching) diamond graphs and the sequence $(L_k^2)_{k\in\mathbb{N}}$ of (2-branching) Laakso graphs. The best known estimate in the literature for the distortion of embeddings of D_n into spaces which are not super-reflexive, due to Pisier [40], is $2+\varepsilon$ for every $\varepsilon > 0$, while the best known estimate for the distortion of embeddings of D_n into $L_1[0,1]$, due to Lee and Rhagavendra [31], is $4/3$.

In Chapter 5 we construct embeddings of $\mathcal{L}_n$ into arbitrary Banach spaces which are not super-reflexive space with disortion $2+\varepsilon$.

Theorem 12. *Suppose X is not super-reflexive. Then, for each $\varepsilon > 0$ and $n \geq 1$, there exists a mapping $f_n\colon \mathcal{L}_n \to X$ such that, for all $a, b \in \mathcal{L}_n$,*

$$\frac{1}{2}d(a,b) - \varepsilon \leq \|f_n(a) - f_n(b)\| \leq d(a,b). \tag{1.2}$$

The embeddings of $\mathcal{L}_n$ which we define depend on the following characterization of not being super-reflexive. Its negation is the characterization of super-reflexivity known as J-convexity.

Theorem 13. *[22, 43] X is not super-reflexive if and only if, for each $m \geq 1$ and $\varepsilon > 0$, there exist $e_1, \ldots, e_m$ in the unit ball of X such that, for each $1 \leq j \leq m$, we have*

$$\|e_1 + \cdots + e_j - e_{j+1} - \cdots - e_m\| \geq m - \varepsilon. \tag{1.3}$$

We then prove a stronger result for $X = L_1[0,1]$.

Theorem 14. *For each $n \geq 1$, there exists a mapping $f_n\colon \mathcal{L}_n \to L_1[0,1]$ such that, for all $a, b \in \mathcal{L}_n$,*

$$\frac{3}{4}d(a,b) \leq \|f_n(a) - f_n(b)\|_1 \leq d(a,b). \tag{1.4}$$

The analogue of Theorem 14 for D_n is proved in [31, Theorem 5.1] with the same distortion of 4/3. Moreover, it is remarked without proof that 4/3 is the best constant for the distortion of embeddings of D_n as $n \to \infty$ [31, p. 359]. In fact, we do not know of any embedding of D_2 into $L_1[0,1]$ with distortion smaller than 4/3.

Our next result shows that $\mathcal{L}_2$ does not embed into $L_1[0,1]$ with distortion smaller than 9/8.

Theorem 15. *Let $f\colon \mathcal{L}_2 \to L_1[0,1]$ satisfy*

$$d(a,b) \le \|f(a) - f(b)\|_1 \le cd(a,b).$$

Then $c \ge 9/8$.

The proof uses the following characterization of isometric embeddability into $L_1[0,1]$.

Theorem 16. *[9, Theorem 6.2.2] Let (M,ρ) be a finite metric space. Then (M,ρ) is isometric to a subset of $\ell_2^2 := (\ell_2, \|\cdot\|_2^2)$ if and only if, for all $k_i \in \mathbb{Z}$ $(1 \le i \le n)$ such that $\sum_{i=1}^n k_i = 0$, we have*

$$\sum_{1 \le i < j \le n} k_i k_j \rho(x_i, x_j) \le 0,$$

where $x_1, \ldots, x_n$ are the distinct elements of M.

In a similar way we can estimate the distortion of metric embeddings of the diamond graph D_2 into $L_1[0,1]$.

Theorem 17. *Let $f\colon D_2 \to L_1[0,1]$ satisfy*

$$d(a,b) \le \|f(a) - f(b)\|_1 \le cd(a,b).$$

Then $c \ge 5/4$.

We shall remark that embedding into $L_1[0,1]$ and the respective estimations for the distortions provide a very useful tool for a number of problems in computer science.

Chapter 2

Duals of Tirilman spaces have unique subsymmetric basic sequences

2.1 Introduction

Definition 18. *[32] A sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space X is called a Schauder basis of X if for every $x \in X$ there is a unique sequence of scalars $\{a_n\}_{n=1}^{\infty}$ so that $x = \sum_{n=1}^{\infty} a_n x_n$.*

An equivalent definition, which we will find the one most useful to work with, is the following condition:

Definition 19. *There is a constant K so that, for every choice of scalars $\{a_n\}_{n=1}^{\infty}$ and integers $n < m$, we have:*

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| \leq K \left\| \sum_{i=1}^{m} a_i x_i \right\|$$

The smallest K which satisfies the above condition is called the basis constant.

In the thesis we will consider the Schauder bases to be normalized.

Symmetric structures play an important role in the theory of Banach spaces. A basic sequence $(x_j)_{j=1}^{\infty}$ is symmetric if the rearranged sequence $(x_{\pi(j)})_{j=1}^{\infty}$ is equivalent to $(x_j)_{j=1}^{\infty}$ for any permutation π of $\mathbb{N}$. Recall that a sequence $(x_j)_{j=1}^{\infty}$ is a basic sequence if it is a (Schauder) basis of its closed linear span; two basic sequences $(x_j)_{j=1}^{\infty}$ and $(y_j)_{j=1}^{\infty}$ are said to be equivalent provided a series $\sum_{j=1}^{\infty} a_j x_j$ converges if and only if $\sum_{j=1}^{\infty} a_j y_j$ does.

The class of subsymmetric basic sequences, that is, those that are unconditional and equivalent to all of their *subsequences* [32], is formally more general than the class of symmetric ones. For a while, these two concepts were believed to be equivalent until Garling [14] provided a counterexample. Later, subsymmetric bases became important on their own within the general theory. For instance, the first arbitrarily distortable Schlumprecht space [44] has a subsymmetric basis which is not symmetric.

Albiac, Ansorena and Wallis [3] used Garling-type spaces to provide the first example of a Banach with a unique subsymmetric basis which is not symmetric. However, as shown in a sequel paper [2], that space contains a continuum of non-equivalent subsymmetric basic sequences. Altshuler [1] (see also Example 3.b.10 in [32]) constructed a space which is not isomorphic to c_0 or ℓ_p for any $1 < p < \infty$ and in which all symmetric basic sequences are equivalent to its symmetric basis. Recently, the first example of a Banach space with *a unique subsymmetric basic sequence which is not symmetric* is given in [7]. That answered a question posed in [27] and [2]. The space under consideration was $Su(T^*)$ [8], the subsymmetric version of T^*. As it became customary, T is the space considered by Figiel and Johnson [13] and its dual T^* is the original space constructed by Tsirelson [47], the first example of a space which does not contain an isomorphic copy of c_0 or ℓ_p, $1 \leq p < \infty$.

Here we will give *more examples of spaces with a subsymmetric but not symmetric basis which contain, up to equivalence, a unique subsymmetric basic sequence.* These examples are based on Tzafriri spaces. Tzafriri [48] had constructed (counter)-examples of spaces with (symmetric bases) showing that the notions of equal-norm type p and equal-norm cotype q are not equivalent to the notions of type p and cotype q for $p, q \neq 2$, respectively. The Tirilman spaces $Ti(p, \gamma)$, where $1 < p < \infty$ and $0 < \gamma < 1$, are modified Tzafriri spaces, which were introduced and studied by Casazza and Shura [8]. They were named after Tzafriri's Romanian surname. We prove that for $1 < p < \infty$ and sufficiently small $0 < \gamma < 1$, the dual space $Ti^*(p, \gamma)$, whose canonical basis is subsymmetric but not symmetric, contains, up to equivalence, a unique subsymmetric basic sequence. That is, all the subsymmetric basic sequences are equivalent to the canonical basis. The method of our proof is parallel to the one in [7]. While there the normalized block bases (x_j) of the canonical basis of $Su(T^*)$ with the property $\|x_j\|_\infty \to 0$ are shown to be asymptotic-c_0 sequences, we show that the similar block bases in $Ti^*(p, \gamma)$ yield asymptotic-ℓ_q sequences, where $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, unlike its dual, $Ti(p, \gamma)$ *has continuum many non-equivalent subsymmetric basic sequences.* This follows immediately from Theorem 21 of [7] which states that if a subsymmetric basis (e_i) is not equivalent to the unit vector basis of c_0 or ℓ_p, then either (e_i) or (e_i^*) admits a continuum of non-equivalent subsymmetric block bases.

2.2 Spaces with a unique subsymmetric basic sequence

Given two basic sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ in Banach spaces X and Y, respectively, we say that $(x_n)_{n=1}^\infty$ *K-dominates* $(y_n)_{n=1}^\infty$ if the bounded linear operator $T(x_n) = y_n$ from $[(x_n)_{n=1}^\infty]$ to $[(y_n)_{n=1}^\infty]$ has norm $\|T\| \leq K$. We say that $(x_n)_{n=1}^\infty$ *dominates* $(y_n)_{n=1}^\infty$ if $(x_n)_{n=1}^\infty$ K-dominates $(y_n)_{n=1}^\infty$ for some $K < \infty$.

A *block basis* with respect to a basic sequence $(x_n)_{n=1}^\infty$ is a sequence $(y_n)_{n=1}^\infty$ of non-zero vectors of the form

$$y_n = \sum_{k=p_n+1}^{p_{n+1}} a_k x_k$$

where $p_1 < p_2 < \cdots$ is an increasing sequence of natural numbers. For a vector x in the closed linear span of $(x_n)_{n=1}^\infty$, its support (with respect to $(x_n)_{n=1}^\infty$) is the set of indices of its non-zero coefficients. For finite sets of natural numbers E and F we say that $E < F$ if $\max(E) < \min(F)$. For a natural number n, we say $n < x$, resp. $n \leq x$, if $n < \min(\text{supp}(x))$, resp. $n \leq \min(\text{supp}(x))$. A basic sequence (x_n) is called *1-subsymmetric* if it is 1-unconditional and isometrically equivalent to its subsequences.

A basic sequence $(x_j)_{j=1}^\infty$ is called *(strongly) asymptotic-ℓ_p*, $1 \leq p < \infty$ if there exist a constant $C > 0$ such that for every $m \in \mathbb{N}$ there is an $M \in \mathbb{N}$ such that for every normalized block basis $(y_j)_{j=1}^m$ of $(x_j)_{j=M}^\infty$ and any set of real numbers (a_i), we have

$$\frac{1}{C}\left(\sum_{i=1}^m |a_i|^p\right)^{\frac{1}{p}} \leq \left\|\sum_{i=1}^m a_i y_i\right\| \leq C\left(\sum_{i=1}^m |a_i|^p\right)^{\frac{1}{p}}.$$

Although we will drop the term 'strongly' when referring to asymptotic-ℓ_p sequences, it is important to note this is a stronger version of the original definition from [35] which was given in a more general setting.

Let $1 < p < \infty$ and $0 < \gamma < 1$. As in the case of Tsirelson space, the norm is defined via an implicit equation. For all $a = (a_i) \in c_{00}$, the linear space of finitely supported real-valued sequences, define

$$\|a\| = \max\left\{\|a\|_\infty, \gamma \sup \frac{\sum_{j=1}^n \|E_j a\|}{n^{\frac{1}{q}}}\right\},$$

where q is such that $\frac{1}{p} + \frac{1}{q} = 1$, and the inner supremum is taken over all n and all finite consecutive sets of natural numbers $1 \leq E_1 < \cdots < E_n$.

The Tirilman space $Ti(p, \gamma)$ is the completion of $(c_{00}, \|\cdot\|)$.

This norm can be computed via the limit of a recursive sequence of norms. More precisely, define an increasing sequence of norms $(\|\cdot\|_m)_{m=1}^{\infty}$ on $\mathbb{R}^{\mathbb{N}}$ as follows:

$$\|a\|_0 = \|a\|_\infty = \sup_{i \in \mathbb{N}} |a_i|$$

and for $m \geq 0$,

$$\|a\|_{m+1} = \max \left\{ \|a\|_m, \ \gamma \sup \frac{\sum_{j=1}^{n} \|E_j a\|_m}{n^{\frac{1}{q}}} \right\} ,$$

where the inner supremum is taken over all families of finite subsets of the natural numbers $1 \leq E_1 < E_2 < \cdots < E_n$ and all $n \in \mathbb{N}$. Then the norm is defined as

$$\|a\| = \lim_{m \to \infty} \|a\|_m \ .$$

It follows from the definition that the unit vectors $(e_n)_{n=1}^{\infty}$ form a 1-subsymmetric basis for $Ti(p, \gamma)$. We shall summarize some of their known properties. The first one is the obvious analogue of Proposition X.d.8 [8] which was proved for $Ti(2, \gamma)$.

Proposition 20. *For every $1 < p < \infty$ and $0 < \gamma < 1$, the canonical basis $(e_n)_{n=1}^{\infty}$ is 1-dominated by every normalized block basis of $(e_n)_{n=1}^{\infty}$.*

Some further properties of $Ti(p, \gamma)$ that were proved in [8] for $Ti(2, \gamma)$ were listed in Theorem 6.1 [42].

Proposition 21. *Let $1 < p < \infty$. Then for sufficiently small $0 < \gamma < 1$ the following hold for $Ti(p, \gamma)$:*

(i) for any normalized successive blocks $(x_j)_{j=1}^{\infty}$ of the basis (e_i), we have

$$\gamma n^{\frac{1}{p}} \leq \left\| \sum_{j=1}^{n} x_j \right\| \leq 3^{\frac{1}{q}} n^{\frac{1}{p}}.$$

(ii) $Ti(p, \gamma)$ does not contain isomorphs of any ℓ_r, $1 \leq r < \infty$ or of c_0. In particular, $Ti(p, \gamma)$ is reflexive.

Remark. We shall apply the above proposition for $\gamma < 3^{-\frac{1}{q}}$.

Actually, we need the more general version of the right-hand inequality of (i), which is the p-analogue of Lemma X.d.4 [8]. We present the modified proof for the sake of completeness.

Proposition 22. *If $0 < \gamma < 3^{-\frac{1}{q}}$ and $(x_j)_{j=1}^n$ are block vectors in $Ti(p,\gamma)$ with consecutive supports, $n \in \mathbb{N}$, then*

$$\left\| \sum_{j=1}^n x_j \right\| \leq 3^{\frac{1}{q}} \left(\sum_{j=1}^n \|x_j\|^p \right)^{\frac{1}{p}}.$$

Proof. Using the definition of the norm via the limit of the recursive sequence of norms, it suffice to show by induction on m that

$$\left\| \sum_{j=1}^n x_j \right\|_m \leq 3^{\frac{1}{q}} \left(\sum_{j=1}^n \|x_j\|^p \right)^{\frac{1}{p}} \tag{2.1}$$

For $m = 0$, (2.1) is obvious. Assume that it is true for some $m \geq 0$ and let

$$1 \leq E_1 < E_2 < \cdots < E_k$$

be a sequence of consecutive finite sets of natural numbers. We split each E_j into at most three disjoint subsets so they satisfy the following conditions:

(a) The first is contained entirely in the support of some x_i,

(b) The second contains completely the supports of several blocks x_i in consecutive order,

(c) The third is contained entirely in the support of another x_i.

Let $F_1 < F_2 < \cdots < F_{k'}$ be re-enumeration of the above formed partitions of the sets E_i, where $k' \leq 3k$. Denote

$$I = \frac{\gamma}{k^{\frac{1}{q}}} \sum_{i=1}^k \left\| E_i \left(\sum_{j=1}^n x_j \right) \right\|_m$$

and observe by the triangle inequality that

$$I \leq \frac{\gamma}{k^{\frac{1}{q}}} \sum_{i=1}^{k'} \left\| F_i \left(\sum_{j=1}^n x_j \right) \right\|_m.$$

Denote

$$A = \left\{ 1 \leq i \leq k' : \begin{array}{l} F_i \text{ completely covers the supports of some } x_j\text{'s,} \\ \text{say } x_{p_i}, x_{p_i+1}, \ldots, x_{q_i}, \text{ where } q_i \geq p_i \end{array} \right\}.$$

For each j, define the set

$$B_j = \{1 \le i \le k' : \; F_i \subseteq \operatorname{supp} x_j\} \;\; ,$$

and let

$$B = \{j : \; x_j \text{ is not covered by any } F_i \text{ for which } i \in A\}\,.$$

For any finite set C denote by $|C|$ the number of its elements. Then we have

$$I \le \frac{\gamma}{k^{\frac{1}{q}}} \; \left(\sum_{i\in A} \left\| \sum_{j=p_i}^{q_i} x_j \right\|_m + \sum_{j\in B} \sum_{i\in B_j} \|F_i x_j\|_m \right)$$

$$\le \frac{\gamma}{k^{\frac{1}{q}}} \; \left(\sum_{i\in A} \left\| \sum_{j=p_i}^{q_i} x_j \right\|_m + \frac{1}{\gamma} \sum_{j\in B} \|x_j\|_{m+1} \, |B_j|^{\frac{1}{q}} \right) .$$

By our inductive hypothesis,

$$I \le \frac{\gamma \; 3^{\frac{1}{q}}}{k^{\frac{1}{q}}} \sum_{i\in A} \left(\sum_{j=p_i}^{q_i} \|x_j\|^p \right)^{\frac{1}{p}} + \frac{1}{k^{\frac{1}{q}}} \sum_{j\in B} \|x_j\|_{m+1} \, |B_j|^{\frac{1}{q}}$$

$$\le \frac{\gamma \; 3^{\frac{1}{q}}}{k^{\frac{1}{q}}} \sum_{i\in A} \left(\sum_{j=p_i}^{q_i} \|x_j\|^p \right)^{\frac{1}{p}} + \frac{1}{k^{\frac{1}{q}}} \sum_{j\in B} \|x_j\| \, |B_j|^{\frac{1}{q}}$$

$$= \frac{1}{k^{\frac{1}{q}}} \left[\sum_{i\in A} \left(\sum_{j=p_i}^{q_i} \|x_j\|^p \right)^{\frac{1}{p}} \left(\gamma \; 3^{\frac{1}{q}} \right) + \sum_{j\in B} \|x_j\| \, |B_j|^{\frac{1}{q}} \right] .$$

By the Cauchy-Schwartz inequality we obtain

$$I \le \frac{1}{k^{\frac{1}{q}}} \left(\sum_{i\in A} \sum_{j=p_i}^{q_i} \|x_j\|^p + \sum_{j\in B} \|x_j\|^p \right)^{\frac{1}{p}} \; \left(\sum_{i\in A} \gamma^q \; 3 + \sum_{j\in B} |B_j| \right)^{\frac{1}{q}} .$$

By our choice of γ, $\gamma < 3^{-\frac{1}{q}}$, so $3\gamma^q < 1$, and thus,

$$I \le \frac{1}{k^{\frac{1}{q}}} \left(\sum_{j=1}^{n} \|x_j\|^p \right)^{\frac{1}{p}} \; \left(|A| + \sum_{j\in B} |B_j| \right)^{\frac{1}{q}} .$$

We have that

$$|A| + \sum_{j\in B} |B_j| \le k' \le 3k.$$

Therefore,

$$I \leq 3^{\frac{1}{q}} \left(\sum_{j=1}^{n} \|x_j\|^p \right)^{\frac{1}{p}}.$$

□

As an immediate corollary we obtain the following

Lemma 23. *Let* $0 < \gamma < 3^{-\frac{1}{q}}$. *Let* (x_j^*) *be a normalized block basis of* (e_j^*) *in the dual space* $Ti^*(\gamma, p)$. *Then for every* n *and every choice of real numbers* $(a_j)_{j=1}^n$, *we have*

$$\left\| \sum_{j=1}^{n} a_j x_j^* \right\| \geq \frac{1}{3^{\frac{1}{q}}} \left(\sum_{j=1}^{n} |a_j|^q \right)^{\frac{1}{q}}.$$

Proof. For any $1 \leq j \leq n$ choose an $x_j \in Ti(\gamma, p)$ with $\|x_j\| = 1$ and $x_j^*(x_j) = 1$. Let $(a_j)_{j=1}^n$ be a set of real numbers. By 1-unconditionality we may assume that $a_j \geq 0$ and $\operatorname{supp} x_j \subseteq \operatorname{supp} x_j^*$. Then by duality,

$$\begin{aligned}
\sum_{j=1}^{n} a_j^q &= \sum_{j=1}^{n} a_j x_j^* (a_j^{\frac{q}{p}} x_j) \\
&= \left(\sum_{j=1}^{n} a_j x_j^* \right) \left(\sum_{j=1}^{n} a_j^{\frac{q}{p}} x_j \right) \\
&\leq \left\| \sum_{j=1}^{n} a_j x_j^* \right\| \left\| \sum_{j=1}^{n} a_j^{\frac{q}{p}} x_j \right\| \\
&\leq \left\| \sum_{j=1}^{n} a_j x_j^* \right\| 3^{\frac{1}{q}} \left(\sum_{j=1}^{n} a_j^q \right)^{\frac{1}{p}},
\end{aligned}$$

which gives the needed inequality. □

Proposition 24 ([42])**.** *Let* $1 < p < \infty$ *and let* $\gamma > 0$ *be sufficiently small. Then* $Ti(p, \gamma)$ *contains no symmetric basic sequence.*

Remark. It was proved in [26] that c_0 is finitely representable in $Ti(2, \frac{1}{2})$ (disjointly w.r.t. (e_j)) which provides an alternative proof that (e_j) is not symmetric.

In fact, this result generalizes to all spaces $Ti(p, \gamma)$ for the range $1 < p < \infty$, $0 < \gamma < 3^{-\frac{1}{q}}$. For the sake of completeness, we shall provide the modified proof in Chapter 4. We postpone this proof until after Chapter 3 for the following reason: the construction of yardsticks in Schlumprecht space, which we use in Chapter 3 is simpler and thus more appropriate to be presented first.

Lemma 25. *Let (e_i) be a 1-unconditional basis of a reflexive Banach space X which is K-dominated by its normalized block bases, where $K \geq 1$. Then (e_i^*) K-dominates all normalized block bases of (e_i^*) in the dual space X^*.*

Proof. Let (x_i^*) be a normalized block-basis of (e_i^*) and let $(a_i)_{i=1}^n$, $n \in \mathbb{N}$, be an arbitrary set of real numbers. (e_i^*) is also 1-unconditional, so we may assume that $a_i \geq 0$ for all $1 \leq i \leq n$. Pick a norming element $w \in X$, $\|w\| = 1$, $\left(\sum_{i=1}^n a_i x_i^*\right)(w) = \|\sum_{i=1}^n a_i x_i^*\|$. Denote $A_i = \mathrm{supp}(x_i^*)$.

The 1-unconditionality of (e_i) allows us to assume that

$$\mathrm{supp}(w) \subseteq \bigcup_{i=1}^n A_i.$$

Let $w_i = w|_{A_i}$ be the restriction of w to the set A_i. Denote $\|w_i\| = c_i$ and

$$B = \{1 \leq i \leq n \colon c_i \neq 0\}.$$

By 1-unconditionality, $c_i \leq 1$, $1 \leq i \leq n$. For each $i \in B$, let $z_i = \frac{w_i}{c_i}$. Clearly $(z_i)_{i=1}^n$ is a normalized block-basis of $(e_i)_{i=1}^\infty$ and

$$w = \sum_{i \in B} c_i z_i.$$

Then,

$$\begin{aligned}
\left\| \sum_{i=1}^n a_i x_i^* \right\| &= \left(\sum_{i=1}^n a_i x_i^* \right) \left(\sum_{i \in B} c_i z_i \right) \\
&= \sum_{i \in B} a_i c_i x_i^*(z_i) \leq \sum_{i \in B} a_i c_i \\
&= \left(\sum_{i \in B} a_i e_i^* \right) \left(\sum_{i \in B} c_i e_i \right) \\
&\leq \left\| \sum_{i \in B} a_i e_i^* \right\| \cdot \left\| \sum_{i \in B} c_i e_i \right\|
\end{aligned}$$

By the K-domination,

$$\left\| \sum_{i \in B} c_i e_i \right\| \leq K \left\| \sum_{i \in B} c_i z_i \right\| = K.$$

Thus,

$$\left\|\sum_{i=1}^{n} a_i x_i^*\right\| \leq K \left\|\sum_{i\in B} a_i e_i^*\right\| \leq K \left\|\sum_{i=1}^{n} a_i e_i^*\right\|.$$

□

Lemma 26. *For any n and any sequence of normalized blocks $(x_j^*)_{j=1}^n$ of $(e_j^*)_{j=1}^\infty$ in $Ti^*(p,\gamma)$,*

$$\left\|\sum_{j=1}^{n} x_j^*\right\| \leq \frac{n^{\frac{1}{q}}}{\gamma}.$$

Proof. By the previous Lemma 25 and Proposition 20, $(x_j^*)_{j=1}^n$ is 1-dominated by $(e_j^*)_{j=1}^n$, so

$$\left\|\sum_{j=1}^{n} x_j^*\right\| \leq \left\|\sum_{j=1}^{n} e_j^*\right\|.$$

The vector $\frac{\gamma}{n^{\frac{1}{q}}}\sum_{j=1}^n e_j^*$ belongs to the unit ball of $Ti^*(p,\gamma)$, see e.g. [34], so $\left\|\sum_{j=1}^n e_j^*\right\| \leq \frac{n^{\frac{1}{q}}}{\gamma}$. □

Lemma 27. *$Ti^*(p,\gamma)$ does not contain an isomorphic copy of ℓ_q (where $\frac{1}{p}+\frac{1}{q}=1$).*

Proof. Assume the contrary. Without loss of generality we may assume that a normalized block basis (x_j^*) of (e_j^*) is C-equivalent to the unit vector basis of ℓ_q. Denote $I_j = \mathrm{supp}(x_j^*)$. Choose norming elements $x_j \in Ti(p,\gamma)$, $\|x_j\| = 1$, $x_j^*(x_j) = 1$. By the 1-unconditionality we may assume that $\mathrm{supp}(x_j) \subseteq I_j \subset \mathbb{N}$ for all $j \in \mathbb{N}$. Clearly, $I_1 < I_2 < \cdots$, and we can denote by P_j the projection on I_j.

Define the projection

$$P(x^*) = \sum_{j=1}^{\infty} \langle P_j(x^*), x_j\rangle x_j^*.$$

Then

$$\begin{aligned}
\|P(x^*)\| &\leq C\left(\sum_{j=1}^{\infty} |\langle P_j(x^*), x_j\rangle|^q\right)^{\frac{1}{q}} \\
&\leq C\left(\sum_{j=1}^{\infty} \|P_j(x^*)\|^q\right)^{\frac{1}{q}} \\
&\overset{\text{Lemma 23}}{\leq} 3^{\frac{1}{q}} C\|x^*\|.
\end{aligned}$$

Thus, the subspace generated by $(x_j^*)_{j=1}^\infty$ is complemented in $Ti^*(p,\gamma)$ which implies that $Ti(p,\gamma)$ contains an isomorphic copy of ℓ_p, a contradiction. $\square$

By Lemma 23 and 26 for all n and all normalized block sequences $(u_i)_{i=1}^n$ in $Ti^*(p,\gamma)$ we have $\|\sum_{i=1}^n u_i\| \overset{K}{\sim} n^{1/q}$ for some K. In [26] it was shown that spaces with such a property are saturated by asymptotic-ℓ_q sequences. An inspection of their proof (of Theorem 3.7) shows that any block sequence (x_i) with $\|x_i\|_\infty \to 0$ is asymptotic-ℓ_q. Thus the next Proposition follows from the proof of Theorem 3.7 in [26]. We reproduce the proof for completeness, which is slightly easier in our case.

Proposition 28. *Let $1 < p < \infty$ and $0 < \gamma < 3^{-1/q}$. Every normalized block sequence $(x_i)_{i=1}^\infty$ in $Ti^*(p,\gamma)$ satisfying $\|x_i\|_\infty \to 0$ is an asymptotic ℓ_q basic sequence where $\frac{1}{p}+\frac{1}{q}=1$.*

Proof. Let $m \in \mathbb{N}, m \geq 2$. Choose $\varepsilon, \delta > 0$, and δ' satisfy

$$\begin{aligned} 0 < \varepsilon &< \frac{1}{4m3^{1/q}}, \\ \delta &= \frac{\varepsilon}{6\gamma^{-1}m}, \\ 0 < \delta' &< \frac{\delta^{q+1}}{\gamma^{-q}m}. \end{aligned} \tag{2.2}$$

Let $M \in \mathbb{N}$ be such that $\|x_i\|_\infty < \delta'$ for all $i \geq M$. Let $(y_i)_{i=1}^m$ be a normalized block basis of $(x_i)_{i\geq M}$. We will show that for all scalars $(a_i)_{i=1}^m$ with $\sum_{i=1}^m |a_i|^q = 1$ we have

$$\frac{1}{3^{1/q}} \leq \left\| \sum_{i=1}^m a_i y_i \right\| \leq 3^{q+1}\gamma^{-q}. \tag{2.3}$$

Fix $(a_i)_{i=1}^m$. The left hand side inequality holds for all normalized block vectors and was shown in Lemma 23.

For each i, write $a_i y_i = \sum_{j=1}^{n_i+1} y_{i,j}$ where $y_{i,j}$'s are successive blocks with $\delta \leq \|y_{i,j}\| < \delta + \delta'$ and $\|y_{i,n_i+1}\| < \delta$. Then by Lemma 23

$$|a_i| = \|a_i y_i\| \geq 3^{-1/q}\Big(\sum_{i=1}^{n_i+1} \|y_{i,j}\|^q\Big)^{1/q} \geq 3^{-1/q}\delta n_i^{1/q}.$$

Thus for all $1 \leq i \leq m$,

$$n_i \leq \frac{3|a_i|^q}{\delta^q}. \tag{2.4}$$

Moreover, by shrinking each $y_{i,j}$ to have norm exactly δ at a cost of δ' we have by Lemma 26 that

$$\|a_i y_i\| \le \gamma^{-1}\delta n_i^{1/q} + n_i\delta' + \delta \le \gamma^{-1}\delta n_i^{1/q} + 2\delta$$

since

$$n_i\delta' + \delta \le \frac{3}{\delta^q}\delta' + \delta \le \frac{3}{\delta^q}\frac{\delta^{q+1}}{\gamma^{-q}m} + \delta \le \frac{\delta}{m} + \delta < 2\delta.$$

If $|a_i| \ge \varepsilon$ then $n_i \ne 0$ and from above

$$\varepsilon \le \|a_i y_i\| \le \gamma^{-1}\delta n_i^{1/q} + 2\delta \le 3\gamma^{-1}\delta n_i^{1/q}$$

since $\gamma^{-1} n_i^{1/q} > 1$. Thus

$$n_i^{1/q} > \frac{\varepsilon\gamma}{3\delta} = 2m.$$

Let

$$N = \sum_{\{i:|a_i|\ge\varepsilon\}} n_i.$$

Then $2m\delta < \delta n_i^{1/q} \le \delta N^{1/q}$, and by above $N\delta' < \delta$. We have, using Lemma 26 again,

$$\begin{aligned}
\Big\|\sum_{\{i:|a_i|\ge\varepsilon\}} a_i y_i\Big\| &\le \gamma^{-1}\delta N^{1/q} + N\delta' + m\delta \\
&\le \gamma^{-1}\delta N^{1/q} + \delta + m\delta \\
&\le \gamma^{-1}\delta N^{1/q} + 2m\delta \\
&\le \gamma^{-1}\delta N^{1/q} + \delta N^{1/q}.
\end{aligned}$$

Thus

$$\Big\|\sum_{\{i:|a_i|\ge\varepsilon\}} a_i y_i\Big\| \le 2\gamma^{-1}\delta N^{1/q}. \tag{2.5}$$

On the other hand, by Lemma 23 we have

$$\Big\|\sum_{\{i:|a_i|\ge\varepsilon\}} a_i y_i\Big\| \ge 3^{-1/q}\Big(\sum_{\{i:|a_i|\ge\varepsilon\}} |a_i|^q\Big)^{1/q} \ge 3^{-1/q}(1-\varepsilon m)^{1/q} \ge \frac{1}{2}3^{-1/q}, \tag{2.6}$$

and

$$\Big\| \sum_{\{i:|a_i|<\varepsilon\}} a_i y_i \Big\| < m\varepsilon \overset{2.2}{<} \frac{1}{4} 3^{-1/q}$$

$$\overset{2.6}{\leq} \frac{1}{2} \Big\| \sum_{\{i:|a_i|\geq\varepsilon\}} a_i y_i \Big\| \overset{2.5}{<} \gamma^{-1}\delta N^{1/q}.$$

Thus by the triangle inequality

$$\begin{aligned}
\Big\| \sum_{i=1}^{m} a_i y_i \Big\|^q &< 3^q \gamma^{-q} \delta^q N \\
&\leq 3^q \gamma^{-q} \delta^q \sum_{\{i:|a_i|\geq\varepsilon\}} n_i \\
&\leq 3^{q+1} \gamma^{-q} \sum_{\{i:|a_i|\geq\varepsilon\}} |a_i|^q \quad \text{by (2.4)} \\
&\leq 3^{q+1} \gamma^{-q}.
\end{aligned}$$

□

Theorem 29. *Let $1 < p < \infty$ and $\gamma > 0$ be sufficiently small. Every subsymmetric basic sequence in the dual space $Ti^*(p, \gamma)$ is equivalent to the subsymmetric canonical basis $(e_j^*)_{j=1}^\infty$ which is not symmetric.*

Proof. By Proposition 24 $(e_j^*)_{j=1}^\infty$ is not symmetric.

Let $(x_j^*)_{j=1}^\infty$ be a normalized subsymmetric basic sequence in $Ti^*(p, \gamma)$. By passing to a subsequence we may assume that $(x_j^*)_{j=1}^\infty$ is a block basis of $(e_j)_{j=1}^\infty$.

If we suppose that $\lim_{j\to\infty} \|x_j^*\|_\infty = 0$, then by combining Lemma 23, Lemma 25 and Proposition 28, we obtain that $(x_j^*)_{j=1}^\infty$ is an asymptotic ℓ_q basic sequence. Then the subsymmetry would imply that $(x_j^*)_{j=1}^\infty$ is equivalent to the unit vector basis of ℓ_q which contradicts Lemma 27.

Thus, by passing again to a subsequence, we may assume that for all $j \in \mathbb{N}$, $\|x_j^*\|_\infty \geq c$ for some $c > 0$. Then $(x_j^*)_{j=1}^\infty$ c-dominates $(e_j^*)_{j=1}^\infty$. On the other hand, by Lemma 25 $(x_j^*)_{j=1}^\infty$ is 1-dominated by $(e_j^*)_{j=1}^\infty$ and therefore, they are equivalent. □

Reflexivity of $Ti(p, \gamma)$ and duality yield the following

Corollary 30. *Let $1 < p < \infty$ and $\gamma > 0$ be sufficiently small. Every subsymmetric basis of a quotient space of $Ti(p, \gamma)$ is equivalent to the canonical basis $(e_j)_{j=1}^\infty$.*

Proposition 31 ([7]). *Let (e_i^*) be a subsymmetric basis which is not equivalent to the unit vector basis of ℓ_p or c_0. Then either (e_i) or (e_i^*) admits a continuum of non-equivalent subsymmetric block bases.*

This, together with Theorem 29, give us the following

Corollary 32. *For $1 < p < \infty$ and sufficiently small γ, the basis (e_i) of $Ti(p, \gamma)$ has a continuum many non-equivalent subsymmetric block bases.*

Chapter 3

On the symmetrized dual of Schlumprecht's space

3.1 Introduction

A central question of the structure theory of Banach spaces is whether every infinite-dimensional space contains an isomorphic copy of a "nice" infinite-dimensional space. The first question in that direction was if any Banach space contained a subspace isomorphic to c_0 or $\ell_p, 1 \leq p < \infty$, or, more generally, a symmetric basic sequence. In 1974 Tsirelson [47] constructed a Banach space without any symmetric sequence. He did it by defining the unit ball of his space as the closed convex hull of a carefully chosen set of vectors. Later, Figiel and Johnson [13] found a nice formula for the norm of the dual of that space and named it "Tsirelson space T". Now everywhere in literature the original space constructed by Tsirelson is referred to as the dual T^*.

In 1991 Schlumprecht [44] constructed another Tsirelson-like space which was the first example of an arbitrarily distortable Banach space. It became the foundation for the celebrated space of Gowers and Maurey [19] which has no unconditional basic sequence. The canonical basis of Schlumprecht's space S is subsymmetric but not symmetric.

The natural symmetrization $S(T)$ of the Tsirelson space T contains a subspace isomorphic to ℓ_1, while the symmetrization $S(T^*)$ of the original Tsirelson space is reflexive, so it does not have the same property [8]. In an unpublished note Schlumprecht showed that the symmetric version of his space S does contain a subspace isomorphic to ℓ_1 (a similar result can be found in [34]).

In this note we prove that, in contrast to the case of $S(T^*)$, the symmetrization $S(S^*)$ of the dual of Schlumprecht's space does contain isomorphically the space ℓ_1.

3.2 Definitions and main result

A sequence $(x_n)_{n=1}^\infty$ in a Banach space is a basic sequence if it is a Schauder basis of its closed linear span. Two normalized basic sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ in a Banach space $(X, \|\cdot\|)$ are said to be C-equivalent, $C \geq 1$, provided that

$$\frac{1}{C}\left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq \left\|\sum_{n=1}^{\infty} a_n y_n\right\| \leq C\left\|\sum_{n=1}^{\infty} a_n x_n\right\|$$

for any choice of real numbers $(a_n)_{n=1}^\infty$.

The following are standard definitions from [32]. A basic sequence $(x_n)_{n=1}^\infty$ is C-unconditional if for any choice of signs $(\epsilon_n)_{n=1}^\infty$, the sequence $(\epsilon_n x_n)_{n=1}^\infty$ is C-equivalent to $(x_n)_{n=1}^\infty$. A Banach space with a C-unconditional basis can be easily renormed so that the basis is 1-unconditional. A basic sequence $(x_n)_{n=1}^\infty$ is K-symmetric if the rearranged sequence $(x_{\pi(n)})_{n=1}^\infty$ is C-equivalent to $(x_n)_{n=1}^\infty$ for any permutation π of the natural numbers. Closely related to symmetry is the notion of subsymmetry. Similarly, a basic sequence $(x_n)_{n=1}^\infty$ is said to be subsymmetric if it is unconditional and any subsequence $(x_{n_j})_{j=1}^\infty$ is equivalent to $(x_n)_{n=1}^\infty$.

Let $(e_i)_{i=1}^\infty$ be an unconditional (say 1-unconditional) Schauder basis of a Banach space $(X, \|\cdot\|)$. One can define a natural symmetrization of X, denoted by $S(X)$ [8] by

$$\left\|\sum_{i=1}^{\infty} a_i e_i\right\|_{S(X)} = \sup_{\pi}\left\|\sum_{i=1}^{\infty} a_i e_{\pi(i)}\right\|$$

where the supremum is taken over all permutations π of $\mathbb{N}$.

Let $(e_i)_{i=1}^\infty$ be the canonical basis of c_{00}, the linear space of all finitely supported real-valued sequences. For $x = \sum_{i=1}^\infty a_i e_i \in c_{00}$, $\operatorname{supp} x$ denotes the set $\{i \in \mathbb{N} : a_i \neq 0\}$.

Let E, F be finite nonempty sets of natural numbers. We write $E < F$ provided $\max E < \min F$. For $x, y \in c_{00}$ we say $x < y$ if $\operatorname{supp} x < \operatorname{supp} y$. For $x = \sum_{i=1}^\infty a_i e_i \in c_{00}$ and E a finite subset of $\mathbb{N}$, $Ex = \sum_{i\in E} a_i e_i$.

We shall now define Schlumprecht's space S. Let $f : [1,\infty) \to [1,\infty)$ be the function

$$f(x) = \log_2(x+1).$$

Then $(S, \|\cdot\|)$ is the completion of c_{00} with respect to the norm $\|\cdot\|$ that satisfies the following implicit equation:

$$\|x\| = \max\left\{\|x\|_\infty, \sup_{E_1<E_2<\cdots<E_n} \frac{1}{f(n)}\sum_{i=1}^{n}\|E_i x\|\right\}$$

Clearly, $(e_i)_{i=1}^{\infty}$ is an 1-unconditional, 1-subsymmetric basis of S, and thus so is the basis $(e_i^*)_{i=1}^{\infty}$ of the dual space S^*.

Like in the case of Tsirelson space one can give an alternative definition through the unit ball of S^*, see [34].

Let $K_0 = \{\pm e_n^* : n \in \mathbb{N}\}$. Define inductively for every $j \geq 0$

$$K_{j+1} = K_j \bigcup \left\{ \frac{1}{f(n)}\sum_{i=1}^{d} x_i^* : \begin{array}{l} x_i^* \in K_i, d \leq n \in \mathbb{N}, \\ n \geq 2, \operatorname{supp} x_1^* < \cdots < \operatorname{supp} x_d^* \end{array} \right\}$$

Finally, let

$$K = \bigcup_{j=0}^{\infty} K_j.$$

The unit ball B_{S^*} of S^* is the closed convex hull of K.

Then for $x \in c_{00}$, $\|x\| = \|x\|_S = \sup\langle x^*, x\rangle$. We shall denote the dual norm again by $\|\cdot\|$, that is, $(S^*, \|\cdot\|)$.

Fact

For every $n \geq 2$,

$$\left\|\frac{1}{f(n)}\sum_{i=1}^{n} e_i^*\right\| = 1$$

Indeed,

$$\frac{1}{f(n)}\sum_{i=1}^{n} e_i^* \in K_1 \subseteq B_{S^*}, \text{ so } \left\|\frac{1}{f(n)}\sum_{i=1}^{n} e_i^*\right\| \leq 1.$$

On the other hand, from [44],

$$\left\|\frac{f(n)}{n}\sum_{i=1}^{n} e_i\right\| = 1,$$

and

$$\left\langle \frac{1}{f(n)}\sum_{i=1}^{n} e_i^*, \frac{f(n)}{n}\sum_{i=1}^{n} e_i \right\rangle = 1$$

We prove now our main result.

Theorem 33. *The symmetrized version of the dual of Schlumprecht's space, $S(S^*)$, contains a subspace isomorphic to ℓ_1.*

Proof. Recall that for $v^* \in S(S^*)$, $v^* = \sum_{i=1}^{\infty} a_i e_i^*$,

$$\|v^*\|_{S(S^*)} = \sup_{\pi} \left\| \sum_{1}^{\infty} a_i e_{\pi(i)}^* \right\|_{S^*}$$

where the supremum is taken over all permutations π of $\mathbb{N}$.

We shall say that two vectors x and y in S have the same distribution if

$$x = \sum_{j=1}^{k} a_j e_{n_j} \text{ and } y = \sum_{j=1}^{k} a_j e_{m_j},$$

where

$$n_1 < n_2 < ... < n_k \text{ and } m_1 < m_2 < ... < m_k.$$

Since the canonical basis of S is 1-subsymmetric, $\|x\| = \|y\|$. The situation is the same for vectors with the same distribution in S^* and $S(S^*)$, since the basis $(e_i^*)_{i=1}^{\infty}$ in S^* is also 1-subsymmetric and $(e_i^*)_{i=1}^{\infty}$ in $S(S^*)$ is similarly 1-subsymmetric.

The main tool in our proof is the "yardstick" construction from [28], Theorem 3, which gives for arbitrary $n \in \mathbb{N}$ a set of disjointly - but not successively - supported vectors (with respect to the canonical basis), which set uniformly equivalent to the unit vector basis of ℓ_∞^n. More precisely, there exists an increasing sequence $(p_k)_{k=0}^{\infty}$ of natural numbers, with $p_0 = 1$, and for $k \geq 0$, $p_{k+1} = p_k d_k$, $d_k \in \mathbb{N}$ and $(d_k)_{k=0}^{\infty}$ rapidly increasing, with the following properties. For every $n \in \mathbb{N}$, there exist n disjointly supported vectors $v_j \in S$, $1 \leq j \leq n$, such that for all $1 \leq k \leq n$, v_k has the same distribution as

$$u_{p_k} = \frac{f(p_k)}{p_k} \sum_{i=1}^{p_k} e_i$$

and

$$\left\| \sum_{j=1}^{n} v_j \right\| \leq 2$$

By [44], $\|v_j\| = 1$. The yardstick construction is defined as follows.

Let

$$\left\{ w_{m_1,m_2,...,m_j} : j \leq n, m_i \leq d_{i-1} \forall i \leq j \right\}$$

be any sequence of natural numbers such that

$$w_{m_1,m_2,...,m_j} < w_{r_1,r_2,...,r_l}$$

if one of the following conditions holds:

- there exists $i \leq max\{j,l\}$ such that $m_s = r_s$ for all $s < i$, and $m_i < r_i$;
- $j < l$ and $m_s = r_s$ for all $s \leq j$.

Then we define for each $1 \leq j \leq n$,

$$v_j = \frac{f(p_j)}{p_j} \sum_{i_1=1}^{d_0} \sum_{i_2=1}^{d_1} \cdots \sum_{i_j=1}^{d_{j-1}} e_{w_{i_1,i_2,\ldots,i_j}} \tag{3.1}$$

By [28] we have

$$\left\| \sum_{j=1}^{n} v_j \right\|_S \leq 2.$$

Now, we define inductively vectors $(v_j^*)_{j=1}^{\infty}$ such that for all $j \in \mathbb{N}$, v_j^* has the same distribution as

$$\frac{1}{f(p_j)} \sum_{i=1}^{p_j} e_i^*$$

and their supports are consecutive with respect to $(e_i^*)_{i=1}^{\infty}$, that is to say,

$$\operatorname{supp} v_1^* < \operatorname{supp} v_2^* < \cdots < \operatorname{supp} v_j^* < \cdots$$

Let $n \in \mathbb{N}$ and $(a_j)_{j=1}^{n}$ be any sequence of n real numbers. Since $(e_i^*)_{i=1}^{\infty}$ is 1-unconditional in both S^* and $S(S^*)$, without loss of generality we may assume for convenience that all $a_j \geq 0$. We want to estimate $\| \sum_{j=1}^{n} a_j v_j^* \|_{S(S^*)}$.

First, by the fact presented in the beginning, $\|v_j^*\|_{S^*} = 1$. Since for each j, all the nonzero coefficients of v_j^* are equal, v_j^* will have the same distribution under any permutation π of $\mathbb{N}$. Since $(e_i^*)_{i=1}^{\infty}$ is 1-subsymmetric in S^*, it follows from the definition of the norm in $S(S^*)$ that $\|v_j^*\|_{S(S^*)} = 1$ for all $j \in \mathbb{N}$.

Now choose a permutation π of $\mathbb{N}$, depending on n, such that π sends v_j^*, $j = 1, 2, ..., n$, to a vector u_j^* such that $\operatorname{supp} u_j^* = \operatorname{supp} v_j$, described in 3.1 for this particular n. We can do this since $\operatorname{supp} v_i^* \cap \operatorname{supp} v_j^* = \emptyset$ for $i \neq j$. Consider $u_n = \sum_{j=1}^{n} v_j$. By [28], $\|u_n\|_S \leq 2$. We shall compute $\langle \sum_{j=1}^{n} a_j u_j^*, u_n \rangle$. Remark that $\sum_{j=1}^{n} a_j u_j^*$ is the image of $\sum_{j=1}^{n} a_j v_j^*$ under the permutation π chosen above. Then

$$\left\langle \sum_{j=1}^{n} a_j u_j^*, u_n \right\rangle = \sum_{j=1}^{n} a_j \langle u_j^*, v_j \rangle$$

$$= \sum_{j=1}^{n} a_j \frac{f(p_j)}{p_j} \frac{1}{f(p_j)} p_j = \sum_{j=1}^{n} a_j$$

Since $\|u_n\|_S \leq 2$, we obtain that

$$\left\|\sum_{j=1}^{n} a_j u_j^*\right\|_{S^*} \geq \frac{1}{2}\sum_{j=1}^{n} a_j$$

By the definition of the norm of $S(S^*)$, we also have

$$\left\|\sum_{j=1}^{n} a_j v_j^*\right\| \geq \frac{1}{2}\sum_{j=1}^{n} a_j$$

Therefore, the sequence $(v_j^*)_{j=1}^{\infty}$ in $S(S^*)$ is 2-equivalent to the unit vector basis of ℓ_1. $\square$

Chapter 4

Yardsticks in Tirilman spaces

It was proved in [26] that c_0 is finitely representable in $Ti\left(2, \frac{1}{2}\right)$ disjointly with respect to the canonical basis (e_i). In this chapter we make the needed adjustments to the proof to obtain a similar result for a wider range of the parameters.

Theorem 34. *Let p and γ be such that $1 < p < \infty$ and $0 < \gamma < 3^{-\frac{1}{q}}$ where $\frac{1}{p} + \frac{1}{q} = 1$. Then c_0 is finitely representable in $Ti\,(p, \gamma)$ disjointly with respect to its canonical basis.*

The first lemma that we will use is from [8]. We present the proof for $1 < p < \infty$.

Lemma 35. *Let $1 < p < \infty$. Then for the norm $\|\cdot\|$ in $Ti\,(p, \gamma)$ and for all $x \in c_{00}$ we have*

$$\|x\| \leq \|x\|_{\ell_p}$$

Proof. We proceed by induction, using the definition of the norm in $Ti\,(p, \gamma)$ as a limit of a sequence of norms $\|\cdot\|_m$.

Clearly, for all $x \in c_{00}$, $\|x\|_\infty \leq \|x\|_{\ell_p}$.

Assume that for $m \geq 1$ we have $\|x\|_{m-1} \leq \|x\|_{\ell_p}$.

Then for every collection of finite subsets of the natural numbers $\mathbb{N}$

$$1 \leq E_1 < E_2 < \cdots < E_k$$

and any natural number $k \geq 2$,

$$\begin{aligned}\frac{\gamma \sum_{j=1}^{k} \|E_j x\|_{m-1}}{k^{\frac{1}{q}}} &\leq \frac{\gamma \sum_{j=1}^{k} \|E_j x\|_{l_p}}{k^{\frac{1}{q}}} \leq \\ &\leq \frac{\gamma \left(\sum_{j=1}^{k} \|E_j x\|_{\ell_p}\right)^{\frac{1}{p}} \left(\sum_{j=1}^{k} 1\right)^{\frac{1}{q}}}{k^{\frac{1}{q}}} \leq \\ &\leq \gamma \|x\|_{\ell_p} \leq \|x\|_{\ell_p}\end{aligned}$$

Taking the supremum over this type of expressions, we reach

$$\|x\|_m \leq \|x\|_{\ell_p}$$

Taking the limit over m, we obtain $\|x\| \leq \|x\|_{\ell_p}$.

$\square$

Lemma 36. *For $(e_i)_{i=1}^{\infty}$ the canonical basis of $Ti\,(p,\gamma)$ and for any $n > \frac{1}{\gamma^p}$ we have*

$$\left\|\sum_{i=1}^{n} e_i\right\| = \gamma n^{\frac{1}{p}}$$

Proof. The lower estimate follows immediately from the definition of the norm (given as an equation). We will focus now on the upper estimate.

Let $E_1 < E_2 < \cdots < E_k$ be such that

$$\left\|\sum_{i=1}^{n} e_i\right\| = \frac{\gamma}{k^{\frac{1}{q}}}\sum_{j=1}^{k}\left\|E_j\left(\sum_{i=1}^{n} e_i\right)\right\|$$

Denote $n_j = |E_j \cap \{1,2,\ldots,n\}|$, and note that

$$\sum_{j=1}^{k} n_j \leq n$$

By Lemma 35 and the Cauchy-Schwarz inequality we have

$$\begin{aligned}\left\|\sum_{i=1}^{n} e_i\right\| &\leq \frac{\gamma}{k^{\frac{1}{q}}}\sum_{j=1}^{k} n_j^{\frac{1}{p}} \leq \\ &\leq \frac{\gamma}{k^{\frac{1}{q}}}\left(\sum_{j=1}^{k} n_j\right)^{\frac{1}{p}} k^{\frac{1}{q}} \leq \\ &\leq \gamma n^{\frac{1}{p}}\end{aligned}$$

$\square$

Lemma 37. *Let us have a $Ti\,(p,\gamma)$ space with $0 < \gamma < 3^{-\frac{1}{q}}$. Let $\delta = \gamma 3^{\frac{1}{q}}$. Assume that $(u_i)_{i=1}^{n}$ is a block basis of (e_i) with $\|u_i\| \leq 1$ for $i \in [1,n]$ (note that the block basis is not presumed to be normalized). Let $E_1 < E_2 < \cdots < E_k$ be finite sets of natural numbers so that for $i \in [1,n]$, $\operatorname{supp} u_i$ intersects at most one E_j. Then*

$$\frac{\gamma}{k^{\frac{1}{q}}}\sum_{j=1}^{k}\left\|E_j\left(\sum_{i=1}^{n} u_i\right)\right\| \leq \delta n^{\frac{1}{p}}$$

Proof. For $j \leq k$ we denote $n_j = |\{i : \operatorname{supp} u_i \cap E_j \neq \emptyset\}|$. Thus $\sum_{j=1}^{k} n_j \leq n$.

By Proposition 22 from Chapter 2 (analog of a lemma from [8]), and the Cauchy-Schwarz inequality,

$$\frac{\gamma}{k^{\frac{1}{q}}} \sum_{j=1}^{k} \left\| E_j \left(\sum_{i=1}^{n} u_i \right) \right\| \leq \frac{\gamma}{k^{\frac{1}{q}}} \sum_{j=1}^{k} 3^{\frac{1}{q}} \sum_{j=1}^{k} n_j^{\frac{1}{p}} \leq$$
$$\leq \frac{\gamma}{k^{\frac{1}{q}}} 3^{\frac{1}{q}} n^{\frac{1}{p}} k^{\frac{1}{q}} = \delta n^{\frac{1}{p}}$$

□

We are now ready to go back to Theorem 34.

Proof. Let $0 < \gamma < 3^{-\frac{1}{q}}$. Then for $\delta = \gamma 3^{\frac{1}{q}}$ we have $\delta < 1$. Let $m \in \mathbb{N}$.

We shall construct disjointly supported normalized vectors $(x_i)_{i=1}^{m}$ in $Ti(p, \gamma)$, $1 < p < \infty$, so that

$$\left\| \sum_{i=1}^{m} x_i \right\| \leq \sum_{i=0}^{\infty} \delta^i + 1$$

By the unconditionality of the basis (e_i) this would complete the proof.

Each x_i will be a normalized average of certain basis vectors, the number of which would rapidly increase with i, and the supports of x_i would be uniformly mixed. The "yardstick" construction of $(x_i)_{i=1}^{m}$ is more complicated than the respective construction in Schlumprecht space that we used in Chapter 3. The different propertise of the functions $\log_n$ and $n^{\frac{1}{q}}$ make the geometry of the Schlumprecht and the Tirilman spaces quite different. While for the "yardstick" construction $(x_i)_{i=1}^{m+1}$ in Schlumprecht space we can add just one more vector x_{m+1} to the already chosen $(x_i)_{i=1}^{m}$, in Tirilman space all of the vectors $(x_i)_{i=1}^{m+1}$ will depend on the given $m + 1$.

Let us fix $m \in \mathbb{N}$ and introduce some notation. Take the following rapidly increasing integers

$$g_1 < g_2 < \cdots < g_m$$

Given these, we define for each $i \leq m$

$$f_i = \prod_{j=1}^{i} g_i$$

We then choose natural numbers $r_{t_1,\dots,t_i}$ for each $i \le m$ and each $t_j \le g_j$ for $1 \le j \le i$ so that

$$r_{t_1,\dots,t_i} < r_{s_1,\dots,s_j}$$

whenever $(t_1,\dots t_i)$ is less than $(s_1,\dots s_j)$ lexicographically, that is to say, $\exists k$, $0 \le k \le i$, s.t. $\forall 1 \le h \le k$ we have $t_h = s_h$ and either $t_{k+1} < s_{k+1}$ or $k = i < j$. For example,

$$r_1 < r_{2,1,4} < r_{2,2} < r_{2,2,4} < r_3$$

We shall say that $(x_i)_{i=1}^m$ corresponds to $(g_1, g_2, \dots, g_m)$ if for every $i < m$ we have

$$x_i = \frac{1}{\gamma f_i^{\frac{1}{p}}} \sum_{j_1=1}^{g_1} \sum_{j_2=1}^{g_2} \cdots \sum_{j_i=1}^{g_i} e_{r_{j_1,j_2,\dots,j_i}}$$

Since (e_i) is 1-subsymmetric, the particular choice of $r_{j_1,j_2,\dots,j_i}$ does not matter, but their order does.

By Lemma 36, $\|x_i\| = 1$ for $i \le m$.

Let $(\epsilon_n)_{n=1}^\infty$ be a sequence of positive numbers such that

$$\sum_{n=1}^{\infty} \epsilon_n < 1 \tag{4.1}$$

We shall prove by induction on m that for every integer $g > \gamma^{-p}$ there exist integers $g < g_2 < \cdots < g_m$ so that for all integers g_1, $\gamma^{-p} < g_1 \le g$, if $(x_i)_{i=1}^m$ corresponds to $(g_1, g_2, \dots, g_m)$ then

$$\left\| \sum_{i=1}^{m} x_i \right\| \le \sum_{i=0}^{m-1} \delta^i + \sum_{i=1}^{m} \epsilon_i \tag{4.2}$$

For ease of use, let us denote

$$M(m) = \sum_{i=0}^{m-1} \delta^i + \sum_{i=1}^{m} \epsilon_i$$

The claim 4.2 is obvious for $m = 1$ so we can assume that it holds for some m. Let $g > \gamma^{-p}$. Choose a natural number d so that

$$g^{\frac{1}{q}} < \epsilon_{m+1} d^{\frac{1}{q}} \tag{4.3}$$

Furthermore, choose another natural number n so that

$$\frac{2dM(m)}{n^{\frac{1}{p}}} < \epsilon_{m+1} \tag{4.4}$$

Let $g_2 = dn$. By the inductive hypothesis, applied to $g_0 = gg_2$, we can find integers $g_0 < g_3 < \cdots < g_{m+1}$ so that if $\gamma^{-p} < s \leq g_0$ and $(y_i)_{i=1}^m$ is a sequence corresponding to $(s, g_3, \ldots, g_{m+1})$, then

$$\left\| \sum_{i=1}^{m} y_i \right\| \leq M(m)$$

Let $\gamma^{-p} < g_1 \leq g$ and let $(x_i)_{i=1}^{m+1}$ corresponds to $(g_1, g_2, \ldots, g_{m+1})$. There exist $k \geq 2$ and $E_1 < \cdots < E_k$ such that

$$\begin{aligned} \left\| \sum_{i=1}^{m+1} x_i \right\| =& \frac{\gamma}{k^{\frac{1}{q}}} \sum_{j=1}^{k} \left\| E_j \left(\sum_{j=1}^{m+1} x_i \right) \right\| \leq \\ \leq& \frac{\gamma}{k^{\frac{1}{q}}} \sum_{j=1}^{k} \|E_j(x_1)\| + \frac{\gamma}{k^{\frac{1}{q}}} \sum_{j=1}^{k} \left\| E_j \left(\sum_{j=2}^{m+1} x_i \right) \right\| \end{aligned}$$

We shall split the proof in two cases.

Case 1: $k \geq d$.
Then

$$\|x_1\|_{\ell_1} = \frac{g_1}{\gamma g_1^{\frac{1}{p}}} = \frac{1}{\gamma} g^{\frac{1}{q}} \leq \frac{1}{\gamma} g^{\frac{1}{q}}$$

By 4.3 we obtain

$$\|x_1\|_{\ell_1} \leq \frac{1}{\gamma} \epsilon_{m+1} d^{\frac{1}{q}}$$

Thus

$$\frac{\gamma}{k^{\frac{1}{q}}} \sum_{j=1}^{k} \|E_j(x_1)\| \leq \frac{\gamma}{k^{\frac{1}{q}}} \|x_1\|_{\ell_1} < \frac{d^{\frac{1}{q}}}{k^{\frac{1}{q}}} \epsilon_{m+1} \leq \epsilon_{m+1}$$

Also, by the definition of the norm we clearly have

$$\frac{\gamma}{k^{\frac{1}{q}}} \sum_{j=1}^{k} \left\| E_j \left(\sum_{i=2}^{m+1} x_i \right) \right\| \leq \left\| \sum_{i=2}^{m+1} x_i \right\|$$

Now $(x_i)_{i=2}^{m+1}$ corresponds to the m-tuple $(g_1g_2, g_3, \ldots, g_{m+1})$ and since

$$g_1g_2 \leq gg_2 = g_0$$

by the inductive hypothesis we obtain

$$\left\| \sum_{i=2}^{m+1} x_i \right\| \leq M(m)$$

Thus

$$\left\| \sum_{i=1}^{m+1} x_i \right\| \leq M(m) + \epsilon_{m+1} < M(m+1)$$

which concludes the proof in this case.

Case 2: $k < d$**.**
For the term x_1 we now use the trivial estimate

$$\frac{\gamma}{k^{\frac{1}{q}}} \sum_{j=1}^{k} \|E_j\,(x_1)\| \leq \|x_1\| = 1$$

To estimate the $\sum_{i=2}^{m+1} x_i$ term, we will represent x_i for $2 \leq i \leq m+1$ the following way:

$$x_i = \sum_{h=1}^{n} x_{i,h}$$

where $(x_{i,h})_{h=1}^{n}$ is an identically distributed block basis. Thus

$$|\operatorname{supp} x_{i,h}| = \frac{f_i}{n} = g_1 d g_3 \ldots g_i$$

By Lemma 36,

$$\|x_{i,h}\| = \gamma \left(\frac{f_i}{n} \right)^{\frac{1}{p}} \frac{1}{\gamma f_i^{\frac{1}{p}}} = \frac{1}{n^{\frac{1}{p}}}$$

for every $2 \leq i \leq m+1$, $1 \leq h \leq n$.

Thus for every $1 \leq h \leq n$, the normalized block basis $\left(n^{\frac{1}{p}} x_{i,h}\right)_{i=2}^{m+1}$ corresponds to $\left(\frac{g_1g_2}{n}, g_3, \ldots, g_{m+1}\right)$ and since

$$\frac{g_1g_2}{n} = g_1 d \leq gg_2 = g_0$$

by the inductive hypothesis we get for all $1 \leq h \leq n$

$$\left\| \sum_{i=2}^{m+1} n^{\frac{1}{p}} x_{i,h} \right\| \leq M(m) \tag{4.5}$$

Set for every $1 \leq h \leq n$

$$z_h = \sum_{i=2}^{m+1} x_{i,h}$$

Then $(z_h)_{h=1}^n$ is an identically distributed block basis of (e_i).
Hence by 4.5 we have

$$\|z_1\| = \|z_2\| = \cdots = \|z_n\| \leq \frac{M(m)}{n^{\frac{1}{p}}}$$

Denote

$$\|z_1\| = \|z_2\| = \cdots = \|z_n\| = a$$

Since $E_1 < \cdots < E_k$ are consecutive blocks, for any $j \leq k$ there exist at most two h's for which $E_j \cap \operatorname{supp} z_h \neq \emptyset$ and $E_{j'} \cap \operatorname{supp} z_h \neq \emptyset$ for some $j' \neq j$.

For every $j \leq k$ let

$$\tilde{E}_j = \bigcup \left\{ \operatorname{supp} z_h : \begin{array}{l} 1 \leq h \leq n, \operatorname{supp} z_h \cap E_j \neq \emptyset, \\ \exists j' \neq j, j' \leq k : \operatorname{supp} z_h \cap E_{j'} \neq \emptyset \end{array} \right\}$$

Set

$$z = \sum_{i=2}^{m+1} x_i$$

Then since $a \leq \frac{M(m)}{n^{\frac{1}{p}}}$, we have

$$\left\| E_j(z) - \tilde{E}_j(z) \right\| \leq \frac{2M(m)}{n^{\frac{1}{p}}}$$

Hence

$$\sum_{j=1}^{k} \|E_j(z)\| \leq \frac{2M(m)k}{n^{\frac{1}{p}}} + \sum_{j=1}^{k} \left\| \tilde{E}_j(z) \right\|$$

Now using 4.4 we obtain

$$\frac{2M(m)k}{n^{\frac{1}{p}}} \leq \frac{2M(m)d}{n^{\frac{1}{p}}} < \epsilon_{m+1}$$

Also, by Lemma 37,

$$\sum_{j=1}^{k} \left\| \tilde{E}_j(z) \right\| \leq a \frac{k^{\frac{1}{q}}}{\gamma} \delta n^{\frac{1}{p}}$$

Thus

$$\frac{\gamma}{k^{\frac{1}{q}}} \sum_{j=1}^{k} \|E_j(z)\| \leq \epsilon_{m+1} + a\delta n^{\frac{1}{p}}$$

Now, since $an^{\frac{1}{p}} \leq M(m)$ and $\delta < 1$, this implies

$$\frac{\gamma}{k^{\frac{1}{q}}} \sum_{j=1}^{k} \|E_j(z)\| < \delta \sum_{i=0}^{m-1} \delta^i + \sum_{i=1}^{m} \epsilon_i + \epsilon_{m+1} = \sum_{i=1}^{m} \delta^i + \sum_{i=1}^{m+1} \epsilon_i$$

From the estimate we have about x_1 of $1 = \delta^0$ we obtain

$$\left\| \sum_{i=1}^{m+1} x_i \right\| \leq M(m+1)$$

which concludes the proof.

□

Remark 38. $(\ell_\infty^n)_{n=1}^\infty$ *are universal for all finite dimensional Banach spaces, that is to say, for every finite dimensional space X there exists an n such that X can be isometrically embedded in ℓ_∞^n. Thus, the finite dimensional structure of $Ti(p,\gamma)$ for $1 < p < \infty$, $\gamma < 3^{-\frac{1}{q}}$ is as rich as possible in the sense that every finite dimensional space is $1+\epsilon$ embeddable in $Ti(p,\gamma)$ for arbitrary $\epsilon > 0$. The last follows from the theorem of James that c_0 is not distortable, and its finite dimensional version.*

Chapter 5

Metric embeddings of Laakso graphs into Banach spaces

5.1 Introduction

James [21] introduced the important property of super-reflexivity: a Banach space X is super-reflexive if every Banach space Y which is finitely representable in X is reflexive.

Definition 39. *An infinite dimensional Banach space Y is said to be finitely representable in an infinite dimensional Banach space X if for every finite-dimensional subspace $E \subset Y$ and every $\epsilon > 0$ there exists a subspace $F \subset X$ and an invertible bounded linear operator*

$$T : E \to F$$

such that

$$\|T\|.\|T^{-1}\| < 1 + \epsilon$$

Definition 40. *A norm $\|\cdot\|$ of a Banach space X is called uniformly convex iff for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every $x, y \in X$ with $\|x\| = 1$ and $\|y\| = 1$, the condition*

$$\|x - y\| \geq \epsilon$$

implies that

$$\left\|\frac{x + y}{2}\right\| \leq 1 - \delta$$

Enflo [12] showed that super-reflexivity of X is equivalent to X having an equivalent uniformly convex norm.

Let us recall the definition of the diamond and Laakso graphs.

Definition 41. *The diamond graph of level* 0 *has two vertices joined by an edge of length* 1 *and is denoted by* D_0. *The diamond graph* D_n *is obtained from* D_{n-1} *in the following way. Each edge* uv *of* D_{n-1} *is replaced by a quadrilateral* u, a, v, b, *with edges* ua, av, vb, bu *of length* 1. *(See Figure 5.1.)*

Definition 41 was introduced in [20].

Definition 42. *The Laakso graph of level* 0 *has two vertices joined by an edge of length* 1 *and is denoted* $\mathcal{L}_0$. *The Laakso graph* $\mathcal{L}_n$ *is obtained from* $\mathcal{L}_{n-1}$ *according to the following procedure. Each edge* $uv \in E(\mathcal{L}_{n-1})$ *is replaced by the graph* $\mathcal{L}_1$ *exhibited in Figure 5.2 in which each edge has length* 1.

Definition 42 was introduced in [30] based on an idea of Laakso [29].

Let $f\colon (M,\rho) \to (N,\sigma)$ be a bilipschitz mapping beween metric spaces. The distortion of f is defined to be the infimum of b/a, where a, b are positive constants such that

$$a\rho(x,y) \le \sigma(f(x), f(y)) \le b\rho(x,y) \qquad (x, y \in M).$$

Bourgain [6] characterized Banach spaces which are not super-reflexive as those for which the binary trees B_n of depth n embed with uniformly bounded distortion. Subsequently, Johnson and Schechtman [25] characterized Banach spaces which are not super-reflexive as those for which the diamond graphs D_n and the Laakso graphs $\mathcal{L}_n$ embed with uniformly bounded distortion. The best known estimate in the literature for the distortion of embeddings of D_n into arbitrary Banach spaces which are not super-reflexive, due to Pisier [40], is $2+\varepsilon$ for every $\varepsilon > 0$, while the best known estimate for the distortion of embeddings of D_n into $L_1[0,1]$, due to Lee and Rhagavendra [31], is $4/3$.

Ostrovskii and Randrianantoanina [38] constructed embeddings of the k-branching diamond graphs $D_{n,k}$ and the k-branching Laakso graphs $L_{n,k}$ into arbitrary Banach spaces which are not super-reflexive with distortion $8+\varepsilon$. Swift [46] constructed embeddings of the family of bundle graphs generated by a finitely-branching bundle graph G into Banach spaces which are not super-reflexive with distortion bounded above by a number not depending on the target space or the branching number of G. In particular, he proved that the finitely branching parasol graphs also embed with distortion $8+\varepsilon$.

In the present article we construct embeddings of $\mathcal{L}_n$ into arbitrary Banach spaces which are not super-reflexive with distortion $2+\varepsilon$ and into $L_1[0,1]$ with distortion $4/3$. We also show that $\mathcal{L}_2$ does not embed into $L_1[0,1]$ with distortion smaller than $9/8$.

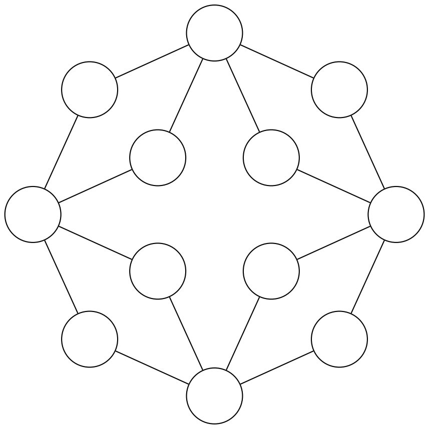

Figure 5.1: The diamond graph D_2.

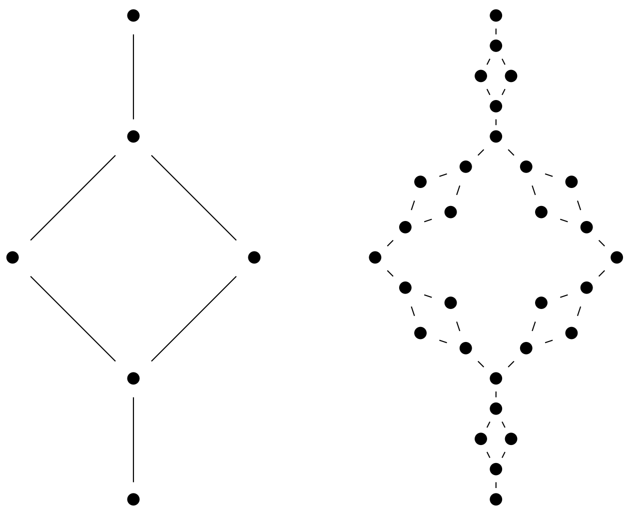

Figure 5.2: The Laakso graphs $\mathcal{L}_1$ and $\mathcal{L}_2$

5.2 Results

The embeddings of $\mathcal{L}_n$ which we define depend on the following characterization of not being super-reflexive. Its negation is the characterization of super-reflexivity known as J-convexity.

Theorem 43. ***A*** *[22, 43] X is not super-reflexive if and only if, for each $m \geq 1$ and $\varepsilon > 0$, there exist $e_1, \ldots, e_m$ in the unit ball of X such that, for each $1 \leq j \leq m$, we have*

$$\|e_1 + \cdots + e_j - e_{j+1} - \cdots - e_m\| \geq m - \varepsilon. \tag{5.1}$$

Remark 44. *It follows easily from Theorem A that if X is not super-reflexive then, for each $n \geq 1$ and $\varepsilon > 0$, B_n embeds into X with distortion $1 + \varepsilon$. This is not true, however, for D_n and $\mathcal{L}_n$ if $n \geq 2$.*

We will make use of the following two consequences of Theorem A.

Lemma 45. *Suppose X is not super-reflexive. Let $(e_i)_{i=1}^m$ be as in Theorem A. If $\max A < \min B$ then*

$$\|\sum_{i \in A} e_i - \sum_{i \in B} e_i\| \geq |A| + |B| - \varepsilon.$$

Proof. This follows at once from (5.1) and the triangle inequality. $\square$

Lemma 46. *Suppose X is not super-reflexive. Let $(e_i)_{i=1}^m$ be as in Theorem A. If $\max A < \min B$ or $\max B < \min A$ then*

$$\|\sum_{i \in A} \varepsilon_i e_i + \sum_{i \in B} e_i\| \geq |B| - \varepsilon.$$

for all choices of signs $\varepsilon_i = \pm 1$.

Proof. Let

$$A^+ = \{i \in A \colon \varepsilon_i = 1\}$$

and let

$$A^- = \{i \in A \colon \varepsilon_i = -1\}.$$

If $|A^+| \geq |A^-|$ then

$$\begin{aligned}\|\sum_{i \in A} \varepsilon_i e_i + \sum_{i \in B} e_i\| &\geq \|\sum_{i \in A^+} e_i + \sum_{i \in B} e_i\| - |A^-| \\ &\geq |A^+| + |B| - \varepsilon - |A^-|\end{aligned}$$

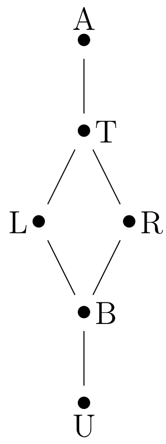

Figure 5.3: The Laakso graph $\mathcal{L}_1$

(by Lemma 45)

$$\geq |B| - \varepsilon.$$

On the other hand, if $|A^-| > |A^+|$ then

$$\begin{aligned}\|\sum_{i\in A}\varepsilon_i e_i + \sum_{i\in B} e_i\| &\geq \|-\sum_{i\in A^-} e_i + \sum_{i\in B} e_i\| - |A^+| \\ &\geq |A^-| + |B| - \varepsilon - |A^+|\end{aligned}$$

(by Lemma 45)

$$\geq 1 + |B| - \varepsilon.$$

□

Theorem 47. *Suppose X is not super-reflexive. Then, for each $\varepsilon > 0$ and $n \geq 1$, there exists a mapping $f_n\colon \mathcal{L}_n \to X$ such that, for all $a, b \in \mathcal{L}_n$,*

$$\frac{1}{2}d(a,b) - \varepsilon \leq \|f_n(a) - f_n(b)\| \leq d(a,b). \tag{5.2}$$

Proof. Let $\varepsilon > 0$ be fixed. For each $n \geq 1$, select vectors $(e_i^n)_{i=1}^{4^n}$ satisfying Theorem A for $m = 4^n$. We define the mappings f_n inductively.

We begin with the base case $n = 1$. Label the vertices of $\mathcal{L}_1$ as shown in Figure 5.3. We define $f_1\colon \mathcal{L}_1 \to X$ as follows:

$$f_1(A) = 0, f_1(T) = e_1^1, f_1(L) = e_1^1 + e_2^1, f_1(R) = e_1^1 + e_3^1$$

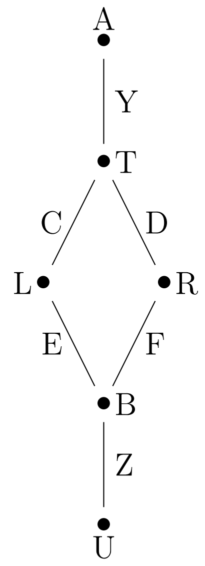

Figure 5.4: The Laakso graph $\mathcal{L}_n$

$$f_1(B) = e_1^1 + e_2^1 + e_3^1, f_1(U) = e_1^1 + e_2^1 + e_3^1 + e_4^1.$$

Using Lemma 45 it is easily checked that f_1 satisfies, for all $a, b \in \mathcal{L}_1$,

$$d(a,b) - \varepsilon \leq \|f_1(a) - f_1(b)\| \leq d(a,b).$$

For example,

$$f_1(L) - f_1(R) = e_2^1 - e_3^1,$$

so

$$2 - \varepsilon \leq \|f_1(L) - f_1(R)\| \leq 2$$

as required.

Now suppose $n \geq 2$. We regard $\mathcal{L}_n$ as being obtained from $\mathcal{L}_1$ by replacing each edge of $\mathcal{L}_1$ by a copy of $\mathcal{L}_{n-1}$. Thus $\mathcal{L}_n$ is composed of 6 copies of $\mathcal{L}_{n-1}$, labelled as Y, C, D, E, F and Z in Figure 5.4.

We have labelled the vertices A, T, L, R, B and U of $\mathcal{L}_n$ which correspond to the vertices of $\mathcal{L}_1$. The correspondence between $\mathcal{L}_{n-1}$ and each of its copies in $\mathcal{L}_n$, namely Y, C, D, E, F, and Z, is the natural 'downward' correspondence in which the vertex A of $\mathcal{L}_{n-1}$ is mapped to the vertices A, T, T, L, R, and B of $\mathcal{L}_n$ respectively. Note that the vertex T of $\mathcal{L}_n$ corresponds to the vertex U of Y and to the vertex A of C and D. There are similar correspondences for L, R and B.

Let $((e_i^n)^*)_{i=1}^{4^n}$ be the coordinate functionals satisfying $(e_i^n)^*(e_j^n) = \delta_{i,j}$ The mapping $f_n \colon \mathcal{L}_n \to X$ will be of the following form:

$$f_n(a) = \sum_{i=1}^{4^n} (e_i^n)^*(f_n(a)) e_i^n, \tag{5.3}$$

where $(e_i^n)^*(f_n(a)) \in \{0,1\}$ and $\operatorname{supp}(f_n(a)) = \{i \colon (e_i^n)^*(f_n(a)) = 1\}$ has size $|\operatorname{supp}(f_n(a))| = d(A,a)$. Note that $d(A,a)$ represents the 'depth' of a in $\mathcal{L}_n$.

To define f_n inductively, we suppose that $f_{n-1} \colon \mathcal{L}_{n-1} \to X$ has already been defined to be of the form (5.3) with n replaced by $n-1$.

Let $\rho \colon \mathcal{L}_{n-1} \to X$ be a 'copy' of f_{n-1} with $(e_i^{n-1})_{i=1}^{4^{n-1}}$ replaced by $(e_i^n)_{i=1}^{4^{n-1}}$. The formal definition is as follows:

$$\rho(a) = \sum_{i=1}^{4^{n-1}} (e_i^{n-1})^*(f_{n-1}(a)) e_i^n.$$

Similarly, let $\theta \colon \mathcal{L}_{n-1} \to X$ be a copy of f_{n-1} with $(e_i^{n-1})_{i=1}^{4^{n-1}}$ replaced by $(e_i^n)_{i=4^{n-1}+1}^{2\cdot 4^{n-1}}$. Formally,

$$\theta(a) = \sum_{i=1}^{4^{n-1}} (e_i^{n-1})^*(f_{n-1}(a)) e_{4^{n-1}+i}^n.$$

Similarly, let $\phi \colon \mathcal{L}_{n-1} \to X$ be a copy of f_{n-1} with $(e_i^{n-1})_{i=1}^{4^{n-1}}$ replaced by $(e_i^n)_{i=2\cdot 4^{n-1}+1}^{3\cdot 4^{n-1}}$. Formally,

$$\phi(a) = \sum_{i=1}^{4^{n-1}} (e_i^{n-1})^*(f_{n-1}(a)) e_{2\cdot 4^{n-1}+i}^n.$$

Finally, let $\sigma \colon \mathcal{L}_{n-1} \to X$ be a copy of f_{n-1} with $(e_i^{n-1})_{i=1}^{4^{n-1}}$ replaced by $(e_i^n)_{i=3\cdot 4^{n-1}+1}^{4^n}$. Formally,

$$\sigma(a) = \sum_{i=1}^{4^{n-1}} (e_i^{n-1})^*(f_{n-1}(a)) e_{3\cdot 4^{n-1}+i}^n.$$

Recall that Y, C, D, E, F and Z are 'copies' of $\mathcal{L}_{n-1}$. Let W be any one of these copies. For $a \in W$, let $\overline{a} \in \mathcal{L}_{n-1}$ denote the element of $\mathcal{L}_{n-1}$ which corresponds to a. Now we define $f_n\colon \mathcal{L}_n \to X$ as follows:

$$f_n(a) = \begin{cases} \rho(\overline{a}), & a \in Y \\ \sum_{i=1}^{4^{n-1}} e_i^n + \theta(\overline{a}), & a \in C \\ \sum_{i=1}^{4^{n-1}} e_i^n + \phi(\overline{a}), & a \in D \\ \sum_{i=1}^{2\cdot 4^{n-1}} e_i^n + \phi(\overline{a}), & a \in E \\ \sum_{i=1}^{4^{n-1}} e_i^n + \sum_{i=2\cdot 4^{n-1}+1}^{3\cdot 4^{n-1}} e_i^n + \theta(\overline{a}), & a \in F \\ \sum_{i=1}^{3\cdot 4^{n-1}} e_i^n + \sigma(\overline{a}), & a \in Z. \end{cases}$$

Note that at the vertices T, L, R and B, which connect the copies of $\mathcal{L}_{n-1}$, f_n is defined twice, but both definitions agree. So f_n is well-defined.

Now we verify (5.2). We begin with the right-hand inequality. If $d(a,b) = 1$, i.e., if a and b are adjacent vertices in $\mathcal{L}_n$, then it is clear from the definition that $\|f_n(a) - f_n(b)\| \le 1$. Since d is the shortest path metric, the right-hand inequality follows at once from the triangle inequality in X.

We now turn to the left-hand inequality. If a and b belong to the same copy of $\mathcal{L}_{n-1}$ (either Y, C, D, E, F or Z) then the left-hand inequality follows from the inductive hypothesis. So suppose that they belong to different copies. There are several cases to consider.

Case 1. Suppose that a is 'above' b in $\mathcal{L}_n$. Then $\operatorname{supp}(f_n(a)) \subseteq \operatorname{supp}(f_n(b))$. Using Lemma 45,

$$\begin{aligned} \|f_n(b) - f_n(a)\| &= \|\sum_{i \in \operatorname{supp}(f_n(b)) \setminus \operatorname{supp}(f_n(a))} e_i^n\| \\ &\ge |\operatorname{supp}(f_n(b))| - |\operatorname{supp}(f_n(a))| - \varepsilon \\ &= d(a,b) - \varepsilon. \end{aligned}$$

Case 2. Suppose $a \in C$, $b \in D$.

$$\begin{aligned} \|f_n(a) - f_n(b)\| &= \|\theta(\overline{a}) - \phi(\overline{b})\| \\ &\ge |\operatorname{supp}(\theta(\overline{a}))| + |\operatorname{supp}(\phi(\overline{b}))| - \varepsilon \end{aligned}$$

(by Lemma 45 since $\max \operatorname{supp}(\theta(a)) < \min \operatorname{supp}(\phi(b))$)

$$\begin{aligned} &= d(T,a) + d(T,b) - \varepsilon \\ &= d(a,b) - \varepsilon. \end{aligned}$$

Case 3. Suppose $a \in C$, $b \in F$. Note that in this case $d(a,b) \le 2 \cdot 4^{n-1}$. Hence

$$f_n(a) - f_n(b) = \theta(\overline{a}) - (\theta(\overline{b}) + \sum_{i=2\cdot 4^{n-1}+1}^{3\cdot 4^{n-1}} e_i^n).$$

Note that $\theta(\overline{a}) - \theta(\overline{b}) = \sum_{i \in A} \varepsilon_i e_i^n$, where $A \subseteq \{i \colon 4^{n-1} + 1 \leq i \leq 2 \cdot 4^{n-1}\}$ and $\varepsilon_i = \pm 1$. Hence, by Lemma 46,

$$\begin{aligned}\|f_n(a) - f_n(b)\| &= \|\sum_{i \in A} \varepsilon_i e_i^n - \sum_{i=2\cdot 4^{n-1}+1}^{3\cdot 4^{n-1}} e_i^n\| \\ &\geq 4^{n-1} - \varepsilon \\ &\geq \frac{1}{2} d(a,b) - \varepsilon.\end{aligned}$$

Case 4. Suppose $a \in D, b \in E$. This is similar to Case 3. Note that $d(a,b) \leq 2 \cdot 4^{n-1}$.

$$\begin{aligned}f_n(a) - f_n(b) &= \phi(\overline{a}) - (\sum_{i=4^{n-1}+1}^{2\cdot 4^{n-1}} e_i^n + \phi(\overline{b})) \\ &= (\phi(\overline{a}) - \phi(\overline{b})) - \sum_{i=4^{n-1}+1}^{2\cdot 4^{n-1}} e_i^n.\end{aligned}$$

Note that $\phi(\overline{a}) - \phi(\overline{b}) = \sum_{i \in A} \varepsilon_i e_i^n$, where $A \subseteq \{i \colon 2 \cdot 4^{n-1} + 1 \leq i \leq 3 \cdot 4^{n-1}\}$ and $\varepsilon_i = \pm 1$. Hence, by Lemma 46,

$$\begin{aligned}\|f_n(a) - f_n(b)\| &= \|\sum_{i \in A} \varepsilon_i e_i^n - \sum_{i=4^{n-1}+1}^{2\cdot 4^{n-1}} e_i^n\| \\ &\geq 4^{n-1} - \varepsilon \\ &\geq \frac{1}{2} d(a,b) - \varepsilon.\end{aligned}$$

Case 5. Suppose $a \in E, b \in F$. This is similar to Case 2. Note that

$$f_n(a) - f_n(b) = (\sum_{i=4^{n-1}+1}^{2\cdot 4^{n-1}} e_i^n - \theta(\overline{b})) - (\sum_{i=2\cdot 4^{n-1}+1}^{3\cdot 4^{n-1}} e_i^n - \phi(\overline{a})).$$

Hence, by Lemma 45,

$$\begin{aligned}\|f_n(a) - f_n(b)\| &\geq (4^{n-1} - |\operatorname{supp}(\theta(\overline{b}))|) + (4^{n-1} - |\operatorname{supp}(\phi(\overline{a}))|) - \varepsilon \\ &= d(b,B) + d(B,a) - \varepsilon \\ &= d(a,b) - \varepsilon.\end{aligned}$$

□

Remark 48. *The analogue of Theorem 47 for D_n is proved in [40, Theorem 13.17, (13.26)] with the same distortion of $2+\varepsilon$.*

We now prove a stronger result for $X = L_1[0,1]$.

Theorem 49. *For each $n \geq 1$, there exists a mapping $f_n \colon \mathcal{L}_n \to L_1[0,1]$ such that, for all $a, b \in \mathcal{L}_n$,*

$$\frac{3}{4} d(a,b) \leq \|f_n(a) - f_n(b)\|_1 \leq d(a,b). \tag{5.4}$$

The proof requires the following elementary lemma.

Lemma 50. *For $0 \leq s, t \leq 1$,*

$$1 + \min(s+t, 2-s-t) \leq \frac{4}{3}(1 + s + t - 2st)$$

with equality if $s = t = 1/2$.

Proof. First suppose $x := s + t \leq 1$. Then $\min(s+t, 2-s-t) = x$ and $st \leq x^2/4$. Hence

$$\begin{aligned}\frac{4}{3}(1+s+t-2st) - (1+(s+t)) &\geq \frac{4}{3}(1 + x - \frac{x^2}{2}) - 1 - x \\ &= \frac{1}{3} + \frac{x}{3} - \frac{2x^2}{3} \\ &\geq 0.\end{aligned}$$

The case $1 \leq s+t$ is similar. □

Proof of Theorem 49. Each f_n will be of the following form:

$$f_n(a) = 4^n 1_{H_n(a)} \qquad (a \in \mathcal{L}_n), \tag{5.5}$$

where $H_n(a) \subseteq [0,1]$ and $|H_n(a)| = 4^{-n} d(A,a)$. We begin with the base case $n = 1$:

$$H_1(A) = \emptyset, H_1(T) = [0,1/4]; H_1(L) = [0,1/2];$$
$$H_1(R) = [0,1/4] \cup [1/2,3/4]; H_1(B) = [0,3/4]; H_1(U) = [0,1].$$

It is easily seen that f_1 is an isometry.

For $n \geq 2$ the definition of f_n is inductive. Suppose that f_{n-1} has been defined to be of the form (5.5). Let θ and ϕ be identically distributed copies of the mapping $a \mapsto H_{n-1}(a)$. Moreover, we require θ and ϕ to be stochastically independent, i.e.,

$$|\theta(a) \cap \phi(b)| = |\theta(a)||\phi(b)| \qquad (a, b \in \mathcal{L}_{n-1}).$$

We use θ and ϕ to define H_n as follows:

$$H_n(a) = \begin{cases} \frac{1}{4}\theta(\overline{a}), & a \in Y \\ [0,1/4] \cup (\frac{1}{4} + \frac{1}{4}\theta(\overline{a})), & a \in C \\ [0,1/4] \cup (\frac{1}{2} + \frac{1}{4}\phi(\overline{a})), & a \in D \\ [0,1/2] \cup (\frac{1}{2} + \frac{1}{4}\theta(\overline{a})), & a \in E \\ [0,1/4] \cup (\frac{1}{4} + \frac{1}{4}\phi(\overline{a})) \cup [\frac{1}{2}, \frac{3}{4}], & a \in F \\ [0,3/4] \cup (\frac{3}{4} + \frac{1}{4}\theta(\overline{a})), & a \in Z. \end{cases}$$

The right-hand inequality of (5.4) follows as in the proof of Theorem 47. For the left-hand inequality we may assume that a and b belong to different copies of $\mathcal{L}_{n-1}$.

Case 1. Suppose that a is 'above' b in $\mathcal{L}_n$. Then $H_n(a) \subseteq H_n(b)$, so

$$d(a,b) = 4^n(|H_n(b)| - |H_n(a)|) = \|f_n(a) - f_n(b)\|_1.$$

Case 2. Suppose $a \in C$, $b \in D$. Then

$$\begin{aligned} \|f_n(a) - f_n(b)\|_1 &= 4^{n-1}(|\theta(\overline{a})| + |\phi(\overline{b})|) \\ &= d(a,T) + d(b,T) \\ &= d(a,b). \end{aligned}$$

Case 3. Suppose $a \in C$, $b \in F$. Note that

$$d(a,b) = 4^{n-1}(1 + \min(|\theta(\overline{a})| + |\phi(\overline{b})|, 2 - |\theta(\overline{a})| - |\phi(\overline{b})|)).$$

Then

$$\begin{aligned} \|f_n(a) - f_n(b)\|_1 &= 4^{n-1}(\|1_{\theta(\overline{a})} - 1_{\phi(\overline{b})}\|_1 + 1) \\ &= 4^{n-1}(|\theta(\overline{a})| + |\phi(\overline{b})| - 2|\theta(\overline{a}) \cap \phi(\overline{b})| + 1) \\ &= 4^{n-1}(|\theta(\overline{a})| + |\phi(\overline{b})| - 2|\theta(\overline{a})||\phi(\overline{b})| + 1) \end{aligned}$$

(since $\theta(\overline{a})$ and $\phi(\overline{b})$ are independent)

$$\geq 4^{n-1}\frac{3}{4}(1 + \min(|\theta(\overline{a})| + |\phi(\overline{b})|, 2 - |\theta(\overline{a})| - |\phi(\overline{b})|))$$

(from Lemma 50 with $s = |\theta(\overline{a})|$ and $t = |\phi(\overline{b})|$)

$$= \frac{3}{4}d(a,b).$$

Case 4. Suppose $a \in D, b \in E$. This is essentially the same as Case 3. As in Case 3, we obtain

$$\|f_n(a) - f_n(b)\|_1 \geq \frac{3}{4} d(a,b).$$

Case 5. Suppose $a \in E, b \in F$. This is very similar to Case 2. Note that

$$\begin{aligned}\|f_n(a) - f_n(b)\|_1 &= 4^{n-1}((1 - |\theta(\overline{a})|) + (1 - |\phi(\overline{b})|)) \\ &= d(a,B) + d(b,B) \\ &= d(a,b).\end{aligned}$$

$\square$

Remark 51. *The analogue of Theorem 49 for D_n is proved in [31, Theorem 5.1] with the same distortion of* $4/3$.

The next result shows that the distortion of any embedding of $\mathcal{L}_2$ into $L_1[0,1]$ is at least $9/8$.

Theorem 52. *Let $f\colon \mathcal{L}_2 \to L_1[0,1]$ satisfy*

$$d(a,b) \leq \|f(a) - f(b)\|_1 \leq c d(a,b).$$

Then $c \geq 9/8$.

The proof uses the following result about hypermetric and negative type inequalities from [9].

Theorem 53. ***B** [9, Lemma 6.1.1] Let (M, ρ) be a finite metric space which is isometric to a subset of $L_1[0,1]$. Then, for all $k_i \in \mathbb{Z}$ $(1 \leq i \leq n)$ such that $\sum_{i=1}^n k_i = 0$ (negative type inequalities) or $\sum_{i=1}^n k_i = 1$ (hypermetric inequalities), we have*

$$\sum_{1 \leq i < j \leq n} k_i k_j \rho(x_i, x_j) \leq 0,$$

where $x_1, \dots, x_n$ are the distinct elements of M.

Proof of Theorem 52. Consider the two choices of weights for $\mathcal{L}_1$ indicated in Figure 5.5 (each weight is shown next to its corresponding vertex). Now define weights for $\mathcal{L}_2$ by assigning the P weights to the C and F copies of $\mathcal{L}_1$, the N weights to the D and E copies, and zero weights to the Y and Z

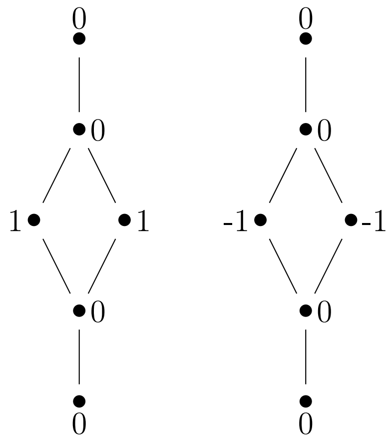

Figure 5.5: Weights P (left) and N (right) for $\mathcal{L}_1$

copies. Let $(k_i)_{i=1}^{30}$ be the enumeration of these weights corresponding to some enumeration of the vertices of $\mathcal{L}_2$. Note that $\sum_{i=1}^{30} k_i = 0$. By Theorem B,

$$\begin{aligned} 72 &= \sum_{i<j, k_i k_j > 0} k_i k_j d(x_i, x_j) \\ &\le \sum_{i<j, k_i k_j > 0} k_i k_j \|f(x_i) - f(x_j)\|_1 \\ &\le \sum_{i<j, k_i k_j < 0} |k_i k_j| \, \|f(x_i) - f(x_j)\|_1 \\ &\le c \sum_{i<j, k_i k_j < 0} |k_i k_j| \, d(x_i, x_j) \\ &= 64c. \end{aligned}$$

So $c \geq 9/8$. □

In a similar way we can estimate the distortion of metric embeddings of the diamond graph D_2 into $L_1[0,1]$.

Theorem 54. *Let $f\colon D_2 \to L_1[0,1]$ satisfy*

$$d(a,b) \leq \|f(a) - f(b)\|_1 \leq c d(a,b).$$

Then $c \geq 5/4$.

Proof. Consider the weights on D_1, denoted again P and N, obtained from Figure 5.5 by removing the A and U vertices of $\mathcal{L}_1$. Now define weights on

D_2 by assigning P to one pair of 'opposite' copies of D_1 and N to the other pair. Let $(k_i)_{i=1}^{12}$ be an enumeration of these weights corresponding to some enumeration of the vertices of D_2. Note that $\sum_{i=1}^{12} k_i = 0$. Using Theorem B as above yields

$$40 = \sum_{i<j, k_i k_j > 0} k_i k_j d(x_i, x_j) \leq c \sum_{i<j, k_i k_j < 0} |k_i k_j| d(x_i, x_j) = 32c.$$

So $c \geq 5/4$. □

Remark 55. *A computer search revealed that $c = 5/4$ is the best estimate of the lower bound for the distortion of D_2 that can be obtained from the negative type and hypermetric inequalities of Theorem B by considering all possible choices of k_i in the range $-10 \leq k_i \leq 10$, and that $c = 9/8$ is the best that can be obtained for $\mathcal{L}_2$ by considering all possible choices of k_i in the range $-1 \leq k_i \leq 1$. Actually, we could not find any embedding of D_2 into $L_1[0,1]$ with distortion smaller than $4/3$, but were not able to prove that $4/3$ is optimal.*

Remark 56. *Since the proofs of Theorems 52 and 54 used only negative type inequalities, by [9, Theorem 6.2.2] they remain valid if $L_1[0,1]$ is replaced by $(\ell_2, \|\cdot\|_2^2)$. This is a stronger result as $L_1[0,1]$ is isometric to a subset of $(\ell_2, \|\cdot\|_2^2)$ (see e.g., [37, p. 20]).*

We thank Mikhail Ostrovskii for his comments and for drawing some references to our attention.

5.3 Code

As mentioned in remark 55, a computer program was used to search for the best possible estimates for the distortion of D_2 and $\mathcal{L}_2$.

The design of the computer program is based on four separate parts: a function to optimize, a parameterized space of possible k_i coefficients, a distance function between the vertices x_i to which the weights k_i correspond, and a search algorithm to go through that space in order to find the biggest possible value of said function. This basic design was adapted for D_2 and $\mathcal{L}_2$, but was also tried on D_3, to a similar success.

The function to optimize was chosen as per the needs of the proof of Theorem 52:

$$f(k_1, k_2, \ldots, k_n) = \frac{\sum_{i<j, k_i k_j > 0} k_i k_j d(x_i, x_j)}{\sum_{i<j, k_i k_j < 0} |k_i k_j|\, d(x_i, x_j)}$$

The parametrized space for the coefficients was chosen to be simply all $k_i \in \mathbb{Z}$ so that $\|k_i\| \leq N$ for a given hardcoded value N.

For the sake of runtime efficiency, the distance function was implemented as a static matrix that was either hardcoded or populated at the beginning of the program run for either of the graphs that were tried.

As for the search algorithm itself, the basic design that was chosen is a full search of all possibilities within the given parameterized space. Other options for algorithms for more efficient searches were considered, however two considerations were taken in mind. The first consideration, of course, is that no computer program would be able to search through all possibilities. We only need the covered space of the search to be "large enough".

And indeed, the example given in Figure 5.5 is not the only one that the computer program returned as a possibility for reaching the optimal function value of 9/8. There were also other non-homogeneous examples. The results of the search seem to suggest that 9/8 is indeed the optimal value that can be found. Very similar results were reached for D_2 and its optimal value of 5/4. This suggested that enlarging the search space beyond what was already achieved would not provide any additional benefit.

A complete mathematical proof whether the values of 5/4 for D_2 and 9/8 for $\mathcal{L}_2$ are the optimal values of the chosen function in the entire space of possible weights seems not impossible to achieve. However it is beyond the scope of the current work.

The second consideration was readability. Full search of all possibilities within the given parameterized space is a very easy to implement algorithm, and vice versa, very easy to understand once written. This way any potential bugs in the computer program are avoided, and one can have reasonable trust that the program is doing what it is supposed to do.

Once this overall structure of the computer program was chosen, it was implemented in the programming language C++ for the diamond graphs D_2 and D_3, and also for the Laakso graph $\mathcal{L}_2$. Additionally, larger coefficient spaces were also tried, with some parallelization implemented so that the search can run on many CPU threads at the same time.

We will provide the code for D_2 and $\mathcal{L}_2$ without any parallelization in the following two sections.

5.3.1 Program code for the diamond graph D_2

For the purposes of the computer program, the vertices in D_2 were numbered from 1 to 12 as per Figure 5.6. The distance function between these vertices was hardcoded directly.

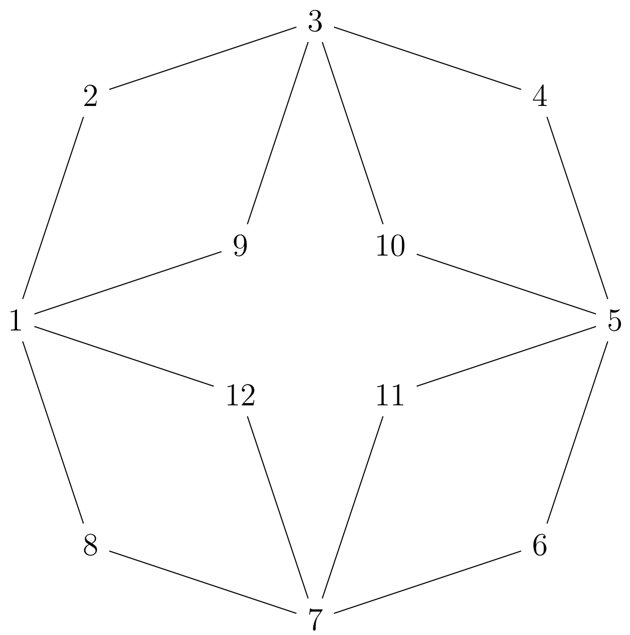

Figure 5.6: Numbering of the vertices in the Diamond graph D_2

```
#include <iostream>
using namespace std;

//Program will iterate k[i] within [-N,N]
#define N 5
//this is the number of vertices, they are numbered from 1 to M
#define M 12

//Distance function for D_2
int d[16][16] = {
  //  1  2  3  4  5  6  7  8  9  10 11 12 13 14 15
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 1, 2, 3, 4, 3, 2, 1, 1, 3, 3, 1, 0, 0, 0},
  {0, 1, 0, 1, 2, 3, 4, 3, 2, 2, 2, 4, 2, 0, 0, 0},
  {0, 2, 1, 0, 1, 2, 3, 4, 3, 1, 1, 3, 3, 0, 0, 0},
  {0, 3, 2, 1, 0, 1, 2, 3, 4, 2, 2, 2, 4, 0, 0, 0},
  {0, 4, 3, 2, 1, 0, 1, 2, 3, 3, 1, 1, 3, 0, 0, 0},
  {0, 3, 4, 3, 2, 1, 0, 1, 2, 4, 2, 2, 2, 0, 0, 0},
  {0, 2, 3, 4, 3, 2, 1, 0, 1, 3, 3, 1, 1, 0, 0, 0},
  {0, 1, 2, 3, 4, 3, 2, 1, 0, 2, 4, 2, 2, 0, 0, 0},
  {0, 1, 2, 1, 2, 3, 4, 3, 2, 0, 2, 4, 2, 0, 0, 0},
  {0, 3, 2, 1, 2, 1, 2, 3, 4, 2, 0, 2, 4, 0, 0, 0},
  {0, 3, 4, 3, 2, 1, 2, 1, 2, 4, 2, 0, 2, 0, 0, 0},
  {0, 1, 2, 3, 4, 3, 2, 1, 2, 2, 4, 2, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
};


//Function for maximization
double f(const int * k){
  //f(k) = p/q;
  int p = 0, q = 0, temp = 0;
  for(int i = 1; i<M; i++){
    for (int j = i+1; j<=M; j++){
      temp = k[i]*k[j];
      if(temp > 0) {
        p += temp*d[i][j];
      }
      else {
        q += (-1)*temp*d[i][j];
      }
    }
  }
  if (p==0 || q == 0) return 0;
  return (double)p/q;
}
```

```
//Print function
void print_coefficients(const double & c, const int * k){
  cout<<"c = "<<c<<" for coefficients";
  for(int i = 1; i<=M; i++)
    cout<<" "<<k[i];
  cout<<endl;
  return;
}

void iterate(double & c, int * k, int curr_coefficient){
  if (curr_coefficient == M){
    //compute for this set of coefficients
    k[M] = 0;
    for(int i = 1; i<M; i++){
      k[M] -= k[i];
    }
    if(k[M]<-N || k[M]>N) return;
    double temp;
    temp = f(k);
    if(temp > c){
      c = temp;
      print_coefficients(c, k);
    }
  }
  else { //iterate this coefficient between -N and N
    //Print progress
    if(curr_coefficient == 3)
      cout << "Now on coefficients "<<k[1]<<","<<k[2]<<endl;

    //Choose an upper bound in order to cut down on iterations
    int upper_bound = N;
    //Without loss of generality first coefficient is nonpositive
    if(curr_coefficient == 1) upper_bound = 0;
    //Without loss of generality we can choose one of the side
    // pair in any D_1 subgraph to not be greater than the other
    else if(curr_coefficient ==  9) upper_bound = k[2];
    else if(curr_coefficient == 10) upper_bound = k[4];
    else if(curr_coefficient == 11) upper_bound = k[6];
    else if(curr_coefficient == 12) upper_bound = k[8];

    //Iterate
    for(k[curr_coefficient] = -N;
        k[curr_coefficient]<=upper_bound;
        k[curr_coefficient]++)
    {
      iterate(c,k,curr_coefficient+1);
    }
  }
  return;
```

```
}

int main(){
  int k[M+1];

  //initializing solution with zeros
  for (int i = 1; i <=M; i++)
    k[i] = 0;
  double c = f(k), temp = 0;
  print_coefficients(c, k);

  //Iterate all k[i] with all integer values between -N and N
  iterate(c,k,1);

  cout<<c<<endl;
  return 0;
}
```

5.3.2 Program code for the Laakso graph $\mathcal{L}_2$

For the purposes of the computer program, the vertices in $\mathcal{L}_2$ were numbered from 1 to 30 as per Figure 5.7. The distance function between these vertices was computed at the start of the run of the computer program based on a hardcoded array containing the edges in $\mathcal{L}_2$.

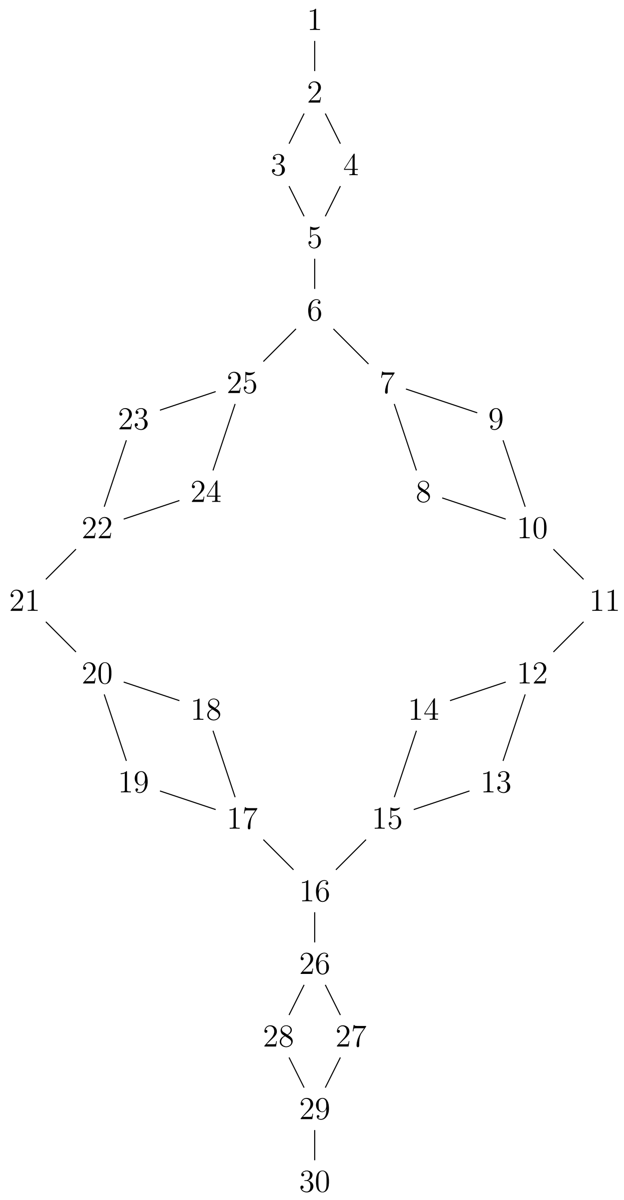

Figure 5.7: Numbering of the vertices in the Laakso graph $\mathcal{L}_2$

```
#include <iostream>
using namespace std;

//Program will iterate k[i] within [-N,N]
#define N 1
//this is the number of vertices, they are numbered from 1 to M
#define M 30

//Distance function for full L_2
int d[M+1][M+1];
//Pairs of edges in L_2
int edges_l_2[36][2] = {
        {1,2},
        {2,3},
        {2,4},
        {3,5},
        {4,5},
        {5,6},
        {6,7},
        {7,8},
        {7,9},
        {8,10},
        {9,10},
        {10,11},
        {11,12},
        {12,13},
        {12,14},
        {13,15},
        {14,15},
        {15,16},
        {16,17},
        {17,18},
        {17,19},
        {18,20},
        {19,20},
        {20,21},
        {21,22},
        {22,23},
        {22,24},
        {23,25},
        {24,25},
        {25,6},
        {16,26},
        {26,27},
        {26,28},
        {27,29},
        {28,29},
        {29,30}
};
```

```
//Function for maximization
double f(const int * k){
  //f(k) = p/q;
  int p = 0, q = 0, temp = 0;
  for(int i = 1; i<M; i++){
    for (int j = i+1; j<=M; j++){
      temp = k[i]*k[j];
      if(temp > 0) {
        p += temp*d[i][j];
      }
      else {
        q += (-1)*temp*d[i][j];
      }
    }
  }
  if (p==0 || q == 0) return 0;
  return (double)p/q;
}

//populate distance matrix
void populate_distance(int d[M+1][M+1], int edges[][2],
                                                        int edges_count){
  int i,j,k;
  //zero out the matrix
  for(i=0; i<=M; i++)
    for(j=0; j<=M; j++)
      d[i][j] = 0;

  //handle initial distances
  for(i=0; i<edges_count; i++){
    d[edges[i][0]][edges[i][1]]=1;
    d[edges[i][1]][edges[i][0]]=1;
  }

  //propagate distances log2(M) times
  for(int iter = 0; iter < 5; iter++)
    for(i = 1; i<=M; i++)
      for(j = i+1; j<=M; j++)
        for(k = 1; k<=M; k++)
          if(d[i][k] != 0 && d[k][j] != 0)
            if((d[i][k]+d[k][j] < d[i][j]) || d[i][j] == 0)
            {
              d[i][j] = d[i][k]+d[k][j];
              d[j][i] = d[i][j];
            }

  return;
}
```

```
//Print function
void print_coefficients(const double & c, const int * k){
  cout<<"c = "<<c<<" for coefficients";
  for(int i = 1; i<=M; i++)
  cout<<" "<<k[i];
  cout<<endl;
  return;
}

void iterate(double & c, int * k, int curr_coefficient){
  if (curr_coefficient == M){
    //compute for this set of coefficients
    k[M] = 0;
    for(int i = 1; i<M; i++){
      k[M] -= k[i];
    }
    if(k[M]<-N || k[M]>N) return;
    double temp;
    temp = f(k);
    if(temp > c){
        c = temp;
        print_coefficients(c, k);
    }
  }
  else { //iterate this coefficient between -N and N
    //Choose lower upper bounds in order to cut down iterations.
    int upper_bound = N;
    //WLOG, first coefficient is non-positive.
    if(curr_coefficient == 1) upper_bound = 0;
    //WLOG, we can choose one of the side pairs
    // in any L_1 subgraph to be not greater than the other.
    else if(curr_coefficient == 4) upper_bound = k[3];
    else if(curr_coefficient == 9) upper_bound = k[8];
    else if(curr_coefficient == 14) upper_bound = k[13];
    else if(curr_coefficient == 19) upper_bound = k[18];
    else if(curr_coefficient == 24) upper_bound = k[23];
    else if(curr_coefficient == 28) upper_bound = k[27];

    //Iterate
    for(k[curr_coefficient] = -N;
        k[curr_coefficient]<=upper_bound;
        k[curr_coefficient]++)
    {
      iterate(c,k,curr_coefficient+1);
    }
  }
  return;
}
```

```
int main(){
  int k[M+1];
  //initialize distance matrix
  populate_distance(d, edges_l_2, 36);

  //initializing solution with zeros
  for (int i = 1; i <=M; i++)
  k[i] = 0;
  double c = f(k), temp = 0;
  print_coefficients(c, k);

  //Iterate all k[i] with all of the integer values
  // between -N and N
  iterate(c,k,1);

  cout<<c<<endl;
  return 0;
}
```

Chapter 6

Conclusion

6.1 Main contributions

1. We prove Theorem 29: Let $1 < p < \infty$ and $\gamma > 0$ be sufficiently small. Every subsymmetric basic sequence in the dual space $Ti^*(p, \gamma)$ is equivalent to the subsymmetric canonical basis $(e_j^*)_{j=1}^{\infty}$ which is not symmetric. In other words, these spaces whose canonical basis is subsymmetric have, up to equivalence, a unique subsymmetric basic sequence which is not symmetric.

Towards the proof of this main result we also obtain the following statements that are interesting on their own:

Lemma 25: Let (e_i) be a 1-unconditional basis of a reflexive Banach space X which is K-dominated by its normalized block bases, where $K \geq 1$. Then (e_i^*) K-dominates all normalized block bases of (e_i^*) in the dual space X^*.

Lemma 27: $Ti^*(p, \gamma)$ does not contain an isomorphic copy of ℓ_q (where $\frac{1}{p} + \frac{1}{q} = 1$).

As applications of the main theorem we get:

Corollary 30: Let $1 < p < \infty$ and $\gamma > 0$ be sufficiently small. Every subsymmetric basis of a quotient space of $Ti(p, \gamma)$ is equivalent to the canonical basis $(e_j)_{j=1}^{\infty}$.

Corollary 32: For $1 < p < \infty$ and sufficiently small γ, the basis (e_i) of $Ti(p, \gamma)$ has a continuum many non-equivalent subsymmetric block bases.

2. We construct "yardsticks" in Tirilman spaces. More precisely, we prove Theorem 34: Let p and γ be such that $1 < p < \infty$ and $0 < \gamma < 3^{-\frac{1}{q}}$ where $\frac{1}{p} + \frac{1}{q} = 1$. Then c_0 is finitely representable in $Ti\,(p, \gamma)$ disjointly with respect to its canonical basis. This provides an alternative proof that the canonical basis of a Tirilman space is not symmetric.

$(\ell_\infty^n)_{n=1}^{\infty}$ are universal for all finite dimensional Banach spaces, that is to say, for every finite dimensional space X there exists an n such that X can be isometrically embedded in ℓ_∞^n. Thus, the finite dimensional structure of $Ti(p,\gamma)$ for $1 < p < \infty$, $\gamma < 3^{-\frac{1}{q}}$ is as rich as possible in the sense that every finite dimensional space is $1+\epsilon$ embeddable in $Ti(p,\gamma)$ for arbitrary $\epsilon > 0$. The last follows from the theorem of James that c_0 is not distortable, and its finite dimensional version.

3. We prove Theorem 33: The symmetrized version of the dual of Schlumprecht's space, $S(S^*)$, contains a subspace isomorphic to ℓ_1.

4. We study bilipschitz embeddings of Laakso graphs into Banach spaces. We estimate the distortion of such embeddings. More precisely, we prove Theorem 47: Suppose X is not super-reflexive. Then, for each $\varepsilon > 0$ and $n \geq 1$, there exists a mapping $f_n \colon \mathcal{L}_n \to X$ such that, for all $a, b \in \mathcal{L}_n$,

$$\frac{1}{2}d(a,b) - \varepsilon \leq \|f_n(a) - f_n(b)\| \leq d(a,b).$$

We prove a stronger result in the particular case when the Banach space is $L[0,1]$, namely Theorem 49: For each $n \geq 1$, there exists a mapping $f_n \colon \mathcal{L}_n \to L_1[0,1]$ such that, for all $a, b \in \mathcal{L}_n$,

$$\frac{3}{4}d(a,b) \leq \|f_n(a) - f_n(b)\|_1 \leq d(a,b)$$

Bilipschitz embedding into $L[0,1]$ are important in Computer Science.

5. We create a computer program to find a lower bound of the distortion of embedding Laakso and diamond graphs into $L[0,1]$. In this way we obtain:

Theorem 52: Let $f \colon \mathcal{L}_2 \to L_1[0,1]$ satisfy

$$d(a,b) \leq \|f(a) - f(b)\|_1 \leq c d(a,b).$$

Then $c \geq 9/8$.

Theorem 54: Let $f \colon D_2 \to L_1[0,1]$ satisfy

$$d(a,b) \leq \|f(a) - f(b)\|_1 \leq c d(a,b).$$

Then $c \geq 5/4$.

Since the proofs of Theorems 52 and 54 used only negative type inequalities, by [9, Theorem 6.2.2] they remain valid if $L_1[0,1]$ is replaced by $(\ell_2, \|\cdot\|_2^2)$. This is a stronger result as $L_1[0,1]$ is isometric to a subset of $(\ell_2, \|\cdot\|_2^2)$.

6.2 Publications related to the thesis

1. Stephen J. Dilworth, Denka Kutzarova, Bünyamin Sarı, Svetozar Stankov, Duals of Tirilman spaces have unique subsymmetric basic sequences, Bulletin of the London Mathematical Society Volume 56, 150-158, https://doi.org/10.1112/blms.12920

2. S. J. Dilworth, Denka Kutzarova, Svetozar Stankov, Metric embeddings of Laakso graphs into Banach spaces, Banach Journal of Mathematical Analysis 16 (2022), no. 4, Paper No. 60, 14 pp., http://doi.org/10.1007/s43037-022-00212-7

3. Svetozar Stankov, On the symmetrized dual of Schlumprecht's space, C. R. Acad. Bulg. Sci., 78, No 1, 2025, https://doi.org/10.7546/CRABS.2025.01.02

6.3 Approbation of the thesis

The results from the thesis have been presented in the following talks:

1. Stephen J. Dilworth, Denka Kutzarova, Svetozar Stankov, Metric embeddings of Laakso graphs into Banach spaces, Week of Mathematics and Informatics 2024, September 23-27, 2024, Duni Royal Resort, Bulgaria, https://www.fmi.uni-sofia.bg/bg/wmi-2024-program

2. Stephen. J. Dilworth, Denka Kutzarova, Svetozar Stankov, Metric embeddings of Laakso graphs into Banach spaces, Annual Scientific Session of Analysis, Geometry and Topology Department, December 5, 2024, IMI-BAS, https://math.bas.bg/event/%d0%be%d1%82%d1%87%d0%b5%d1%82%d0%bd%d0%b0-%d1%81%d0%b5%d1%81%d0%b8%d1%8f-%d0%bd%d0%b0-%d1%81%d0%b5%d0%ba%d1%86%d0%b8%d1%8f-%d0%b0%d0%bd%d0%b0%d0%bb%d0%b8%d0%b7-%d0%b3%d0%b5%d0%be%d0%bc%d0%b5%d1%82/

6.4 Declaration of originality

The author declares that the thesis contains original results obtained by him in cooperation with the scientific advisor. The usage of results of other scientists is accompanied by suitable citations.

6.5 Acknowledgements

First I would like to express my sincere gratitude to the Analysis, Geometry and Topology Department of the Institute of Mathematics and Informatics of the Bulgarian Academy of Sciences for the significant help I have gotten from them over the years - not only during my PhD program but ever since high school. Of course, I owe my thanks to the entire Institute of Mathematics and Informatics as a whole. The Operations Research, Probability and Statistics Department bears specific mention, not only for helping me with starting my PhD program but for all the guidance over the years.

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